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Some Alternative Theorems of Set-Valued Maps and their Applications

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Abstract. We establish some theorems for a certain minimization problem whose constraints are presented by set-valued maps. For this, we prove two alternative theorems for set-valued maps. By using those theorems, we show some theorems for this minimization problem.

Key Words. Mathematical programming, set-valued analysis, convex analysis, convexity of set-valued maps, continuity of set-valued maps, alternative theorem.

1. Introduction and Preliminaries

我々は，集合値写像を用いて表される次の問題 (P) を考える：

$$\text{(P)} \qquad \text{minimize} \quad f(x) \qquad \text{subject to} \quad F(x) \cap (-P) \neq \emptyset$$

ただし，X : 実ベクトル空間，C : X の空でない凸集合，Y : 実線形位相空間，P : Y の凸錐，$f : C \longrightarrow \boldsymbol{R}$，$F : C \rightsquigarrow Y$.

この問題 (P) は，従来の不等式制約型の問題：

$$\text{(P}'\text{)} \qquad \text{minimize} \quad f(x) \qquad \text{subject to} \quad g_i(x) \leq 0, \quad i = 1,\ 2,\ \ldots,\ n$$

(ただし，$g_i : C \longrightarrow \boldsymbol{R}$，$i = 1, 2, \ldots, n$) を含み，さらに，問題 (P′) に定式化されないような問題も (P) においては扱うことができる.

本論文における目的は，問題 (P) について考察することである．具体的には，

(1) (P) の双対問題 (D) を考える.

(2) (P) と (D) の値が等しくなるような条件を求める.

などを考察するが，このときに非常に重要な役割を果たすのが，二者択一の定理 (alternative theorem) である．二者択一の定理の古典的な例としては，Gordan の定理，Farkas の定理などがあり，いずれも応用する上で，非常に有用な定理である.

そこで，二者択一の条件を定式化し，どのような条件の下で，二者択一の定理が成立するのかを観察していく.

まず，(P) の双対問題 (D) を次のように定義する.

$$\begin{array}{lll} \text{(D)} & \text{maximize} & \phi(y^*) \\ & \text{subject to} & y^* \in P^+ \end{array}$$

ただし，$\phi(y^*) \equiv \inf_{(x,y)\in \mathrm{Graph}(F)} \{f(x) + \langle y^*, y\rangle\}$，$P^+ \equiv \{y^* \in Y^* | \langle y^*, y\rangle \geq 0,\ \forall y \in P\}$.
このとき，次が成立する．

Proposition 1.1. (Weak Duality)

$$\mathrm{val(D)} \leq \mathrm{val(P)}.$$

この等号を成立させる条件の一つが，関数の凸性である．従って，次の章においては，集合値関数の凸性を定義する．

2. Convexity of Set-Valued Maps and their Relations

この章では，集合値写像の凸性をいくつか定義し，それらの間にある関係について述べていく．集合値写像の凸性は，ベクトル値関数の凸性を基にして定義する．その拡張の方法は，いくつかの方法がある．[4]

Definition 2.1. A set-valued map $F : C \rightsquigarrow Y$ is said to be

(i) *convex* if for every $x_1, x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, there exists $y \in F(\lambda x_1 + (1-\lambda)x_2)$ such that $y \leq_P \lambda y_1 + (1-\lambda)y_2$;

(ii) *convexlike* if for every $x_1, x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, there exists $(x, y) \in \mathrm{Graph}(F)$ such that $y \leq_P \lambda y_1 + (1-\lambda)y_2$;

(iii) *properly quasiconvex* if for every $x_1, x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, there exists $y \in F(\lambda x_1 + (1-\lambda)x_2)$ such that either $y \leq_P y_1$ or $y \leq_P y_2$;

(iv) *quasiconvex* if for every $x_1, x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, if $y \in Y$ satisfies $y_1 \leq_P y$ and $y_2 \leq_P y$, then there exists $y' \in F(\lambda x_1 + (1-\lambda)x_2)$ such that $y' \leq_P y$;

(v) *naturally quasiconvex* (c.f. [7]) if for every $x_1, x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, there exists $y \in F(\lambda x_1 + (1-\lambda)x_2)$ and $\eta \in [0,1]$ such that $y \leq_P \eta y_1 + (1-\eta)y_2$;

(vi) $*$-*quasiconvex* (c.f. [3]) if for each $y^* \in P^+$, function $x \longmapsto \inf_{y\in F(x)} \langle y^*, y\rangle$ is quasiconvex on C.

