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TOEPLITZ 作用素 及び HANKEL 作用素 の HYPONORMALITY について

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以下の結果は, 今年の秋の学会で既に報告したものであるが, ここでは, その証明も含めて詳しく報告する.

A bounded measurable function $\varphi \in L^\infty$ on the circle induces the multiplication operator on L^2 called the Laurent operator L_φ given by

$$L_\varphi f = \varphi f \text{ for } f \in L^2.$$

And the Laurent operator induces in a natural way twin operators on H^2 called Toeplitz operator T_φ given by

$$T_\varphi f = PL_\varphi f \text{ for } f \in H^2,$$

where P is the orthogonal projection from L^2 onto H^2 and Hankel operator H_φ given by

$$H_\varphi f = J(I-P)L_\varphi f \text{ for } f \in H^2,$$

where J is the unitary operator on L^2 defined by

$$J(z^{-n}) = z^{n-1}, \ n = 0, \pm 1, \pm 2, \cdots .$$

Lemma 1. For $f \in L^2$, let $f^*(z) = \overline{f(\overline{z})}$. Then $\|f^*\|_2 = \|f\|_2$ and $f^* \in L^2$. Particularly, if $f \in H^2$, then $f^* \in H^2$ also.

Lemma 2. For $\varphi \in L^\infty$, $\|\varphi^*\|_\infty = \|\varphi\|_\infty$ and $\varphi^* \in L^\infty$. Particularly, if φ is inner, then φ^* is also inner.

Lemma 3. For $\varphi \in H^\infty$, $J(I-P)L_{\varphi^*} = T_\varphi{}^* J(I-P)$.

Concerning these twin operators, the following results are well known.

Proposition 1. ([1]) $A \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $T_z{}^* A T_z = A$. And, in particular, $A \in \mathcal{B}(H^2)$ is analytic Toeplitz operator (i.e., $A = T_\varphi$ for some $\varphi \in H^\infty$) if and only if $T_z A = A T_z$.

Proposition 2. ([4]) Let q be a non-constant inner function, and let Q be the orthogonal projection from L^2 onto $K = H^2 \ominus T_q H^2$. If $A \in \mathcal{B}(K)$ commutes with QL_zQ, then there is a function ψ in H^∞ such that $\|\psi\|_\infty = \|A\|$ and $A = QL_\psi Q$.

Remark 1. In Proposition 2, we may assume that q is a zero function or an inner function. Because, in the case where $q = 0$, Proposition 2 reduces to Proposition 1 and, in the case where q is a constant inner function, we may take $\psi = 0$ because $A = O$.

Proposition 3. H_φ has the following properties ;

(1) $T_z{}^* H_\varphi = H_\varphi T_z$
(Hence $\mathcal{N}_{H_\varphi} = \{x \in H^2 \ ; \ H_\varphi x = o\}$ is invariant under T_z and $\mathcal{N}_{H_\varphi} = \{o\}$ or $\mathcal{N}_{H_\varphi} = T_q H^2$, where q is inner)

(2) $H_\varphi{}^* = H_{\varphi^*}$

(3) $H_{\alpha\varphi+\beta\psi} = \alpha H_\varphi + \beta H_\psi, \quad \alpha,\ \beta \in \mathbb{C}$

(4) $H_\varphi = O$ if and only if $(I-P)\varphi = o$ (i.e., $\varphi \in H^\infty$)

(5) $\|H_\varphi\| = \inf\{\|\varphi + \psi\|_\infty \ ; \ \psi \in H^\infty\}$

Now we state here the relations between these twin operators.

Proposition 4. $H_\psi{}^* H_\varphi = T_{\overline{\psi}\varphi} - T_{\overline{\psi}} T_\varphi$ and

$$H_{\overline{\varphi}}{}^* H_{\overline{\varphi}} - H_\varphi{}^* H_\varphi = T_\varphi{}^* T_\varphi - T_\varphi T_\varphi{}^*.$$

Proposition 5. For any $\psi \in H^\infty$, $H_\varphi T_\psi = H_{\varphi\psi}$ and $T_\psi{}^* H_\varphi = H_\varphi T_{\psi^*}$.

Concerning the operator inequality of Hankel operators, we have the following.

Theorem 1. The following assertions are equivalent.

(1) $H_{\varphi_1}H_{\varphi_1}{}^* \leq \lambda^2 H_{\varphi_2}H_{\varphi_2}{}^*$ for some $\lambda \geq 0$.

(2) There exists a function $h \in H^\infty$ such that $\|h\|_\infty \leq \lambda$ for some $\lambda \geq 0$ and that $H_{\varphi_1} = H_{\varphi_2}T_h$.

