数理解析研究所講究録
946 巻 1996 年 133-140

HYPERBOLIC BESOV FUNCTIONS

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ABSTRACT. In this paper we study bounded holomorphic functions in the unit disk D with given growth of hyperbolic derivative and application of these functions to composition operators.

1. Introduction

Let $D=\{z:|z|<1\}$ be the unit disk in $\mathbf{C}$ with pseudohyperbolic metric

$$\rho(a,b)=\left|\frac{a-b}{1-\bar{a}b}\right|$$

and with hyperbolic metric

$$\sigma(a,b)=\frac{1}{2}\log\frac{1+\rho(a,b)}{1-\rho(a,b)}.$$

Here $dA(z)$ is a normalized area measure on D and $d\lambda(z)$ is hyperbolic area measure

$$d\lambda(z)=\frac{dA(z)}{(1-|z|^2)^2}.$$

Let B be the family of bounded holomorphic functions $f(z)$, $|f(z)|<1$, in D and $f^*(z)=\dfrac{|f'(z)|}{1-|f(z)|^2}$ be the hyperbolic derivative of $f(z)$. Let $f_a(z)=f(\dfrac{z+a}{1+\bar{a}z})$, $a\in D$.

Definition. *For $1\leq p\leq\infty$ hyperbolic analytic Besov class B_p^h consists of functions $f(z)\in B$ which satisfy the condition*

$$\|f\|_{B_p^h}=\left(\iint\limits_D((1-|z|^2)f^*(z))^p\,d\lambda(z)\right)^{\frac{1}{p}}<\infty. \tag{1}$$

Classes B_p^h, $1\leq p\leq\infty$, are Möbius invariant. By Schwarz-Pick lemma

$$(1-|z|^2)f^*(z)\leq 1$$

for any $f \in B$, i.e. $\mathcal{B}^h_\infty = B$, but Möbius transforms of D are not p-hyperbolic Besov functions for $1 \le p < \infty$. It is easy to see that any function $f(z)$, $|f(z)| \le k < 1$, belongs to all classes B^h_p, $1 \le p \le \infty$. Class B^h_2 is a class of hyperbolic Dirichlet functions.

Examples.

1. Let $S_\alpha = \{z = x + iy : |x|^\alpha + |y|^\alpha < 1\}$, $0 < \alpha \le 1$, and $\varphi_\alpha : D \to S_\alpha$ then $\varphi_1(z) \notin B^h_2$ and $\varphi_1(z) \in B^h_p$ if $p > 2$. If $\alpha < 1$ then $\varphi_\alpha(z) \in B^h_2$.

2. It is known [4] that for hyperbolic Lipschitz functions $\sigma\Lambda_\alpha$, $0 < \alpha \le 1$, i.e. functions which satisfy the condition $\sup\limits_{|u-v|\le\tau} \sigma(f(u), f(v)) \le K\tau^\alpha$, the necessary and sufficient condition that $f(z) \in \sigma\Lambda_\alpha$ is $(1-|z|^2)f^*(z) = O((1-|z|^2)^\alpha)$ as $|z| \to 1$.

Thus we can see that $\sigma\Lambda_{\frac{1}{p}} \subset B^h_p$.

In chapter 2 we establish Lipschitz type properties of p-hyperbolic Besov functions and prove that p-hyperbolic Besov functions don't have angular derivatives for finite values p. In chapter 3 we show that composition of Bloch functions and p-hyperbolic Besov functions are p-analytic Besov functions.

2. Main properties

Classes B^h_p satisfy nesting property $B^h_p \subset B^h_q$ for $p < q$. It follows from the Schwarz-Pick lemma and inequality

$$\iint_D (1-|z|^2)^{q-2}(f^*(z))^q \, dA(z)$$

$$= \iint_D ((1-|z|^2)f^*(z))^{q-p}(1-|z|^2)^{p-2}(f^*(z))^p \, dA(z)$$

$$\le \iint_D (1-|z|^2)^{p-2}(f^*(z))^p \, dA(z).$$

Theorem 1. *If bounded function $f(z)$ satisfies the condition*

$$\iint_D \iint_D \frac{\rho(f(z), f(w))^p}{|1-z\bar{w}|^4} \, dA(z)\, dA(w) < \infty$$

then $f(z) \in B^h_p$, $1 \le p < \infty$.

Proof.