ただし，$y_1 \leq_P y_2 \iff y_2 - y_1 \in P$.
これらの集合値写像の凸性に関して，次が成立する．[4]

Proposition 2.1. *The following statements hold:*

(i) *F is convex if and only if* $\mathrm{Graph}(F) + \{\theta_X\} \times P$ *is a convex set;*

(ii) *F is convexlike if and only if* $F(C) + P$ *is a convex set;*

(iii) *F is quasiconvex if and only if for all* $y \in Y$, *the set* $F^{-1}(y - P)$ *is a convex set.*

ただし，$F^{-1}(M) \equiv \{x \in C | F(x) \cap M \neq \emptyset\}$; $F^{+1}(M) \equiv \{x \in C | F(x) \subset M\}$.

Proposition 2.2. *The following statements hold:*

(i) *every convex map is also convexlike;*

(ii) *every convex map is also naturally quasiconvex;*

(iii) *properly quasiconvex map is also naturally quasiconvex;*

(iv) *naturally quasiconvex map is also quasiconvex;*

(v) *naturally quasiconvex map is also* $*$*-quasiconvex.*

Theorem 2.1. *Assume that* Y *is a locally convex space and* $F(x) + P$ *is closed convex for all* $x \in C$. *If* F *is* $*$*-quasiconvex, then* F *is also naturally quasiconvex.*

Theorem 2.2. *If We assume that* P *is closed and* F *is upper semicontinuous and convex valued. If* F *is naturally quasiconvex then it is convexlike.*

3. Alternative Theorems for Some Set-Valued Maps

この章では，２つの二者択一の定理を示す．これらの定理は，最適化問題を解く上で，非常に重要であり，この論文の主定理の主な道具である．まず，最初の二者択一の定理で使われる条件を述べる．

(A1) $Q \neq \emptyset$;

(A2) Q is open;

(A3) F is convexlike,

where $Q \equiv \{y \in Y | \langle y^*, y\rangle > 0, \ \forall y^* \in P^+ \setminus \{\theta_{Y^*}\}\}$.

Remark 3.1. *It is easy to show that* $\mathrm{int}P \subset Q$, *and if* $\mathrm{int}P \neq \emptyset$, $\mathrm{int}P = Q$. *Also, assumption* **(A2)** *is fulfilled when the function* $(y^*, y) \mapsto \langle y^*, y\rangle$ *is continuous in* $\sigma(Y^*, Y) \times \mathcal{O}_Y$, *where* $\mathcal{O}_Y$ *is the topology of* Y. *We recall that this continuity is satisfied if* Y *is a normed space.*

このとき，次の定理を得る．

Theorem 3.1. *Under the assumptions* **(A1)**, **(A2)**, *and* **(A3)**, *exactly one of the following statements* (i) *and* (ii) *is true :*

(i) *there exists* $x_0 \in C$ *such that* $F(x_0) \cap (-Q) \neq \emptyset$;

(ii) *there exists* $y_0^* \in P^+ \backslash \{\theta_{Y^*}\}$ *such that for any* $(x, y) \in \mathrm{Graph}(F)$, $\langle y^*, y \rangle \geq 0$.

Remark 3.2. *If F is a vector-valued map and* $\mathrm{int}P \neq \emptyset$, *then Theorem 3.1 becomes Lemma 2.1 of [2].*

次に，2つめの二者択一の定理を述べる．そこで使われる条件を述べるために，まず，集合値写像のある連続性を定義する．

Definition 3.1. A set-valued map $F : C \rightsquigarrow Y$ is said to be *$*$-lower semicontinuous* ($*$-l.s.c.) at $x \in C$ if for any $y^* \in P^+$, the function $z \longmapsto \inf_{y \in F(z)} \langle y^*, y \rangle$ is lower semicontinuous at x. F is said to be $*$-lower semicontinuous if and only if it is $*$-lower semicontinuous at every point of C.

Remark 3.3. *Every upper-semicontinuous set-valued map is also $*$-lower semicontinuous.*

(B1) X is a topological vector space;

(B2) Y is a locally convex space;

(B3) P^+ has a w*-compact convex base D;

(B4) F is $*$-quasiconvex on C;

(B5) F is $*$-lower semicontinuous on C.

Remark 3.4. *In* **(B3)**, P^+ *has a w*-compact convex base* D, *means that there exists a w*-compact convex subset* D *of* Y^* *such that* $\theta_{Y^*} \notin D$ *and* $P^+ = \bigcup_{\lambda \geq 0} \lambda D$. *Assumption* **(B3)** *is satisfied when* $\mathrm{int}P \neq \emptyset$, *see [3].*

このとき，次の定理を得る．

Theorem 3.2. *Under the assumptions* **(B1)**, **(B2)**, **(B3)**, **(B4)**, *and* **(B5)**, *exactly one of the following statements* (i) *and* (ii) *is true:*

(i) *there exists* $x_0 \in C$ *such that for any* $y^* \in P^+ \backslash \{\theta_{Y^*}\}$, $\inf_{y \in F(x_0)} \langle y^*, y \rangle < 0$;

(ii) *there exists* $y_0^* \in P^+ \backslash \{\theta_{Y^*}\}$ *such that for any* $x \in C$, $\inf_{y \in F(x)} \langle y_0^*, y \rangle \geq 0$.

