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An Equilibrium Theorem for Subdifferential

by

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1 Introduction

我々は ある制約条件のもとで目的関数 f を最小化する数理計画問題に対して，特に凸関数 $f: X \to \mathbb{R}$ における凸最適化問題を考える．このとき，この問題を考えることは 制約集合と目的関数により表された関数の劣微分が X の共役空間 X^* の null vector θ^* を含む問題を考えることに置き換えられる．従ってそのような問題を考えることは，凸最適化問題を考える上で重要な問題であり ここでは $\theta^* \in \partial f(x)$ なる解 x の存在について考察する．

2 Equilibrium Theorem

この論文での 主定理を示すにおいて Ky Fan's Inequality が重要な役割を果たす．

Theorem 2.1 (Ky Fan's Inequality) *Let K be a w-compact convex subset of a Banach space X and $\varphi : X \times X \longrightarrow \mathbb{R}$ be a function satisfying :*

(i) $\forall y \in K,\ x \to \varphi(x, y)$ *is w-lower semicontinuous ;*

(ii) $\forall x \in K,\ y \to \varphi(x, y)$ *is concave ;*

(iii) $\varphi(y, y) \leqq 0,$ *for all* $y \in K$.

Then, there exists $\bar{x} \in K$ *such that* $\forall y \in K,\ \varphi(\bar{x}, y) \leqq 0$.

ここで，$\mathfrak{L}(X, X^*)$ は X から X^* への linear, bounded な関数の全体を表し，$T_K(x)$ は x における K の tangent cone のことで，$T_K(x) := \mathrm{cl}(\cup_{h>0}(K - x)/h)$ であるとする．

Definition 2.1 *$\partial f(x)$ is said to satisfy tangential condition with respect to $A \in \mathfrak{L}(X, X^*)$ and a subset $K \subset X$ if*

$$\forall x \in K, \qquad \partial f(x) \cap \mathrm{cl}(AT_K(x)) \neq \phi. \tag{2.1}$$

ただし，

$$\partial f(x) := \{\xi \in X^* | f(x) - f(y) \leqq \langle x - y\ ,\ \xi\rangle\ ,\ \textit{for all}\ y \in X\}.$$

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Theorem 2.2 Kを *Banach* 空間 X の弱コンパクト凸集合とし，実関数 f : $X \to \mathbb{R}$ を X 上で連続かつ凸な写像とする．さらに，$A \in \mathfrak{L}(X, X^*)$ が次を満たすとする．

$$\forall x \in K, \qquad \partial f(x) \cap \mathrm{cl}(AT_K(x)) \neq \phi.$$

このとき，

$$\exists \bar{x} \in K, \qquad s.t. \qquad \theta^*_{X^*} \in \partial f(\bar{x}), \tag{2.2}$$

が成立する．

Proof. We proceed by contradiction, assuming that the conclusion is false. Hence, for any $x \in K$, θ^* does not belong to $\partial f(x)$. Since the sets $\partial f(x)$ are w*-closed and convex, the Hahn-Banach Separation Theorem implies

$$\exists p_x \in X \backslash \{\theta_X\} \qquad such\ that \qquad \sigma(\partial f(x), p_x) < 0,$$

where θ_X is the null vector of X. We set $\Gamma_p := \{\, x \in X \mid \sigma(\partial f(x), p) < 0 \,\}$. Then K is covered by the subsets Γ_p when p ranges over X. These subsets are weak open. So, K can be covered by n such weak open subsets Γ_{p_i}.

Let us consider a continuous partition of unity $\{\alpha_i\}_{i=1,\ldots,n}$ associated with $\{\Gamma_{p_i}\}_{i=1,2,\cdots,n}$ and introduce the function $\varphi\ K \times K \longrightarrow \mathbb{R}$ defined by

$$\varphi(x,y) := \sum_{i=1}^{n} \alpha_i(x) \langle A^* p_i \, , \, x - y \rangle \, .$$

Being continuous with respect to x and affine with respect to y, the assumptions of Theorem 2.1 are satisfied. Hence there exists $\bar{x} \in K$ such that

$$\forall y \in K, \qquad \varphi(\bar{x}, y) = \langle A^* p^* \, , \, \bar{x} - y \rangle \leqq 0,$$

where we have set $p^* := \sum_{i=1}^{n} \alpha_i(\bar{x}) p_i$. In other words, $-A^* p^*$ belongs to the normal cone $N_K(\bar{x})$. The dual tangential condition implies that

$$\sigma(\partial f(\bar{x}), p^*) \geqq 0.$$

But this inequality is false. To see that, we let I be the subset of the indices i such that $\alpha_i(\bar{x}) > 0$. I is non-empty since $\sum_{i=1}^{n} \alpha_i(\bar{x}) = 1$. If i belongs to I, then $\bar{x}$ belongs to Γ_{p_i} and consequently

$$\begin{aligned} \sigma(\partial f(\bar{x}), \bar{p}) &= \sigma(\partial f(\bar{x}), \sum_{i=1}^{n} \alpha_i(\bar{x}) p_i) \\ &\leqq \sum_{i=1}^{n} \alpha_i(\bar{x}) \sigma(\partial f(\bar{x}), p_i) \\ &< 0. \end{aligned}$$

Thus, we have obtained a contradiction and prove our theorem. □

Definition 2.2 X を *Banach* 空間とする．このとき，$x \in X$ に対して，X^* の部分集合

$$J(x) := \{\, x^* \in X^* \mid \langle x \, , \, x^* \rangle = \|x\|^2 = \|x^*\|^2 \,\} \tag{2.3}$$

