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REPRESENTING AND INTERPOLATING SEQUENCES FOR HARMONIC BERGMAN FUNCTIONS ON THE UPPER HALF-SPACES

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1. Introduction

The upper half-space $H=H_n$ is the open subset of $\mathbb{R}^n$ given by

$$H=\{(z',z_n)\in\mathbb{R}^n : z'\in\mathbb{R}^{n-1}, z_n>0\},$$

where we have written a typical point $z\in\mathbb{R}^n$ as $z=(z',z_n)$. For $1\leq p<\infty$, we will write b^p for the harmonic Bergman space consisting of all harmonic functions u on H such that

$$||u||_p=\left\{\int_H |u(w)|^p\,dw\right\}^{1/p}<\infty.$$

Being closed subspaces of $L^p=L^p(H)$, the spaces b^p are Banach spaces. There is a reproducing kernel $R(z,w)$ such that

$$u(z)=\int_H u(w)R(z,w)\,dw$$

for all $u\in b^p$ and $z\in H$. The explicit formula for $R(z,w)$ is given by (see [3])

$$R(z,w)=\frac{4}{n\sigma_n}\frac{n(z_n+w_n)^2-|z-\overline{w}|^2}{|z-\overline{w}|^{n+2}}.$$

Here, we use the notation $\overline{w}=(w',-w_n)$ for $w\in H$ and σ_n denotes the volume of the unit ball of $\mathbb{R}^n$. The kernel $R(z,w)$ has the following properties:

- $R(z,w)=R(w,z)$
- $R(z,\cdot)$ is a bounded harmonic function on H.
- $R(z,\cdot)\in b^p$ iff $1<p<\infty$.

Associated with the kernel $R(z,w)$ is the integral operator

$$Rf(z)=\int_H f(w)R(z,w)\,dw$$

which takes L^p-functions into harmonic functions on H. In fact, $R:L^2\to b^2$ is the Hilbert space orthogonal projection and $R:L^p\to b^p$ is a bounded projection for

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$1 < p < \infty$. See [9]. Ramey and Yi [9] have also shown that there are many other nonorthogonal bounded projections. To be more explicit, put

$$R_k(z,w) = \frac{(-2)^k}{k!} w_n^k D_{w_n}^k R(z,w) \qquad (k = 0,1,2,\cdots)$$

where D_{w_n} denotes the differentiation with respect to the last component of w. Note that $R_0(z,w) = R(z,w)$. This kernel $R_k(z,w)$ also has the following reproducing property as does $R(z,w)$: If $1 \le p < \infty$ and $u \in b^p$, then

$$u(z) = \int_H u(w) R_k(z,w)\, dw \tag{1.1}$$

for every $z \in H$. Associated with the kernel $R_k(z,w)$ is the integral operator R_k defined by the formula

$$R_k f(z) = \int_H f(w) R_k(z,w)\, dw$$

whenever the above integral makes sense. For $k \ge 1$, the kernel $R_k(z,w)$ behaves better than the kernel $R(z,w)$ in the sense that $R_k : L^p \to b^p$ is a bounded projection for every $1 \le p < \infty$ (see [9]).

The purpose of this lecture is to announce recent joint work [5] with Yi concerning the following properties of b^p-functions:
1. The property of b^p-functions that can be represented as sums based on reproducing kernels along a sequence with weighted ℓ^p-coefficients, which can be viewed as discrete versions of the reproducing formula (1.1).
2. The "dual" property of the above b^p-representation property. This property is the interpolation perperty of b^p-functions.
3. The limiting cases of the above two properties of b^p-functions. These are the representation and interpolation properties of harmonic (little) Bloch functions.

2. Some Geometry

In the hyperbolic geometry of H, the arclength element is $|d\vec{x}|/x_n$ and geodesics are (i) vertical lines and (ii) semi-circles centered on and orthogonal to $\mathbb{R}^{n-1}$. Thus, one can verify that the hyperbolic distance between two points z, $w \in H$ is

$$\log \frac{1+\rho(z,w)}{1-\rho(z,w)}$$

where

$$\rho(z,w) = \frac{|z-w|}{|z-\overline{w}|}.$$

It turns out that this ρ itself is a distance function on H, which we shall call the *pseudohyperbolic* distance. See [7] for the case of the upper half-plane. Note that ρ is horizontal translation invariant and dilation invariant. In particular,

$$\rho(z,w) = \rho\big(\phi_a(z), \phi_a(w)\big) \qquad (z,w \in H) \tag{2.1}$$

where ϕ_a $(a \in H)$ denotes the function defined by

$$\phi_a(z) = \left(\frac{z' - a'}{a_n}, \frac{z_n}{a_n}\right)$$

for $z = (z', z_n) \in H$.

For $z \in H$ and $0 < \delta < 1$, let $E_\delta(z)$ denote the pseudohyperbolic ball centered at z with radius δ. Note that $\phi_z\big(E_\delta(z)\big) = E_\delta(z_0)$ by the invariance property (2.1). Here and later, $z_0 = (0,1) \in H$ is a fixed reference point. Also, a straightfoward calculation shows that

$$E_\delta(z) = B\left(\left(z', \frac{1+\delta^2}{1-\delta^2} z_n\right), \frac{2\delta}{1-\delta^2} z_n\right)$$

so that $B(z, \delta z_n) \subset E_\delta(z) \subset B(z, 2\delta(1-\delta)^{-1} z_n)$ where $B(z,r)$ denotes the euclidean ball centered at z with radius r.