Remark 3.5. *If F is a vector-valued map, then Theorem 3.2 becomes Theorem 2.1 of [3].*

4. Applications to Optimization Problem

この章では，最初に与えた問題 (P) に対して，前章における Theorem 3.1, Theorem 3.2 を適用して，その双対問題 (D) との関連を調べていく．まず，1章で述べた Weak Duality を証明する．

Proof of Proposition 1.1. For each $y^* \in P^+$,

$$\begin{aligned}\mathrm{val(P)} &= \inf_{x \in F^{-1}(-P)} f(x)\\ &\geq \inf_{x \in F^{-1}(-P)} \{f(x) + \langle y^*, y\rangle\} \quad (\forall y \in F(x) \cap (-P))\\ &\geq \inf_{(x,y) \in \mathrm{Graph}(F)} \{f(x) + \langle y^*, y\rangle\}\\ &= \phi(y^*).\end{aligned}$$

Hence,

$$\mathrm{val(P)} \geq \sup_{y^* \in P^+} \{\phi(y^*)\} = \mathrm{val(D)}.$$

This completes the proof. □

次に，主問題 (P) の値が，その双対問題 (D) の値に一致するための条件について考察していく．まず，問題 (P) に対して，拡張された Slater condition を定義する．

(AS) $F^{-1}(-Q) \neq \emptyset$;

(BS) there exists $x_0 \in C$ such that for any $y^* \in P^+ \setminus \{\theta_{Y^*}\}$, $\inf_{y \in F(x_0)} \langle y^*, y\rangle < 0$.

Remark 4.1. *If F is a vector-valued map, then condition* **(BS)** *becomes the generalized Slater condition in [3]. Moreover* int$P \neq \emptyset$, *then condition* **(AS)** *becomes the Slater condition in [2].*

条件 **(AS)** と **(BS)** の間には，次のような関係がある．

Proposition 4.1. *For each problem* (P),

(i) *if* **(AS)** *is satisfied, then* **(BS)** *is also satisfied;*

(ii) *if* **(BS)** *is satisfied and for each* $x \in C$, $F(x) + P$ *is closed convex, then* **(AS)** *is also satisfied;*

(iii) *if conditions* **(BS)**, **(A1)**, **(A2)**, *and* **(A3)** *are satisfied, then* **(AS)** *is also satisfied;*

また，次のように条件 **(A3')**, **(B4')**, **(B5')**, を置き直す．

(A3') (f, F) is convexlike;

(B4') (f, F) is $*$-quasiconvex on C;

(B5') (f, F) is $*$-lower semicontinuous on C, where (f, F) is the set-valued map from C to $\boldsymbol{R} \times Y$ defined by $(f, F)(x) \equiv (\{f(x)\}, F(x))$ for each $x \in C$. In this case, we consider $\boldsymbol{R}_+ \times P$ as the convex cone in **(A3')**, and $(\boldsymbol{R}_+ \times P)^+ = \boldsymbol{R}_+ \times P^+$ as the positive polar cone in **(B4')** and **(B5')**.

さらに，次の条件 **(B6)** を定義する．

(B6) $F(x) + P$ is closed convex for any $x \in C$.

Remark 4.2. *From Theorem 2.1, we have the following: under assumption* **(B6)***, condition* **(B4')** *holds if and only if* (f, F) *is naturally quasiconvex on* C.

このとき，Theorem 3.1, Theorem 3.2 より，次の主定理を得る．

Theorem 4.1. *For problem* (P)*, assume that one of the following assumptions:*

(i) **(AS)**, **(A1)**, **(A2)**, *and* **(A3')** *are satisfied;*

(ii) **(BS)**, **(B1)**, **(B2)**, **(B3)**, **(B4')**, **(B5')**, *and* **(B6)** *are satisfied.*

Then val(P) = val(D)*, and there exists* $y_0^* \in P^+$ *such that* $\phi(y_0^*) = \text{val(D)}$*. Moreover, if there exists* $x_0 \in C$ *such that* val(P) $= f(x_0)$ *and* $x_0 \in F^{-1}(-P)$*, then* $\langle y_0^*, y \rangle = 0$ *for all* $y \in F(x_0) \cap (-P)$.

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