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マトロイドと凸解析 (Matroid and Convex Analysis)[1]

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1 Introduction

The analogy between convex/concave functions and submodular/supermodular functions attracted research interest in the 80's. Fujishige [4] formulated Edmonds' intersection theorem into a Fenchel-type min-max duality theorem. Frank [3] showed a separation theorem for a pair of submodular/supermodular functions, with integrality assertion for the separating hyperplane in the case of integer-valued functions. This theorem can also be regarded as being equivalent to Edmonds' intersection theorem. A precise statement, beyond analogy, about the relationship between convex functions and submodular functions was made by Lovász [5]. Namely, a set function is submodular if and only if the so-called Lovász extension of that function is convex. This penetrating remark also established a direct link between the duality for convex/concave functions and that for submodular/supermodular functions. The essence of the duality for submodular/supermodular functions is now recognized as the discreteness (integrality) assertion in addition to the duality for convex/concave functions.

In spite of these developments, our understanding of the relationship between convexity and submodularity seems to be only partial. In the convex analysis, a convex function is minimized over a convex domain of definition, which can be described by a system of inequalities in (other) convex functions. In the polyhedral approach to matroid optimization, a linear function is optimized over a (discrete) domain of definition, which is described by a system of inequalities involving submodular functions. The relationship between convexity and submodularity we have understood so far is concerned only with the domain of definitions and not with the objective functions. In the literature, however, we can find a number of results on the optimization of nonlinear functions over the base polytope of a submodular system. In particular, the minimization of a separable convex function over a base polytope has been considered by Fujishige (1980) and Groenevelt (1985), and the submodular flow problem with a separable convex objective function has been treated by Fujishige (1991). Our present knowledge does not help us understand these results in relation to convex analysis.

Quite independently of the developments in the theory of submodular functions, Dress and Wenzel [1], [2] have recently introduced the concept of a valuated matroid, as a quantitative generalization of matroid. A matroid $(V, \mathcal{B})$, defined in terms of the family of bases $\mathcal{B} \subseteq 2^V$, is characterized by the simultaneous exchange property:

[1]This is an extended abstract of [8].

For $X, Y \in \mathcal{B}$ and $u \in X - Y$ there exists $v \in Y - X$ such that $X - u + v \in \mathcal{B}$ and $Y + u - v \in \mathcal{B}$.

A valuation of $(V, \mathcal{B})$ is a function $\omega : \mathcal{B} \to \mathbf{R}$ which enjoys the quantitative extension of this exchange property:

(MV) For $X, Y \in \mathcal{B}$ and $u \in X - Y$ there exists $v \in Y - X$ such that $X - u + v \in \mathcal{B}$, $Y + u - v \in \mathcal{B}$ and $\omega(X) + \omega(Y) \le \omega(X - u + v) + \omega(Y + u - v)$.

It has turned out recently that the valuated matroids afford a nice combinatorial framework to which the optimization algorithms for matroids can be generalized. Variants of greedy algorithms work for maximizing a matroid valuation, as has been shown by Dress-Wenzel [1] as well as by Dress-Terhalle (1995) and Murota (1995). The weighted matroid intersection problem has been extended by Murota [6] to what is called the valuated matroid intersection problem.

This direction of research can be further extended by considering a function $\omega : B \to \mathbf{R}$ defined on the set of integral points of an integral base polytope such that

(EXC) For $x, y \in B$ and $u \in \mathrm{supp}^+(x - y)$ there exists $v \in \mathrm{supp}^-(x - y)$ such that $x - \tilde{u} + \tilde{v} \in B$, $y + \tilde{u} - \tilde{v} \in B$ and $\omega(x) + \omega(y) \le \omega(x - \tilde{u} + \tilde{v}) + \omega(y + \tilde{u} - \tilde{v})$,

where $\mathrm{supp}^+(x - y) = \{u \in V \mid x(u) > y(u)\}$, $\mathrm{supp}^-(x - y) = \{v \in V \mid x(v) < y(v)\}$ and $\tilde{u}$ denotes the characteristic vector of $u \in V$. We recall the following folk theorem, where

(B2) For $x, y \in B$ and for $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that $x - \tilde{u} + \tilde{v} \in B$ and $y + \tilde{u} - \tilde{v} \in B$.

Theorem 1.1 *Let B be a finite nonempty subset of $\mathbf{Z}^V$. B satisfies* (B2) *if and only if there exists an integer-valued supermodular function $g : 2^V \to \mathbf{Z}$ with $g(\emptyset) = 0$ such that $B = \mathbf{Z}^V \cap \{x \in \mathbf{R}^V \mid x(X) \ge g(X)\ (\forall X \subset V), x(V) = g(V)\}$.*

Functions with property (EXC) arise naturally in combinatorial optimization; for example, a linear function on a matroid, a separable concave function on the integral base polytope of a submodular system, and the maximum cost of a network flow that meets the boundary requirement. It is remarked that a general concave function on a base polytope does not satisfy (EXC) when restricted to $\mathbf{Z}^V$.

