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A linear programming instance with many crossover events*

Shinji Mizuno† 水野 眞治
Nimrod Megiddo‡
Takashi Tsuchiya§ 土谷 隆

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Abstract

Crossover events for a linear programming problem were introduced by Vavasis and Ye and provide important insight into the behavior of the path of centers. The complexity of a layered-step interior-point algorithm presented by them depends on the number of disjoint crossover events and the coefficient matrix A, but not on b and c. In this short note, we present a linear programming instance with more than $(1/8)n^2$ crossover events.

Key words: linear programming, layered-step interior-point method, path of centers, crossover events

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†The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106, Japan. Research performed while visiting IBM Almaden Research Center

‡IBM Research Division, Almaden Research Center, 650 Harry Road, San Jose, CA 95120, and School of Mathematical Sciences, Tel Aviv University, Tel Aviv, Israel

§The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106, Japan. Research performed while visiting IBM Almaden Research Center

1 Introduction

Interior-point methods for linear programming have been developed tremendously since the presentation of Karmarkar [1]. Recently Vavasis and Ye [2] proposed a layered-step primal-dual interior-point algorithm, whose number of operations has an upper bound that depends only on the coefficient matrix A and not on b and c.

Crossover events for a linear programming problem were introduced by Vavasis and Ye [2] and provide important insight into the behavior of the path of centers. The number of operations of the layered-step interior-point algorithm depends on the number of disjoint crossover events. Although the number depends on b and c, they prove that it is bounded by $(1/2)n(n-1)$. The question of whether there could be more than n crossover events was left open in [2]. If one could prove that the number is bounded by $O(n)$, the complexity of the layered-step interior-point algorithm could be reduced by a factor of n. In this short note, we present a linear programming instance with more than $(1/8)n^2$ disjoint crossover events. We believe that the instance helps much for understanding the behavior of the path of centers.

2 Crossover events

Let $n \geq m > 0$ be integers. For an $m \times n$ matrix A and vectors $b \in R^m$ and $c \in R^n$, an instance $LP(A, b, c)$ denotes a primal-dual pair of linear programming problems

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \geq 0 \end{array}$$

and

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \quad s \geq 0. \end{array}$$

We assume that the primal-dual pair has feasible interior point (x, y, s) (i.e., $x > 0$ and $s > 0$). For each $\mu > 0$, we denote by $(x(\mu), y(\mu), s(\mu))$ the solution of the system

$$\begin{aligned} Ax = b, &\quad x \geq 0, \\ A^T y + s = c, &\quad s \geq 0, \\ Xs = \mu e, & \end{aligned}$$

where e is the vector of 1's and $X = \mathrm{diag}(x)$ is the diagonal matrix such that $Xe = x$. The solution is called a center and the set of centers $P = \{(x(\mu), y(\mu), s(\mu)) : \mu > 0\}$ is called a path of centers.

In this note, we define crossover events for the path of centers for simplicity and clarification of the definition, although Vavasis and Ye [2] defined them for a neighborhood of the path. Moreover their definition depends on a point generated by an algorithm, while ours depends only on the instance $LP(A, b, c)$.

For a given $g > 10$ and $(x, y, s) \in P$, we partition the index set $\{1, 2, \ldots, n\}$ into layers $J_1, J_2, \ldots, J_p$ as follows: Let π be a permutation that sorts components of s in nondecreasing order:

$$s_{\pi(1)} \leq s_{\pi(2)} \leq \cdots \leq s_{\pi(n)}.$$

Let $J_1 = \{\pi(1), \ldots, \pi(i)\}$ be the set of successive indices of π such that the ratio-gap $s_{\pi(j+1)}/s_{\pi(j)}$ is less than or equal to g for each $j = 1, \cdots, i-1$ but $s_{\pi(i+1)}/s_{\pi(i)}$ is greater than g. Then put $\pi(i+1), \pi(i+2), \ldots$ in J_2, until another ratio-gap greater than g is encountered, and so on. Let J_p be the last set which contains $\pi(n)$.