Function $g(z) = \dfrac{f(z)-f(0)}{1-f(z)\overline{f(0)}}$ is holomorphic in D. Using an expansion of $g(z)$ on Taylor series we can obtain

$$f^*(0) = \left| \iint_D \bar{z} g(z) \, dA(z) \right|.$$

Let $g = f \circ \varphi_a(z)$, where $\varphi_a(z) = \frac{z+a}{1+\bar{a}z}$, $a \in D$. Then

$$((1-|a|^2)f^*(a))^p \leq \iint\limits_D (\rho(f \circ \varphi_a(z), f(a)))^p \, dA(z)$$

and thus

$$\iint\limits_D ((1-|a|^2)f^*(a))^p \, d\lambda(a)$$

$$\leq \iint\limits_D d\lambda(a) \iint\limits_D (\rho(f_a(z), f(a)))^p \, dA(z)$$

$$= \iint\limits_D \iint\limits_D \frac{\rho(f(z), f(w))^p}{|1-z\bar{w}|^4} \, dA(z) \, dA(w) < \infty.$$

Theorem 2. *If $f(z) \in B_p^h$, $1 < p < \infty$, then*

$$\iint\limits_D \iint\limits_D \frac{\sigma(f(z), f(w))^p}{|1-z\bar{w}|^4} \, dA(z) \, dA(w) < \infty.$$

Proof.

At first we estimate hyperbolic distance between $f(z)$ and $f(0)$.

$$\sigma(f(z), f(0)) = |\int_0^1 f^*(tz) z \, dt|$$

$$= \left| \int_0^1 \frac{f^*(tz)}{1-t^2|z|^2}(1-t^2|z|^2) z \, dt \right| \leq \left| \int_0^1 \frac{f^*(tz)}{1-t^2|z|^2} \, dt \right|$$

$$\leq \left(\int_0^1 \frac{|z|^q}{(1-t^2|z|^2)^{\frac{1+q}{2}}} \, dt \right)^{\frac{1}{q}} \left(\int_0^1 \frac{((1-t^2|z|^2)f^*(tz))^p}{(1-t^2|z|^2)^{\frac{1}{2}}} \, dt \right)^{\frac{1}{p}}$$

$$= \left(\frac{2|z|^{q-1}}{q-1} \left(\frac{1}{(1-|z|)^{\frac{q-1}{2}}} - 1 \right) \right)^{\frac{1}{q}} \left(\int_0^1 \frac{((1-t^2|z|^2)f^*(tz))^p}{(1-t^2|z|^2)^{\frac{1}{2}}} \, dt \right)^{\frac{1}{p}}$$

$$\leq C \left(\frac{|z|}{\sqrt{1-|z|}} \int_0^1 \frac{((1-t^2|z|^2)f^*(tz))^p}{\sqrt{1-t^2|z|^2}} \, dt \right)^{\frac{1}{p}}.$$

Thus, there exists such C that

$$\sigma(f(z), f(0))^p \leq \frac{C|z|}{\sqrt{1-|z|}} \int_0^1 \frac{((1-t^2|z|^2)f^*(tz))^p}{\sqrt{1-t^2|z|^2}} \, dt \quad , \quad p > 1.$$

Then

$$\iint\limits_D \sigma(f(z), f(0))^p \, dA(z) \leq$$

$$\leq C \iint\limits_D dA(z) \int_0^1 \frac{|z|((1-t^2|z|^2)f^*(tz))^p}{\sqrt{1-|z|^2}\sqrt{1-t^2|z|^2}} \, dt$$

$$= C \int_0^1 \iint\limits_{tD} \frac{|z|((1-t^2|z|^2)f^*(tz))^p}{\sqrt{1-|z|^2}} \cdot \frac{dA(z)\,\frac{dt}{t^3}}{\sqrt{1-\frac{|z|^2}{t^2}}} = \text{(by Fubini's theorem)}$$

$$= C \iint\limits_D \frac{|z|((1-t^2|z|^2)f^*(z))^p}{\sqrt{1-|z|^2}} \, dA(z) \int_{|z|}^1 \frac{dt}{t^2\sqrt{t}\sqrt{t^2-|z|^2}}$$

$$\leq 2C \iint\limits_D ((1-|z|^2)f^*(z))^p \, dA.$$

We obtain that for $1 < p < \infty$ and for some C_1

$$\iint\limits_D \sigma(f(z), f(0))^p \, dA(z) \leq C_1 \iint\limits_D ((1-|z|^2)f^*(z))^p \, dA(z).$$