The property (B2) is known to be (cryptomorphically) equivalent to sub/supermodularity (see Theorem 1.1). With the correspondence between convexity and submodularity in mind, we may then say that (B2) prescribes a certain "convexity" of the domain of definition of the function ω. The main theme of this paper is to demonstrate that the property (EXC) can be interpreted as "concavity" of the objective function in the context of combinatorial optimization. Three central questions are the following:

- We know two characterizations of the base polytope of a sub/supermodular system, namely, the exchange property (B2) for the points in the polytope and the sub/supermodularity for (the inequalities describing) the faces of the polytope. The property (EXC) is a quantitative generalization of (B2). Then what is the generalization of sub/supermodularity that corresponds to (EXC)?

$$\begin{array}{ccc} [\textit{Domain}] & & [\textit{Function}] \\ \text{(B2)} & \Longrightarrow & \text{(EXC)} \\ \Updownarrow & & \Updownarrow \\ \text{Sub/supermodularity} & \Longrightarrow & \textbf{What ?} \end{array} \tag{1.1}$$

  An answer is given in Theorem 4.2.

- Is a function with (EXC) can be extended to a concave function in the usual sense, just as a submodular function can be extended to a convex function through the Lovász extension? Theorem 3.4 gives a positive answer to this.

- Is there any duality for functions with the property (EXC) that corresponds to the duality for convex/concave functions? The main concern here will be the discreteness (integrality) assertion for a pair of integer-valued such functions. This amounts to a generalization of the potential characterization of the optimality due to Iri-Tomizawa (1976) and the weight splitting theorem of Frank (1981) for the weighted matroid intersection.

2 Functions with the Exchange Property

Let $B \subseteq \mathbf{Z}^V$ be a finite nonempty set with (B2). We are concerned with a function $\omega : B \to \mathbf{R}$ that satisfies (EXC), a variant of Steinitz's exchange property. First we give some fundamental properties of such ω. (A convention: $\omega(x) = -\infty$ for $x \notin B$).

For $p : V \to \mathbf{R}$ we define $\omega[p] : B \to \mathbf{R}$ by $\omega[p](x) = \omega(x) + \langle p, x \rangle$.

Theorem 2.1 *$\omega[p]$ satisfies* (EXC).

For $x, y \in B$ we consider a transportation problem on a bipartite graph $G(x, y)$, which has $(V^+, V^-) = (\mathrm{supp}^+(x-y), \mathrm{supp}^-(x-y))$ as the vertex bipartition and $\hat{A} = \{(u,v) \mid u \in V^+, v \in V^-, x - \tilde{u} + \tilde{v} \in B\}$ as the arc set. Each arc (u, v) is associated with "arc weight" $\omega(x, u, v) = \omega(x - \tilde{u} + \tilde{v}) - \omega(x)$. We define

$$\hat{\omega}(x,y) = \max\left\{ \sum_{(u,v)\in\hat{A}} \omega(x,u,v)\lambda(u,v) \;\middle|\; \lambda(u,v) \geq 0 \qquad ((u,v) \in \hat{A}), \right.$$
$$\left. \sum_{v\in V^-} \lambda(u,v) = x(u) - y(u) \ (u \in V^+), \ \sum_{u\in V^+} \lambda(u,v) = y(v) - x(v) \ (v \in V^-) \right\}.$$

It is known that such $\lambda \in \mathbf{R}^{\hat{A}}$ exists, so that $\hat{\omega}(x,y)$ is defined to be a finite value. The "upper-bound lemma" reads as follows.

Theorem 2.2 ([7, Lemma 2.4]) *For $x, y \in B$, $\omega(y) \leq \omega(x) + \hat{\omega}(x,y)$.*

It follows from this that the local optimality implies the global optimality.

Theorem 2.3 ([7]) *Let $x \in B$. Then $\omega(x) \geq \omega(y)$ ($\forall\ y \in B$) if and only if*

$$\omega(x,u,v) \leq 0 \qquad (u, v \in V). \tag{2.1}$$

Just as the maximizers of a concave function form a convex set, the family of the maximizers of ω, denoted $\mathrm{argmax}\,(\omega)$, enjoys a nice property. By $\overline{\mathrm{argmax}\,(\omega)}$ is meant the convex hull of $\mathrm{argmax}\,(\omega)$.