を対応させる写像 J のことを *duality mapping* と呼ぶ．

Lemma 2.1 X を *reflexive* な *Banach* 空間とする．このとき，$\|x^*\| = 1$ である $x^* \in B^*$ に対して次が成立する．

$$T_{B^*}(x^*) = \bigcap_{y \in J^{-1}(x^*)} \{ p \in X^* \mid \langle p\,,\, y \rangle \leqq 0 \}. \tag{2.4}$$

ただし，B^* は X^* の *unit ball* であり，J^{-1} は *duality mapping* J の逆写像である．

Proof. For any $v^* \in T_{B^*}(x^*)$, there exists a sequence of elements $v_n^* \in (\cup_{h>0}(B^* - x^*)/h)$ converging to v^*. Hence, for any n, there exists $h_n > 0$ and $b_n^* \in B^*$ such that $v_n^* = (b_n^* - x^*)/h_n$. Since $\langle v_n^*\,,\, y \rangle \leqq 0$ for any $y \in J^{-1}(x^*)$, $\langle v^*\,,\, y \rangle \leqq 0$ for any $y \in J^{-1}(x^*)$. Hence,

$$v^* \in \bigcap_{y \in J^{-1}(x^*)} \{ p \in X^* \mid \langle p\,,\, y \rangle \leqq 0 \}.$$

Assume that there exists $w_0^* \notin T_{B^*}(x^*)$ such that $\langle w_0^*\,,\, y \rangle \leqq 0$ for any $y \in J^{-1}(x^*)$. Since the sets $T_{B^*}(x^*)$ are closed and convex, the Hahn-Banach Separation Theorem implie

$$\exists z \in X \backslash \{\theta\}, \qquad \exists a \in \mathbb{R}, \quad s.t. \quad \langle w_0^*\,,\, z \rangle > a > \langle v^*\,,\, z \rangle, \qquad \forall v^* \in T_B^*(x^*).$$

So, we have z belongs to the normal cone $N_{B^*}(x^*)$ and $a > 0$. We set $z' := z/\|z\|$, then we have $z' \in J^{-1}(x^*)$. Hence,

$$\langle w_0^*\,,\, z' \rangle > \frac{a}{\|z\|} > 0.$$

However from assumption $\langle w_0^*\,,\, z' \rangle \leqq 0$. Thus, we have obtained a contradiction and proved our theorem. □

Theorem 2.3 K を *reflexive Banach* 空間 X の閉凸集合とし，実関数 $f : X \to \mathbb{R}$ を X 上で連続かつ凸な写像とし，このとき次の 1∼ 3 を満たす $A \in \mathfrak{L}(X, X^*)$ が存在したとする：

1. $\forall x \in K, \quad \partial f(x) \cap \mathrm{cl}(AT_K(x)) \neq \phi$;
2. $A(B_X) = B^*$;
3. A^{-1} *exists* .

更に 次の仮定

$$\limsup_{\substack{\|x\| \to \infty \\ x \in K}} \sup_{y \in J^{-1}(Ax)} f'(x; y) < 0, \tag{2.5}$$

を満たしていれば，(2.2) が成立する．ここで，$f'(x; d)$ は f の x での d 方向の方向微分であり，$f'(x; d) := \lim_{h \to 0_+} \frac{f(x+hd) - f(x)}{h}$ である．

Proof. Assumption (2.5) implies that there exists $\varepsilon > 0$ and $a > 0$ such that

$$\sup_{\|Ax\| \geqq a} \sup_{y \in J^{-1}(Ax)} \sigma\Big(\partial f(x), y\Big) \leqq -\varepsilon, \qquad and \quad AK \cap \mathrm{int}(A(aB)) \neq \phi. \tag{2.6}$$

By Lemma 2.1, we know that for any Ax belongs to $A(aB)$ with $\|Ax\| = a$ then,

$$T_{A(aB)}(Ax) = \bigcap_{y \in J^{-1}(Ax)} \{\, p \in X^* \mid \langle p\,,\, y\rangle \leqq 0\,\}. \tag{2.7}$$

Hence, from (2.6) and (2.7), it follows that $\forall Ax \in AK \cap A(aB)$,

$$\partial f(x) \subset \mathrm{cl}(AT_{aB}(x)) = T_{A(aB)}(Ax). \tag{2.8}$$

Next, since θ_{X^*} belong to $\mathrm{int}(K + aB)$ from (2.6), we know that

$$\forall Ax \in AK \cap A(aB), \qquad T_{AK \cap A(aB)}(Ax) = T_A K(Ax) \cap T_{A(aB)}(Ax).$$

Hence, by assumtion 3

$$\forall x \in K \cap aB, \qquad \partial f(x) \cap \mathrm{cl}(AT_{K \cap aB}(x)) \neq \phi.$$

So, $K \cap aB$ satisfies the tangential condition (2.1) and obviously to prove that convex and w-compact set.

Hence, by Theorem 2.2 there exists a solution $\bar{x} \in K$ of the inclusion $\theta^* \in \partial f(\bar{x})$. □

Corollary 2.1 X を *Hilbert* 空間，実関数 $f : X \to \mathbb{R}$ は X 上で連続かつ凸である写像とし，$A \in \mathfrak{L}(X, X^*)$ は次を満たしているとする：

$$\limsup_{\|x\| \to \infty} f'(x; Ax) < 0. \tag{2.9}$$

このとき (2.2) が成立する．

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