To prove this theorem, we need the following.

Lemma 4. ([3]) For A, $B \in \mathcal{B}(\mathcal{H})$, the following assertions are equivalent.

(1) $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$.

(2) There exists a $C \in \mathcal{B}(\mathcal{H})$ uniquely such that $A = BC$ and that

$$(a)\ \ \|C\|^2 = \inf\{\mu\ ;\ AA^* \leq \mu BB^*\}$$
$$(b)\ \ \mathcal{N}_A = \mathcal{N}_C \quad \text{and} \quad (c)\ \ C\mathcal{H} \subseteq [B^*\mathcal{H}]^\sim.$$

Proof of Theorem 1. If $H_{\varphi_1}H_{\varphi_1}{}^* \leq \lambda^2 H_{\varphi_2}H_{\varphi_2}{}^*$ for some $\lambda \geq 0$, then, by Lemma 4, there exists a $A \in \mathcal{B}(H^2)$ uniquely such that $H_{\varphi_1} = H_{\varphi_2}A$ and that

$$(a)\ \ \|A\|^2 = \inf\{\mu : H_{\varphi_1}H_{\varphi_1}{}^* \leq \mu H_{\varphi_2}H_{\varphi_2}{}^*\} \leq \lambda^2$$
$$(b)\ \ \mathcal{N}_{H_{\varphi_1}} = \mathcal{N}_A \quad \text{and} \quad (c)\ \ AH^2 \subseteq [H_{\varphi_2}{}^*H^2]^{\sim L^2}.$$

And then, by Proposition 3 (1), $\mathcal{N}_{H_{\varphi_2}} = T_qH^2$, where q is a zero function or an inner function and, by Proposition 5, we have, for any $\psi \in H^\infty$,

$$\begin{aligned} A^*T_\psi{}^*H_{\varphi_2}{}^* &= A^*H_{\varphi_2}{}^*T_{\psi^*} = H_{\varphi_1}{}^*T_{\psi^*} \\ &= T_\psi{}^*H_{\varphi_1}{}^* = T_\psi{}^*A^*H_{\varphi_2}{}^* \end{aligned}$$

and hence

$$(A^*T_\psi{}^* - T_\psi{}^*A^*)[H_{\varphi_2}{}^*H^2]^{\sim L^2} = \{o\}. \tag{i}$$

Since

$$\langle (T_qA - AT_q)H^2,\ H_{\varphi_2}{}^*H^2\rangle = \langle H^2,\ (T_qA - AT_q)^*H_{\varphi_2}{}^*H^2\rangle = 0 \ \text{ by (i)},$$

$(T_qA - AT_q)H^2 \subseteq \mathcal{N}_{H_{\varphi_2}} = T_qH^2$ and $\mathcal{N}_{H_{\varphi_2}}$ is invariant under A and hence $[H_{\varphi_2}{}^*H^2]^{\sim L^2}$ is invariant under A^*. Since $[H_{\varphi_2}{}^*H^2]^{\sim L^2}$ is invariant under $T_z{}^*$ by Proposition 3 (2) and (1) and since

$$(A^*T_z{}^* - T_z{}^*A^*)[H_{\varphi_2}{}^*H^2]^{\sim L^2} = \{o\} \ \text{ by (i)},$$

$A^*|[H_{\varphi_2}{}^*H^2]^{\sim L^2}$ commutes with $T_z{}^*|[H_{\varphi_2}{}^*H^2]^{\sim L^2}$ and hence $(A^*|[H_{\varphi_2}{}^*H^2]^{\sim L^2})^*$ commutes with $QL_zQ = (T_z{}^*|[H_{\varphi_2}{}^*H^2]^{\sim L^2})^*$, where Q is the orthogonal projection from L^2 onto $[H_{\varphi_2}{}^*H^2]^{\sim L^2}$. And, by Proposition 2 and Remark 1, there is a function h in H^∞ such that

$$\|h\|_\infty = \|(A^*|[H_{\varphi_2}{}^*H^2]^{\sim L^2})^*\| = \|A^*|[H_{\varphi_2}{}^*H^2]^{\sim L^2}\| \le \|A^*\| = \|A\| \le \lambda$$

and $(A^*|[H_{\varphi_2}{}^*H^2]^{\sim L^2})^* = QL_hQ$. And then, for any $f \in H^2$, we have

$$\begin{aligned} H_{\varphi_1}{}^*f = A^*H_{\varphi_2}{}^*f = QL_h{}^*H_{\varphi_2}{}^*f = QT_h{}^*H_{\varphi_2}{}^*f \\ = H_{\varphi_2}{}^*T_{h^*}f = T_h{}^*H_{\varphi_2}{}^*f \quad \text{by Proposition 5} \end{aligned}$$

and $H_{\varphi_1}{}^* = T_h{}^*H_{\varphi_2}{}^*$ and hence $H_{\varphi_1} = H_{\varphi_2}T_h$.