Lemma 2.4 *If $\omega : B \to \mathbf{R}$ has the property* (EXC), *then* $\mathrm{argmax}\,(\omega)$ *satisfies* (B2), *that is, $\overline{\mathrm{argmax}\,(\omega)}$ is an integral base polytope.*

This lemma implies furthermore that $\overline{\mathrm{argmax}\,(\omega[p])}$ is an integral base polytope for each $p : V \to \mathbf{R}$, since $\omega[p]$ also satisfies (EXC) by Theorem 2.1. This turns out to be a key property for (EXC) as follows (the proof is nontrivial).

Theorem 2.5 *Let $\omega : B \to \mathbf{R}$, where $B \subseteq \mathbf{Z}^V$ is a finite nonempty set with* (B2). *Then ω satisfies* (EXC) *if and only if $\overline{\mathrm{argmax}\,(\omega[p])}$ is an integral base polytope for each $p : V \to \mathbf{R}$.*

3 Conjugate Function and Concave Extension

In line with the standard method in the convex analysis, we introduce the concept of conjugate function. For a function $g : B \to \mathbf{R}$ in general we define $g^\circ : \mathbf{R}^V \to \mathbf{R}$ by

$$g^\circ(p) = \min\{\langle p, x\rangle - g(x) \mid x \in B\}. \tag{3.1}$$

We call g° the concave conjugate function of g. Since $|B|$ is finite, g° is a polyhedral concave function, taking finite values for all p. Furthermore we define $\hat{g} : \mathbf{R}^V \to \mathbf{R}$ by

$$\hat{g}(b) = \inf\{\langle p, b\rangle - g^\circ(p) \mid p \in \mathbf{R}^V\}. \tag{3.2}$$

Obviously, $\hat{g}$ is a concave function, which we call the concave closure of g. By a standard result from the convex analysis we see

$$\hat{g}(b) = \begin{cases} \max\{\sum_{y\in B} \lambda_y g(y) \mid b = \sum_{y \in B} \lambda_y y,\ \lambda \in \Lambda(B)\} & (b \in \overline{B}) \\ -\infty & (b \notin \overline{B}) \end{cases} \tag{3.3}$$

where $\lambda = (\lambda_y \mid y \in B) \in \mathbf{R}^B$, $\overline{B}$ denotes the convex hull of B, and $\Lambda(B) = \{\lambda \in \mathbf{R}^B \mid \sum_{y \in B} \lambda_y = 1,\ \lambda_y \geq 0\ (y \in B)\}$. Define

$$\operatorname{argmax}(g) = \{x \in B \mid g(x) \geq g(y)\ (\forall y \in B)\}, \tag{3.4}$$

$$\operatorname{argmax}(\hat{g}) = \{b \in \overline{B} \mid \hat{g}(b) \geq \hat{g}(c)\ (\forall c \in \overline{B})\}. \tag{3.5}$$

Lemma 3.1 *(1)* $\hat{g}(x) \geq g(x)$ *for* $x \in B$.
(2) $\max\{\hat{g}(b) \mid b \in \overline{B}\} = \max\{g(x) \mid x \in B\}$. *(3)* $\operatorname{argmax}(\hat{g}) = \overline{\operatorname{argmax}(g)}$.

For $p : V \to \mathbf{R}$ (or $p \in \mathbf{R}^V$) we define $g[p] : B \to \mathbf{R}$ and $\hat{g}[p] : \overline{B} \to \mathbf{R}$ by

$$g[p](x) = g(x) + \langle p, x \rangle, \qquad \hat{g}[p](b) = \hat{g}(b) + \langle p, b \rangle. \tag{3.6}$$

Lemma 3.2 *(1)* $(g[p_0])^\circ(p) = g^\circ(p - p_0)$. *(2)* $(g[p_0])\hat{\ }(b) = \hat{g}[p_0](b)$.

We reveal a precise relationship between the exchangeabilty (EXC) and the concavity. By Lemma 3.1(1) we know that $\hat{\omega} : \overline{B} \to \mathbf{R}$ is a concave function such that $\hat{\omega}(x) \geq \omega(x)$ for $x \in B$. The exchangeabilty (EXC) guarantees the equality here as follows.

Lemma 3.3 *If* $\omega : B \to \mathbf{R}$ *has the property* (EXC), *then* $\hat{\omega}(x) = \omega(x)$ *for* $x \in B$.

Theorem 3.4 (Extension Theorem) *Let* $\omega : B \to \mathbf{R}$, *where* $B \subseteq \mathbf{Z}^V$ *is a finite nonempty set with* (B2). *Then* ω *satisfies* (EXC) *if and only if it can be extended to a concave function* $\overline{\omega} : \overline{B} \to \mathbf{R}$ *such that* $\operatorname{argmax}(\overline{\omega}[p])$ *is an integral base polytope for each* $p : V \to \mathbf{R}$.