A change of variables and some properties of Bergman kernal (see e.g. [5]) give

$$\iint\limits_D \iint\limits_D \frac{\sigma(f(z), f(w))^p}{|1-z\bar{w}|^4} \, dA(z) \, dA(w)$$

$$= \iint\limits_D d\lambda(z) \iint\limits_D \sigma^p(f \circ \varphi_z(w), f(z)) dA(w)$$

$$\leq C \iint\limits_D d\lambda(z) \iint\limits_D ((1-|w|^2) f \circ \varphi_z^*(w))^p dA(w)$$

$$= \iint\limits_D d\lambda(z) \iint\limits_D (1-|w|^2)^p f^*(w)^p |k_z(w)|^2 dA(w)$$

$$= \iint\limits_D (1-|w|^2)^p f^*(w)^p dA(w) \iint\limits_D \frac{dA(z)}{|1-z\bar{w}|^4}$$

$$= \iint\limits_D (1-|w|^2)^p f^*(w)^p d\lambda(w).$$

Theorem 3. *For any $1 < p \le \infty$ there exists a constant C_p such that for any function $f(z) \in \mathcal{B}_p^h$ and $z, w \in D$, $\sigma(f(z), f(w) \le 1$,*

$$\sigma(f(z), f(w)) \le C_p \|f\|_{B_p^h} \sigma(z,w)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof.

Since $\|.\|_{B_p^h}$, σ, ρ are Möbius invariant it suffices to prove that there is a constant C_p such that

$$\rho(f(z), f(0)) \le C_p \|f\|_{B_p^h} \sigma(z,0)^{\frac{1}{q}}$$

for all $f \in B_p^h$ and all $z \in D$.

Let $\varphi_f(z) = \dfrac{f(z) - f(0)}{1 - \overline{f(0)} f(z)}$. Since $\varphi_f(z)$ is holomorphic function in D then for any $f \in B$ and $z \in D$ following Zhu [5]

$$\varphi_f(z) = \iint_D \frac{1 - |w|^2}{\bar{w}(1 - z\bar{w})^2} \varphi'_f(w)\, dA(w).$$

Thus there exists a constant C_1 such that (see [2], [3])

$$\rho(f(z), f(0)) = |\varphi_f(z)| \le \iint_D \frac{1 - |w|^2}{|w||1 - z\bar{w}|^2} |\varphi'_f(w)|\, dA(w)$$

$$\le C_1 \iint_D \frac{1 - |w|^2}{|1 - z\bar{w}|^2} f^*(w)(1 - |\varphi_f(w)|^2)\, dA(w)$$

$$\le C_1 \iint_D (1 - |w|^2)\, f^*(w) \left(\frac{1 - |w|^2}{|1 - z\bar{w}|^2} \right)^2 d\lambda(w).$$

By Hölder inequality and 1.4.10 of [3] we get

$$\rho(f(z), f(0)) \le C_1 \|f\|_{B_p^h} \left(\iint_D \frac{(1 - |w|^2)^{2q-2}}{|1 - z\bar{w}|^{2q}}\, dA(w) \right)^{\frac{1}{q}} \le C_2\, \|f\|_{B_p^h} \sigma(z,0)^{\frac{1}{q}}.$$

When points a and b lie closely one to another the behaviours of hyperbolic distance $\sigma(a,b)$ and pseudohyperbolic distance $\rho(a,b)$ are the same. Thus for sufficiently small τ there is a constant C such that for any $z, w \in D$, $\sigma(z,w) \le \tau$,

$$\sigma(f(z), f(w)) \le C\, \|f\|_{B_p^h} \sigma(z,w)^{\frac{1}{q}}.$$

Let $\varphi : D \to D$ analytic and $\Gamma = \{|z| = 1\}$.

Definition. [1] *Bounded function $\varphi(z)$ has a finite angular derivative at ζ on unit circle Γ if there is η on Γ so that $\frac{\varphi(z)-\eta}{z-\zeta}$ has a finite nontangantial limit as $z \to \zeta$.*

By the Julia-Caratheodory Theorem [1] function $\varphi(z)$ has a finite angular derivative at ζ if and only if

$$\liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} < \infty.$$

Let ζ be a point on the unit circle Γ, $0 < \delta < 1$, $0 < \varepsilon < \frac{\pi}{2}$ and $\Delta_\zeta(\delta,\varepsilon) = \{z : |z-\zeta| < \delta, |arg(\zeta) - arg(z)| < \varepsilon\}$.