As a special case of Theorem 1, we have the following.

Theorem 2. H_φ is hyponormal (i.e., $H_\varphi H_\varphi{}^* \le H_\varphi{}^*H_\varphi$) if and only if $H_\varphi = H_\varphi{}^*T_h$ for some $h \in H^\infty$ such that $\|h\|_\infty \le 1$.

Proof. Since $H_\varphi{}^*H_\varphi = H_{\varphi^*}H_{\varphi^*}{}^*$ by Proposition 3 (2), the hyponormality of H_φ is equivalent that there exists a function $h \in H^\infty$ such that $\|h\|_\infty \le 1$ and that $H_\varphi = H_{\varphi^*}T_h = H_\varphi{}^*T_h$ by Theorem 1 and by Proposition 3 (2).

Corollary 1. Every hyponormal Hankel operator is normal.

Proof. If H_φ is hyponormal, then $H_\varphi = H_\varphi{}^*T_h$ for some $h \in H^\infty$ such that $\|h\|_\infty \le 1$ by Theorem 2 and, by Propositions 3 (2) and 5,

$$H_{\varphi^*} = H_\varphi{}^* = T_h{}^*H_\varphi = H_\varphi T_{h^*} = H_{\varphi^*}{}^*T_{h^*}.$$

Since $h^* \in H^\infty$ and $\|h^*\|_\infty = \|h\|_\infty$ by Lemmas 1 and 2, $H_\varphi{}^* = H_{\varphi^*}$ is also hyponormal by Theorem 2. Therefore H_φ is normal.

By Proposition 4, T_φ is hyponormal if and only if $H_\varphi{}^*H_\varphi \le H_{\overline{\varphi}}{}^*H_{\overline{\varphi}}$ and, by Proposition 3 (2), $H_{\varphi^*}H_{\varphi^*}{}^* \le H_{\overline{\varphi}^*}H_{\overline{\varphi}^*}{}^*$ and hence, by Theorem 1,

$$H_{\varphi^*} = H_{\overline{\varphi}^*}T_h$$

for some function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$ and, by using Proposition 3 (2) again, we have the following result.

Theorem 3. T_φ is hyponormal if and only if $H_\varphi = T_h{}^* H_{\overline{\varphi}}$ for some function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$.

Corollary 2. If T_φ is hyponormal, then T_{φ^*} is also hyponormal.

Proof. If T_φ is hyponormal, then, by Theorems 3 and by Proposition 5,

$$H_\varphi = T_h{}^* H_{\overline{\varphi}} = H_{\overline{\varphi}} T_{h^*}$$

for some function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$ and, by Proposition 3 (2),

$$H_{\varphi^*} = H_\varphi{}^* = T_{h^*}{}^* H_{\overline{\varphi}}{}^* = T_{h^*}{}^* H_{\overline{\varphi}^*} = T_{h^*}{}^* H_{\overline{\varphi^*}}$$

and hence, by Theorem 3, T_{φ^*} is also hyponormal because $h^* \in H^\infty$ and $\|h^*\|_\infty = \|h\|_\infty \leq 1$ by Lemmas 1 and 2.

For φ in L^2, we can define the **generalised Hankel operator** H_φ as follows ;

$$H_\varphi f = J(I-P)L_\varphi f \quad \text{for} \quad f \in \mathcal{D}(H_\varphi),$$
$$\text{where} \qquad \mathcal{D}(H_\varphi) = \{f \in H^2 \ : \ \varphi f \in L^2\}.$$

H_φ is generally unbounded and, for its definition domain $\mathcal{D}(H_\varphi)$,

$$H^\infty \subseteq \mathcal{D}(H_\varphi)$$

and we have the following.

Theorem 4. For $\varphi \in L^\infty$, let

$$\varphi = f + \varphi(0) + \overline{g},$$

where f and g in ${H_0}^2$. Then, for any $\psi \in H^\infty$, we have

$$H_\varphi \psi = H_{\overline{g}} \psi.$$

Proof. $H_\varphi\psi = J(I-P)(f\psi + \varphi(0)\psi + \overline{g}\psi) = J(I-P)(\overline{g}\psi) = H_{\overline{g}}\psi$.