(Proof) "only if": We can take $\overline{\omega} = \hat{\omega}$, which is an extension of ω by Lemma 3.3 and meets the requirement by $\operatorname{argmax}(\hat{\omega}[p]) = \operatorname{argmax}((\omega[p])\hat{\ }) = \overline{\operatorname{argmax}(\omega[p])}$ and Theorem 2.5.

"if": Obviously we have $\max(\overline{\omega}[p]) \equiv \max\{\overline{\omega}[p](b) \mid b \in \overline{B}\} \geq \max\{\omega[p](x) \mid x \in B\} \equiv \max(\omega[p])$, since $\overline{\omega}[p](x) = \omega[p](x)$ for $x \in B$. On the other hand, $\operatorname{argmax}(\overline{\omega}[p])$ contains an integral point, which belongs to $\mathbf{Z}^V \cap \overline{B} = B$. Therefore we have $\max(\overline{\omega}[p]) = \max(\omega[p])$ and $\mathbf{Z}^V \cap \operatorname{argmax}(\overline{\omega}[p]) = \operatorname{argmax}(\omega[p])$. Since $\operatorname{argmax}(\overline{\omega}[p])$ is an integral base polytope by the assumption, it follows from Theorem 2.5 that ω satisfies (EXC). □

4 Supermodularity in Conjugate Function

In Theorem 1.1 we have seen that the exchange property (B2) of B is equivalent to the supermodularity of the function g describing the face of the polytope $\overline{B}$. As the property (EXC) for ω can be regarded as a quantitative extension of (B2) for B, it is natural to seek for an extension of the above correspondence between the exchangeability and the sub/supermodularity (see (1.1)). Theorem 4.2 below says that (EXC) for ω is equivalent to "local supermodularity" of the concave conjugate function ω°.

4.1 Exchangeability (B2) and supermodularity

We reformulate Theorem 1.1 into a form that is suitable for our subsequent extension. We assume $B \subseteq \mathbf{Z}^V$ is a finite nonempty set such that $B = \mathbf{Z}^V \cap \overline{B}$.

We define $\psi^\circ : \mathbf{R}^V \to \mathbf{R}$ by

$$\psi^\circ(p) = \min\{\langle p, x\rangle \mid x \in B\}. \tag{4.1}$$

Note that ψ° is the concave conjugate function of $\psi \equiv 0$ (on B) in the sense of (3.1), and also that $-\psi^\circ(-p)$ agrees with the support function of $\overline{B}$. Obviously, $\psi^\circ(p)$ is concave, $\psi^\circ(0) = 0$, and positively homogeneous, i.e., $\psi^\circ(\lambda p) = \lambda\psi^\circ(p)$ for $\lambda > 0$.

Suppose B satisfies (B2). We first observe that the function $g : 2^V \to \mathbf{R}$ defined by $g(X) = \psi^\circ(\chi_X)$ $(X \subseteq V)$ is supermodular. In fact, we have

$$g(X) = \min\{\langle \chi_X, x\rangle \mid x \in B\} = \min\{x(X) \mid x \in B\}$$

and this is how the supermodular function g in Theorem 1.1 is constructed. Secondly, the value of $\psi^\circ(p)$ at arbitrary p can be expressed as a linear combination of $\psi^\circ(\chi_X)$ $(X \subseteq V)$. In fact, the greedy algorithm for minimizing a linear function over the base polytope, say $B(g)$, of the supermodular system $(2^V, g)$ shows

$$\min\{\langle p, x\rangle \mid x \in B(g)\} = \sum_{j=1}^{n}(p_j - p_{j+1})g(V_j), \tag{4.2}$$

where, for given $p \in \mathbf{R}^V$, the elements of V are indexed as $\{v_1, v_2, \cdots, v_n\}$ (with $n = |V|$) in such a way that

$$p(v_1) \geq p(v_2) \geq \cdots \geq p(v_n);$$

$p_j = p(v_j)$, $V_j = \{v_1, v_2, \cdots, v_j\}$ for $j = 1, \cdots, n$, and $p_{n+1} = 0$. Noting $\overline{B} = B(g)$ we obtain

$$\psi^\circ(p) = \sum_{j=1}^{n}(p_j - p_{j+1})\psi^\circ(\chi_{V_j}). \tag{4.3}$$

Conversely, suppose $\psi^\circ(p)$ defined from B by (4.1) satisfies the two conditions:

(C1) [supermodularity] $g(X) = \psi^\circ(\chi_X)$ is supermodular.

(C2) [greediness] $\psi^\circ(p) = \sum_{j=1}^{n}(p_j - p_{j+1})\psi^\circ(\chi_{V_j})$.

Then Theorem 1.1 shows that B satisfies (B2).