For two indices $i, j \in \{1, 2, \ldots, n\}$, we denote

$$i \prec j \text{ at } (x, y, s)$$

if there exists $k \in \{1, 2, \ldots, p-1\}$ such that $i \in J_1 \cup \cdots \cup J_k$ and $j \in J_{k+1} \cup \cdots \cup J_p$. We also denote

$$i \preceq j \text{ at } (x, y, s)$$

if there exists $k \in \{1, 2, \ldots, p\}$ such that $i \in J_1 \cup \cdots \cup J_k$ and $j \in J_k \cup \cdots \cup J_p$.

Definition 1 *For an instance $LP(A, b, c)$ and a constant $g > 10$, we say that the triple (μ, i, j) defines a* **crossover event**, *if $\{i, j\} \subseteq \{1, 2, \ldots, n\}$,*

$$i \preceq j \textit{ at the point } (x(\mu), y(\mu), s(\mu)) \in P,$$

there exists a positive $\mu' < \mu$ such that

$$j \prec i \textit{ at any } (x(\mu''), y(\mu''), s(\mu'')) \in P \textit{ where } \mu'' \in (0, \mu'], \tag{1}$$

and $s(\mu'')_j \to 0$ as $\mu'' \to 0$.

The definition above is slightly different from the one given in Vavasis and Ye [2], but the essential meaning is the same. It is not hard to see that whenever our crossover event occurs, theirs does too. So our definition is sufficient to give a lower bound for their crossover events.

We say that two crossover events (μ_1, i_1, j_1) and (μ_2, i_2, j_2) with $\mu_1 > \mu_2$ are **disjoint** if the value μ_1' that is required for the first one satisfies $\mu_1' \geq \mu_2$. Vavasis and Ye proved that the number of disjoint crossover events is bounded by $n(n-1)/2$.

3 The linear programming instance

Let m be a positive integer and let $n = 2m$. We denote the $m \times m$ identity matrix by I. Let $\epsilon > 0$ be a sufficiently small real number. Then we define an instance $LP(A, b, c)$ by

$$A = (I, I), b = (\epsilon, \epsilon^2, \ldots, \epsilon^m)^T, c = (0, \ldots, 0, \epsilon^m, \epsilon^{2m}, \ldots, \epsilon^{m^2})^T,$$

Table 1: Approximate values at a center

μ	1	$\cdots$	$(2/3)\epsilon^{i(m+1)}$	$\cdots$	0
x_i	$(1/2)\epsilon^i$	$(1/\lambda_1)\epsilon^i$	$(2/3)\epsilon^i$	$(1/\lambda_2)\epsilon^i$	ϵ^i
x_{i+m}	$(1/2)\epsilon^i$	$(1/\lambda_3)\epsilon^i$	$(1/3)\epsilon^i$	$(1/\lambda_4)\mu\epsilon^{-im}$	0
s_i	$2\epsilon^{-i}$	$\lambda_1\mu\epsilon^{-i}$	ϵ^{im}	$\lambda_2\mu\epsilon^{-i}$	0
s_{i+m}	$2\epsilon^{-i}$	$\lambda_3\mu\epsilon^{-i}$	$2\epsilon^{im}$	$\lambda_4\epsilon^{im}$	ϵ^{im}

where the number of 0's in c is m. Then for any $\mu > 0$, the center $(x(\mu), y(\mu), s(\mu))$ is the solution of the system

$$\begin{aligned} x_i + x_{i+m} &= \epsilon^i \quad \text{for} \quad i = 1, 2, \ldots, m \\ y_i + s_i &= 0 \quad \text{for} \quad i = 1, 2, \ldots, m \\ y_i + s_{i+m} &= \epsilon^{im} \quad \text{for} \quad i = 1, 2, \ldots, m \\ x_i s_i &= \mu \quad \text{for} \quad i = 1, 2, \ldots, 2m \end{aligned}$$