Remark 2. It is known that

$$L^\infty \neq H^\infty \oplus \overline{H_0{}^\infty}.$$

By Theorem ~~5.18~~ 4, H_φ is a bounded extension of $H_{\overline{g}}|H^\infty$. Moreover we see that it is also a bounded extension of $H_{\overline{g}}$.

In fact, since $u \in \mathcal{D}(H_{\overline{g}})$ implies $\overline{g}u \in L^2$,

$$fu = \varphi u - \varphi(0)u - \overline{g}u \in L^2$$

because $\varphi \in L^\infty$ and hence $fu \in H^2$. Therefore

$$H_{\overline{g}}u = H_\varphi u \qquad \text{for} \quad u \in \mathcal{D}(H_{\overline{g}})$$

and so H_φ is a bounded extension of $H_{\overline{g}}$.

By the same reason, $H_{\overline{\varphi}}$ is a bounded extension of $H_{\overline{f}}$.

As a special case of Theorem 1, we have the following.

Theorem 5. For $\varphi = f + \varphi(0) + \overline{g} \in L^\infty$, where f and g in $H_0{}^2$ and for some $\lambda \geq 0$, the following assertions are equivalent.

(1) $H_\varphi{}^* H_\varphi \leq \lambda^2 H_{\overline{\varphi}}{}^* H_{\overline{\varphi}}$.

(2) $g = T_{h^*}{}^* f + c$ for some constant c and some function $h \in H^\infty$ such that $\|h\|_\infty \leq \lambda$.

Proof. If $g = T_{h^*}{}^* f + c$ for some constant c and some function $h \in H^\infty$ such that $\|h\|_\infty \leq \lambda$, then

$$c = g - T_{h^*}{}^* f = P(g - \overline{h^*} f) = P(\overline{\overline{g} - h^* \overline{f}})$$

and $\overline{g} - h^*\overline{f} \in H^2$ and hence, by Theorem 4, for any $\psi \in H^\infty$,

$$\|H_\varphi\psi\| = \|H_{\overline{g}}\psi\| = \|J(I-P)L_{h^*}\overline{f}\psi\| = \|T_h{}^* J(I-P)\overline{f}\psi\| \quad \text{by Lemma 3}$$
$$\leq \|T_h{}^*\| \, \|J(I-P)\overline{f}\psi\| = \|h\|_\infty \|H_{\overline{f}}\psi\| = \|h\|_\infty \|H_{\overline{\varphi}}\psi\|.$$

And since $[H^\infty]^{\sim L^2} = H^2$,

$$H_\varphi{}^* H_\varphi \leq \|h\|_\infty^2 H_{\overline{\varphi}}{}^* H_{\overline{\varphi}} \leq \lambda^2 H_{\overline{\varphi}}{}^* H_{\overline{\varphi}}.$$

Conversely, if $H_\varphi{}^* H_\varphi \leq \lambda^2 H_{\overline{\varphi}}{}^* H_{\overline{\varphi}}$, then, by Theorem 1 and by Proposition 3 (2), there exists a function h in H^∞ such that $\|h\|_\infty \leq \lambda$ and that $\varphi^* - \overline{\varphi}^* h \in H^\infty$ and hence $\varphi - \overline{\varphi} h^* \in H^\infty$ by Lemmas 1 and 2. Since

$$\varphi - \overline{\varphi} h^* = (f + \varphi(0) - \overline{\varphi(0)} h^* - g h^*) + (\overline{g} - \overline{f} h^*),$$

we have $\overline{g} - \overline{f} h^* \in H^2$ because $h^* \in H^\infty$. And then $\overline{\overline{g} - h^* \overline{f}} \in [H_0^2]^\perp$ and $P(\overline{\overline{g} - h^* \overline{f}}) = c$ (constant) and hence

$$c = P(g - \overline{h^*} f) = g - P L_{h^*}{}^* f = g - T_{h^*}{}^* f.$$

Corollary 3. ([2]) For $\varphi = f + \varphi(0) + \overline{g} \in L^\infty$, where f and g in $H_0{}^2$, the following assertions are equivalent.

(1) T_φ is hyponormal.

(2) $g = T_{h^*}{}^* f + c$ for some constant c and some function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$.

Proof. Since T_φ is hyponormal if and only if $H_\varphi{}^* H_\varphi \leq H_{\overline{\varphi}}{}^* H_{\overline{\varphi}}$ by Proposition 4, we have the conclusion by setting $\lambda = 1$ in Theorem 5.

References

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