We say that a positively homogeneous function $h : \mathbf{R}^V \to \mathbf{R}$ is "matroidal" if it satisfies (C1) and (C2) with ψ° replaced by h. By a result of Lovász [5] such h is necessarily concave. The above observations are summarized in the following theorem.

Theorem 4.1 *Let $B \subseteq \mathbf{Z}^V$ be a finite nonempty set with $B = \mathbf{Z}^V \cap \overline{B}$. Then B satisfies* (B2) *if and only if ψ° is "matroidal" (satisfying* (C1) *and* (C2)*).*

4.2 Exchangeability (EXC) and supermodularity

We now consider the concave conjugate function

$$\omega^\circ(p) = \min\{\langle p, x\rangle - \omega(x) \mid x \in B\} \tag{4.4}$$

of $\omega : B \to \mathbf{R}$ defined on a finite nonempty set $B \subseteq \mathbf{Z}^V$ with the property (B2). As opposed to ψ°, ω° is not a positively homogeneous function though it is concave.

Since $\omega^\circ(p)$ is a concave function, we can think of its subdifferential in the ordinary sense in the convex analysis. Namely, the subdifferential of ω° at $p_0 \in \mathbf{R}^V$, denoted $\partial\omega^\circ(p_0)$, is defined by $\partial\omega^\circ(p_0) = \{b \in \mathbf{R}^V \mid \omega^\circ(p) - \omega^\circ(p_0) \leq \langle p - p_0, b\rangle \ (\forall p \in \mathbf{R}^V)\}$. Using this we define a positively homogeneous concave function $\hat{L}(\omega^\circ, p_0) : \mathbf{R}^V \to \mathbf{R}$ by

$$\hat{L}(\omega^\circ, p_0)(p) = \inf\{\langle p, b\rangle \mid b \in \partial\omega^\circ(p_0)\}, \tag{4.5}$$

which we call the localization of ω° at p_0 (provided $\partial\omega^\circ(p_0) \neq \emptyset$). Note that

$$\omega^\circ(p) \leq \omega^\circ(p_0) + \hat{L}(\omega^\circ, p_0)(p - p_0) \tag{4.6}$$

and that $\omega^\circ(p)$ is equal to the right-hand side in the neighborhood of p_0.

The following theorem allows us to say that the exchange property (EXC) is nothing but "a collection of local supermodularity", just as the exchange property (B2) corresponds to supermodularity.

Theorem 4.2 (Local Supermodularity Theorem) *Let $\omega : B \to \mathbf{R}$, where $B \subseteq \mathbf{Z}^V$ is a finite nonempty set with* (B2). *Then ω satisfies* (EXC) *if and only if the localization $\hat{L}(\omega^\circ, p_0)$ of ω° is "matroidal" (satisfying* (C1) *and* (C2)*) at each point p_0.*

(Proof) It is not difficult to see $\hat{L}(\omega^\circ, p_0)(p) = \min\{\langle p, x\rangle \mid x \in \mathrm{argmax}\,(\omega[-p_0])\}$. By Theorem 4.1, this is "matroidal" if and only if $\mathrm{argmax}\,(\omega[-p_0])$ satisfies (B2), whereas the latter condition for all p_0 is equivalent to (EXC) by Theorem 2.5. □

As a corollary we obtain the following theorem.

Theorem 4.3 *If $\omega_1 : B_1 \to \mathbf{R}$ and $\omega_2 : B_2 \to \mathbf{R}$ satisfy* (EXC), *then the supremum convolution $\omega_1 \Box \omega_2 : B_1 + B_2 \to \mathbf{R}$ satisfies* (EXC), *where*

$$(\omega_1 \Box \omega_2)(x) = \sup\{\omega_1(x_1) + \omega_2(x_2) \mid x_1 + x_2 = x, x_1 \in B_1, x_2 \in B_2\}.$$

(Proof) It follows from Theorem 4.2 that both $\hat{L}(\omega_1{}^\circ, p_0)$ and $\hat{L}(\omega_2{}^\circ, p_0)$ are "matroidal" for each p_0. This implies $\hat{L}(\omega_1{}^\circ, p_0) + \hat{L}(\omega_2{}^\circ, p_0) = \hat{L}(\omega_1{}^\circ + \omega_2{}^\circ, p_0) = \hat{L}((\omega_1 \Box \omega_2)^\circ, p_0)$ is also "matroidal" for each p_0. Finally we use the other direction of Theorem 4.2. □

5 Duality

Using the standard Fenchel duality framework of convex analysis, we derive a min-max duality formula for a pair of functions with the exchange property (EXC). The content of the min-max relation lies in the integrality assertion that both the primal (maximization) problem and the dual (minimization) problem have the integral optimum solutions when the given functions with (EXC) are integer-valued. This min-max formula is a succinct unified statement of the two groups of more or less equivalent theorems, (i) Edmonds' polymatroid intersection theorem , Fujishige's Fenchel-type duality theorem [4], and Frank's discrete separation theorem for a pair of sub/supermodular functions [3] and (an extension of) (ii) Iri-Tomizawa's potential characterization of the optimality for the independent assignment problem, Fujishige's generalization thereof to the independent flow problem and Frank's weight splitting theorem for the matroid intersection problem. The min-max formula can be reformulated also as discrete separation theorems, which are distinct from Frank's.