When the value of μ decreases from 1 to 0, approximate values of x_i, x_{i+m}, s_i and s_{i+m} at a center are shown in Table 1 for each $i = 1, 2, \ldots, m$, where λ_j $(j = 1, 2, 3, 4)$ are real numbers between 1 and 3. From Table 1, we can compute the layers at the centers $(x(\mu), y(\mu), s(\mu))$ as follows: If $\mu \in [\epsilon^{m+1}, 1]$ then $p = m$ and

$$J_1 = \{1, m+1\}, J_2 = \{2, m+2\}, \cdots, J_m = \{m, 2m\},$$

if $\mu \in [10g\epsilon^{m+2}, (10g)^{-1}\epsilon^{m+1}]$ then $p = m + 1$ and

$$J_1 = \{1\}, J_2 = \{m+1\}, J_3 = \{2, m+2\}, \cdots, J_{m+1} = \{m, 2m\},$$

if $\mu = \epsilon^{m+2}$ then $p = m$ and

$$J_1 = \{1\}, J_2 = \{2, m+1, m+2\}, \cdots, J_m = \{m, 2m\},$$

and so on. If $\mu = \epsilon^{2m+1}$ then $p = m + 1$ and

$$J_1 = \{1\}, J_2 = \{2, m+2\}, \cdots, J_m = \{m, 2m\}, J_{m+1} = \{m+1\}.$$

It is easy to see that if $j \in \{1, 2, \ldots, m\}$ then

$$j \prec m+1 \text{ at } (x(\mu''), y(\mu''), s(\mu'')) \in P \text{ where } \mu'' \in (0, \epsilon^{m+j+1}]$$

and $s(\mu'')_j \longrightarrow 0$ as $\mu'' \longrightarrow 0$. Hence while μ decreases from ϵ^{m+1} to ϵ^{2m+1}, there are m disjoint crossover events $(\epsilon^{m+1}, m+1, 1)$ with $\mu' = \epsilon^{m+2}$, $(\epsilon^{m+2}, m+1, 2)$ with $\mu' = \epsilon^{m+3}$, $\cdots$, $(\epsilon^{2m}, m+1, m)$ with $\mu' = \epsilon^{2m+1}$. Similarly, while μ decreases from $\epsilon^{2(m+1)}$ to ϵ^{3m+2}, there are $m-1$ crossover events, and so on. As a total, the number of disjoint crossover events is

$$m + (m-1) + \cdots + 1 = (1/2)m(m+1) \geq (1/8)n^2.$$

4 Concluding remarks

We have presented a linear programming instance with more than $(1/8)n^2$ crossover events. This result indicates that the path of centers consists of more than $(1/8)n^2$ parts each of which defines a partition of $\{1, 2, \ldots, n\}$ as layers. As discussed in Vavasis and Ye [2], we can also see that the path consists of almost straight parts and curved parts, and the number of such parts is bounded by twice of the number of crossover events. From Table 1, we can observe that the projection $\{(x_i, x_{i+m}, s_i, s_{i+m}) : (x, y, s) \in P\}$ of the path of centers consists of two almost straight parts and one curve part for μ around $\epsilon^{i(m+1)}$. Thus the path of centers appears to consist of $(n/2) + 1$ straight parts and $(n/2)$ curved parts, which are much less than the number of crossover events. So if we could trace each almost straight part of the path in a constant number of steps, it would yield a very efficient algorithm for linear programming.

References

[1] N. Karmarkar, A new polynomial-time algorithm for linear programming, Combinatorica **4** (1984) 373–395.

[2] S. Vavasis and Y. Ye, A primal-dual accelerated interior point method whose running time depends only on A, Technical Report, School of Computer Sciences, Cornell University, Ithaca, NY (1994).