Theorem 4. *Functions of B_p^h, $1 < p < \infty$, don't have angular derivatives.*

Proof.
Suppose that $\varphi(z) \in B_p^h$, $1 < p < \infty$, and $\varphi(z)$ has an angular derivative at a point $\zeta \in \Gamma$. Then by the Julia-Caratheodory Theorem $\dfrac{1-|\varphi(z)|}{1-|z|} \sim |\varphi'(\zeta)|$ for sufficiently small δ and $z \in \Delta_\zeta(\delta,\varepsilon)$ and moreover, $|\varphi'(z)| \approx |\varphi'(\zeta)|$ for $z \in \Delta_\zeta(\delta,\varepsilon)$. Thus

$$\iint_D ((1-|z|^2)\varphi^*(z))^p \, d\lambda(z) \geq \iint_{\Delta_\zeta(\delta,\varepsilon)} ((1-|z|^2)\varphi^*(z))^p \, d\lambda(z) \approx \iint_{\Delta_\zeta(\delta,\varepsilon)} d\lambda(z) = \infty.$$

3. Composition operators

Let $\mathcal{B}$ be Bloch space of holomorphic functions in D. By the definition (e.g. [5]) $f(z) \in \mathcal{B}$ if

$$\lim_{|z|\to 1} (1-|z|^2)|f'(z)| < \infty.$$

Holomorphic in D function $f(z)$ belongs to the analytic Besov space B_p, $1 < p < \infty$, if

$$\|f\|_{B_p} = \left(\iint_D (1-|z|^2)^p |f'(z)|^p d\lambda(z) \right)^{\frac{1}{p}} < \infty$$

and $B_\infty = \mathcal{B}$.

Let X be a normed subspace of $\mathcal{B}$ and $\varphi(z)$ be a holomorphic self map of the unit disk D. We say that composition operator $C_\varphi : \mathcal{B} \to X$ is compact if

$$\lim_{r\to 1-0} \|f \circ \varphi - f \circ (\chi_r \varphi)\|_X = 0$$

where $\chi_r(z)$ is a characteristic function of the disk $D_r = rD$

$$\chi_r(z) = \begin{cases} 1, & |z| \leq r \\ 0, & |z| > r. \end{cases}$$

Theorem 5. *For every $\varphi(z) \in B_p^h$, $1 < p < \infty$, and any $f \in \mathcal{B}$ composition $f \circ \varphi \in B_p$. Composition operator C_φ is a compact.*

Proof.
Let $f \in \mathcal{B}$ and $\sup\limits_{z\in D}(1-|z|^2)|f'(z)| = M_f$. Let $\varphi(z) \in B_p^h$ and $\|\varphi(z)\|_{B_p^h} = M_\varphi$. If $g(z) = f \circ \varphi(z)$ then

$$\|g(z)\|^p_{B_p^h} = \iint\limits_D (1-|z|^2)^p |g'(z)|^p \, d\lambda(z) =$$

$$= \iint\limits_D (1-|z|^2)^p |f'_\varphi(z)|^p |\varphi'(z)|^p \, d\lambda(z) =$$

$$= \iint\limits_D (1-|z|^2)^p (\varphi^*(z))^p (1-|\varphi(z)|^2)^p |f'_\varphi(z)|^p \, d\lambda(z) \le M_f^p \cdot M_\varphi^p < \infty.$$

Now we prove compactness of operator C_φ.

$$\|f(\varphi(z)) - f(\chi_r \cdot \varphi(z))\|^p_{B_p} =$$

$$= \iint\limits_{D\setminus D_r} (1-|z|^2)^p |(f\circ\varphi(z))'|^p \, d\lambda(z) \le$$

$$\le M_f^p \iint\limits_{D\setminus D_r} (1-|z|^2)^p \varphi^*(z))^p \, d\lambda(z).$$

The last integral tends to zero as $r \to 1$ because $\varphi(z) \in B_h^p$.

Theorem 6. *If for every Bloch function $f(z)$ the composition $f \circ \varphi$ is a p-analytic Besov function, $1 < p < \infty$, then $\varphi(z) \in B_p^h$.*

Proof.
Ramey and Ullrich [2] constructed such Bloch functions f and g that

$$|f'(z)| + |g'(z)| \ge \frac{1}{1-|z|^2}.$$

Then for every $p > 1$

$$|f'(z)|^p + |g'(z)|^p \ge \frac{K_p}{(1-|z|^2)^p}.$$

and thus

$$K_p \|\varphi\|^p_{B_p^h} \le \|f\circ\varphi\|^p_{B_p} + \|g\circ\varphi\|^p_{B_p} < \infty.$$

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