Let B_1 and B_2 be finite nonempty subsets of $\mathbf{Z}^V$, each enjoying the exchange property (B2). For $\omega : B_1 \to \mathbf{R}$ and $\zeta : B_2 \to \mathbf{R}$, we define the conjugate functions ω° and $\zeta^\bullet$ by (3.1) and $\zeta^\bullet(p) = \max\{\langle p, x\rangle - \zeta(x) \mid x \in B_2\}$ with reference to B_1 and B_2, respectively, and also the concave/convex closure functions $\hat{\omega}$ and $\check{\zeta}$ by (3.2) and $\check{\zeta}(b) = \sup\{\langle p, b\rangle - \zeta^\bullet(p) \mid p \in \mathbf{R}^V\}$, respectively. We sometimes use the following convention: $\omega(x) = -\infty \quad (x \notin B_1)$, $\zeta(x) = +\infty \quad (x \notin B_2)$.

We define a primal-dual pair of problems and a relaxation as follows.

[Primal problem] Maximize $\Phi(x) = \omega(x) - \zeta(x) \quad (x \in B_1 \cap B_2)$.
[Dual problem] Minimize $\Psi(p) = \zeta^\bullet(p) - \omega^\circ(p) \quad (p \in \mathbf{R}^V)$.
[Relaxed primal problem] Maximize $\tilde{\Phi}(b) = \hat{\omega}(b) - \check{\zeta}(b) \quad (b \in \overline{B_1} \cap \overline{B_2})$.

The following identity is known as the Fenchel duality:

$$\max\{\hat{\omega}(b) - \check{\zeta}(b) \mid b \in \overline{B_1} \cap \overline{B_2}\} = \inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{R}^V\}, \tag{5.1}$$

which holds true independently of (EXC). Here we assume the convention that the maximum taken over an empty family is equal to $-\infty$. With this convention, the above formula implies that $\overline{B_1} \cap \overline{B_2} \neq \emptyset$ if the infimum on the right-hand side is finite.

Combining (5.1) with the obvious inequalities (cf. Lemma 3.1(1)): $\omega(x) \leq \hat{\omega}(x)$ $(x \in B_1)$, $\zeta(x) \geq \check{\zeta}(x)$ $(x \in B_2)$, we obtain the following weak duality.

Lemma 5.1 *For any functions $\omega : B_1 \to \mathbf{R}$ and $\zeta : B_2 \to \mathbf{R}$,*

$$\max\{\omega(x) - \zeta(x) \mid x \in B_1 \cap B_2\}$$
$$\leq \max\{\hat{\omega}(b) - \check{\zeta}(b) \mid b \in \overline{B_1} \cap \overline{B_2}\} = \inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{R}^V\}.$$

Naturally, we are interested in whether the equality holds in the weak duality above. The next theorem shows that this is indeed the case if ω and $-\zeta$ enjoy (EXC).

Theorem 5.2 *Let $\omega : B_1 \to \mathbf{R}$ and $\zeta : B_2 \to \mathbf{R}$ be such that ω and $-\zeta$ satisfy* (EXC).
(1) [Primal integrality]

$$\max\{\omega(x) - \zeta(x) \mid x \in B_1 \cap B_2\}$$
$$= \max\{\hat{\omega}(b) - \check{\zeta}(b) \mid b \in \overline{B_1} \cap \overline{B_2}\} = \inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{R}^V\}.$$

To be more precise,
(P1) $\inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{R}^V\} \neq -\infty$ *if and only if* $B_1 \cap B_2 \neq \emptyset$,
(P2) *If* $B_1 \cap B_2 \neq \emptyset$, *all these values are finite and equal.*
(2) [Dual integrality] If ω and ζ are integer-valued, the infimum can be taken over integral vectors, i.e., $\max\{\omega(x) - \zeta(x) \mid x \in B_1 \cap B_2\} = \inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{Z}^V\}$.

Before giving the proof, we observe that the essence of the first half of Theorem 5.2 lies in the integrality of the relaxed primal problem. Since $B_i = \mathbf{Z}^V \cap \overline{B_i}$ $(i = 1, 2)$, we have $B_1 \cap B_2 = \mathbf{Z}^V \cap (\overline{B_1} \cap \overline{B_2})$. Hence, if the relaxed primal problem has an integral optimal solution, say b, then b belongs to $B_1 \cap B_2$. Furthermore, $\omega(b) = \hat{\omega}(b)$ and $\zeta(b) = \check{\zeta}(b)$ by Lemma 3.3. Hence follows Theorem 5.2(1).

The proof of Theorem 5.2 relies on Frank's discrete separation theorem for a pair of sub/supermodular functions and a recent theorem of the present author.

Theorem 5.3 (Discrete Separation Theorem [3]) *Let $f : 2^V \to \mathbf{R}$ and $g : 2^V \to \mathbf{R}$ be submodular and supermodular functions, respectively, with $f(\emptyset) = g(\emptyset) = 0$. If $g(X) \leq f(X)$ $(X \subseteq V)$, there exists $x^* \in \mathbf{R}^V$ such that*

$$g(X) \leq x^*(X) \leq f(X) \qquad (X \subseteq V). \tag{5.2}$$

Moreover, if f and g are integer-valued, there exists such $x^* \in \mathbf{Z}^V$.

Theorem 5.4 ([7, Theorem 4.1]) *Assume that $\omega_1 : B_1 \to \mathbf{R}$ and $\omega_2 : B_2 \to \mathbf{R}$ satisfy* (EXC) *and let $x^* \in B_1 \cap B_2$. Then*

$$\omega_1(x^*) + \omega_2(x^*) \geq \omega_1(x) + \omega_2(x) \qquad (x \in B_1 \cap B_2)$$

if and only if there exists $p^* \in \mathbf{R}^V$ such that

$$\omega_1[-p^*](x^*) \geq \omega_1[-p^*](x) \quad (x \in B_1), \qquad \omega_2[p^*](x^*) \geq \omega_2[p^*](x) \quad (x \in B_2).$$

Moreover, if ω_1 and ω_2 are integer-valued, there exists such $p^* \in \mathbf{Z}^V$.

Remark 5.1 When ω_1 and ω_2 are affine functions, the above theorem agrees with the optimality criterion for the weighted intersection problem. When $B_1, B_2 \subseteq \{0,1\}^V$, on the other hand, the above theorem reduces to the optimality criterion [6, I, Theorem 4.2] for the valuated matroid intersection problem. If, in addition, ω_1 is affine and $\omega_2 = 0$, this criterion recovers Frank's weight splitting theorem for the weighted matroid intersection problem, which is in turn equivalent to Iri-Tomizawa's potential characterization of the optimality for the independent assignment problem. □

We now prove (P1) of Theorem 5.2(1). Recall Theorem 1.1 and let g_1 be the supermodular function describing B_1 and f_2 be the submodular function describing B_2. We have $g_1(\emptyset) = f_2(\emptyset) = 0$. We also introduce (cf. (4.1))

$$\psi_1{}^{\circ}(p) = \min\{\langle p, x\rangle \mid x \in B_1\}, \qquad \psi_2{}^{\bullet}(p) = \max\{\langle p, x\rangle \mid x \in B_2\}.$$

Lemma 5.5

$$\inf\{\zeta^{\bullet}(p) - \omega^{\circ}(p) \mid p \in \mathbf{R}^V\} \neq -\infty \tag{5.3}$$

$$\iff \psi_2{}^{\bullet}(p) \geq \psi_1{}^{\circ}(p) \qquad (p \in \mathbf{R}^V) \tag{5.4}$$

$$\iff f_2(X) \geq g_1(X) \quad (X \subseteq V), \qquad f_2(V) = g_1(V). \tag{5.5}$$

Moreover, (5.3) $\iff \inf\{\zeta^{\bullet}(p) - \omega^{\circ}(p) \mid p \in \mathbf{Z}^V\} \neq -\infty$.

(Proof) Since $|\omega^{\circ}(p) - \psi_1{}^{\circ}(p)| \leq \max_{x\in B_1} |\omega(x)|$, $|\zeta^{\bullet}(p) - \psi_2{}^{\bullet}(p)| \leq \max_{x \in B_2} |\zeta(x)|$, and $\psi_1{}^{\circ}(p)$ and $\psi_2{}^{\bullet}(p)$ are positively homogeneous, we have $\inf\{\zeta^{\bullet}(p) - \omega^{\circ}(p) \mid p \in \mathbf{R}^V\} \neq -\infty \iff \inf\{\psi_2{}^{\bullet}(p) - \psi_1{}^{\circ}(p) \mid p \in \mathbf{R}^V\} \neq -\infty \iff \psi_2{}^{\bullet}(p) \geq \psi_1{}^{\circ}(p)$ $(p \in \mathbf{R}^V)$. By Theorem 4.1 it suffices to consider the last inequality for $p = \chi_X$ $(X \subseteq V)$. A straightforward calculation using (4.3) shows this is further equivalent to (5.5). □

If (5.5) is true, we can apply Theorem 5.3 to obtain $x^* \in B_1 \cap B_2$. The converse is obvious, since $B_1 \cap B_2 \neq \emptyset$ implies (5.5). [End of proof of (P1)]

Next, we prove (P2) of Theorem 5.2(1). By Lemma 5.1 we see that (P2) is equivalent to the existence of $x^* \in B_1 \cap B_2$ and $p^* \in \mathbf{R}^V$ such that $\omega(x^*) - \zeta(x^*) = \zeta^{\bullet}(p^*) - \omega^{\circ}(p^*)$. Put $\omega_1 = \omega$ and $\omega_2 = -\zeta$ and denote by x^* a common base that maximizes $\omega_1(x) + \omega_2(x)$. By Theorem 5.4 we have $\omega_1[-p^*](x^*) = \max\{\omega_1[-p^*](x) \mid x \in B_1\}$, $\omega_2[p^*](x^*) = \max\{\omega_2[p^*](x) \mid x \in B_2\}$ for some $p^* \in \mathbf{R}^V$. This implies $\omega(x^*) - \zeta(x^*) = \omega_1(x^*) + \omega_2(x^*) = \omega_1[-p^*](x^*) + \omega_2[p^*](x^*) = \max_{x\in B_1} \omega_1[-p^*](x) + \max_{x \in B_2} \omega_2[p^*](x) = \max_{x \in B_1}(-\langle p^*, x\rangle + \omega(x)) + \max_{x\in B_2}(\langle p^*, x\rangle - \zeta(x)) = \zeta^{\bullet}(p^*) - \omega^{\circ}(p^*)$.

The second half of Theorem 5.2 follows from the second half of Theorem 5.4 that guarantees the existence of integral p^*. [End of proof of Theorem 5.2]

The min-max identity of Theorem 5.2 yields a pair of separation theorems, one for the primal pair (ω, ζ) and the other for the dual (conjugate) pair $(\omega^{\circ}, \zeta^{\bullet})$. It is emphasized that these separation theorems do not exclude the case of $B_1 \cap B_2 = \emptyset$.

Theorem 5.6 (Primal Separation Theorem) *Let* $\omega : B_1 \to \mathbf{R}$ *and* $\zeta : B_2 \to \mathbf{R}$ *be such that* ω *and* $-\zeta$ *satisfy* (EXC). *If* $\omega(x) \le \zeta(x)$ $(x \in B_1 \cap B_2)$*, there exist* $\alpha^* \in \mathbf{R}$ *and* $p^* \in \mathbf{R}^V$ *such that* $\omega(x) \le \alpha^* + \langle p^*, x\rangle \le \zeta(x) \quad (x \in \mathbf{Z}^V)$.
[This is a short-hand expression for

$$\omega(x) \le \alpha^* + \langle p^*, x\rangle \quad (x \in B_1), \qquad \alpha^* + \langle p^*, x\rangle \le \zeta(x) \quad (x \in B_2). \quad]$$

Moreover, if ω *and* ζ *are integer-valued, there exist such* $\alpha^* \in \mathbf{Z}$ *and* $p^* \in \mathbf{Z}^V$.

Theorem 5.7 (Dual Separation Theorem) *Let* $\omega : B_1 \to \mathbf{R}$ *and* $\zeta : B_2 \to \mathbf{R}$ *be such that* ω *and* $-\zeta$ *satisfy* (EXC). *If* $\omega^\circ(p) \le \zeta^\bullet(p)$ $(p \in \mathbf{R}^V)$*, there exist* $\beta^* \in \mathbf{R}$ *and* $x^* \in B_1 \cap B_2$ *such that* $\omega^\circ(p) \le \beta^* + \langle p, x^*\rangle \le \zeta^\bullet(p) \quad (p \in \mathbf{R}^V)$.
Moreover, if ω *and* ζ *are integer-valued, there exists such* $\beta^* \in \mathbf{Z}$.

Remark 5.2 The dual separation theorem for $\omega = 0$ and $\zeta = 0$ reduces to the discrete separation theorem (Theorem 5.3) for sub/supermodular functions. In fact, the assumption reduces to (5.4), which is equivalent to (5.5), and we have $\beta^* = 0$. □

Finally we schematically summarize the relationship among the duality theorems. It is emphasized that the "equivalence" relies on Lemma 3.3 and Lemma 5.5.

Primal separation
(Theorem 5.6)
$\Updownarrow$
Min-max duality (Theorem 5.2) $\begin{cases} \text{(P1)} \iff \text{Discrete separation (Theorem 5.3)} \\ \text{(P2)} \iff \text{Weighted intersection (Theorem 5.4)} \end{cases}$
$\Updownarrow$
Dual separation
(Theorem 5.7)

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