Let $\{z_m\}$ be a sequence in H and $0 < \delta < 1$. We say that $\{z_m\}$ is *δ-separated* if the balls $E_\delta(z_m)$ are pairwise disjoint or simply say that $\{z_m\}$ is *separated* if it is δ-separated for some δ. Pseudohyperbolic balls (with the same radii) centered along a separated sequence cannot intersect too often in the following sense.

Lemma 2.1. *Let $\alpha > 0$ and assume $0 < (1+\alpha)\eta < 1$. If $\{z_m\}$ is an η-separated sequence, then there is a constant $M = M(n, \alpha, \eta)$ such that more than M of the balls $E_{\alpha\eta}(z_m)$ contain no point in common.*

Also, we say that $\{z_m\}$ is a *δ-lattice* if it is $\delta/2$-separated and $H = \cup E_\delta(z_m)$. Note that any "maximal" $\delta/2$-separated sequence is a δ-lattice. The following covering lemma is the main tool in proving our results.

Lemma 2.2. *Fix a $1/2$-lattice $\{a_m\}$ and let $0 < \delta < 1/8$. If $\{z_m\}$ is a δ-lattice, then we can find a rearrangement $\{z_{ij} | i = 1, 2, \ldots, j = 1, 2, \ldots, N_i\}$ of $\{z_m\}$ and a pairwise disjoint covering $\{D_{ij}\}$ of H with the following properties:*

(*a*) $E_{\delta/2}(z_{ij}) \subset D_{ij} \subset E_\delta(z_{ij})$

(*b*) $E_{1/4}(a_i) \subset \cup_{j=1}^{N_i} D_{ij} \subset E_{5/8}(a_i)$

(*c*) $z_{ij} \in E_{1/2}(a_i)$

for all $i = 1, 2, \cdots$, and $j = 1, 2, \ldots, N_i$.

Note. By property (*c*) of the above lemma and Lemma 2.1, the sequnce N_i cannot grow arbitrarily. In fact, we have $N_i = O(\delta^{-n})$.

3. Representing Sequence

For a motivation, consider a sequence $\{z_m\}$ of distinct points in H with $z_m \to \partial H \cup \{\infty\}$ and pick a pairwise disjoint covering $\{E_m\}$ of H such that $z_m \in E_m$. For an integer $k \geq 0$ and $u \in b^p$, we see from the reproducing property (1.1)

$$u(z) = \sum \int_{E_m} u(w) R_k(z, w)\, dw.$$

Let q be the conjugate exponent of p. Then, the series

$$\sum u(z_m) |E_m|^{1/p} \cdot |E_m|^{1/q} R_k(z, z_m) \tag{3.1}$$

can be considered as an approximating Riemann sum of the above integral. Here, we use the notation $|E|$ for the volume of a Borel set $E \subset H$. Note that the sum

$$\sum |u(z_m)|^p |E_m|$$

can be viewed as an approximating Riemann sum of $||u||_p^p$.

Let $\{z_m\}$ be a sequence in H. Let $1 \leq p < \infty$ and $k \geq 0$ be an integer. For $(\lambda_m) \in \ell^p$, let $Q_k(\lambda_m)$ denote the series defined by

$$Q_k(\lambda_m)(z) = \sum \lambda_m z_{mn}^{n(1-1/p)} R_k(z, z_m) \qquad (z \in H). \tag{3.2}$$

Here, we restrict $k \geq 1$ for $p = 1$. For a sequence $\{z_m\}$ good enough, $Q_k(\lambda_m)$ will be harmonic on H. We say that $\{z_m\}$ is a *b^p-representing sequence of order k* if $Q_k(\ell^p) = b^p$.

Of course, the motivation for the series (3.2) is the approximating Riemann sum (3.1) where E_m is pretended to be the ball $E_\delta(z_m)$ for some fixed δ. However, it might not be clear from the very definition that the series (3.2) defines a b^p-function under the separation condition. The following proposition makes this clear.

Proposition 3.1. *Let $1 \leq p < \infty$ and $k \geq 0$ be an integer. Suppose $\{z_m\}$ is a δ-separated sequence. Let Q_k be the associated operator as in* (3.2). *Then, for $1 < p < \infty$, $Q_k : \ell^p \to b^p$ is bounded for each $k \geq 0$. Also, $Q_k : \ell^1 \to b^1$ is bounded for each $k \geq 1$.*

We now state our b^p-representation result under the lattice density condition. We first consider the case $1 < p < \infty$.

Theorem 3.2. *Let $1 < p < \infty$ and let $k \geq 0$ be an integer. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ-lattice with $\delta < \delta_0$ and let $Q_k : \ell^p \to b^p$ be the associated linear operator as in* (3.2). *Then there is a bounded linear operator $P_k : b^p \to \ell^p$ such that $Q_k P_k$ is the identity on b^p. In particular, $\{z_m\}$ is a b^p-representing sequence of order k.*

The b^1-representation theorem takes exactly the same form as the above b^p-representation theorem except for the restriction $k \geq 1$. This restriction is caused by the fact that the operator R is not L^1-bounded.

Theorem 3.3. *Let $k \geq 1$ be an integer. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ-lattice with $\delta < \delta_0$ and let $Q_k : \ell^1 \to b^1$ be the associated linear operator as in* (3.2)*. Then there is a bounded linear operator $P_k : b^1 \to \ell^1$ such that $Q_k P_k$ is the identity on b^1. In particular, $\{z_m\}$ is a b^1-representing sequence of order k.*

4. Interpolating Sequence

We have seen that the representation property amounts to the "onto" property of the operator Q_k. Considering their adjoint operators we are led to the interpolation property. For example, consider a δ-separated sequence $\{z_m\}$ and let $k = 0$ for simplicity. The associated operator Q_0 is then bounded from ℓ^p into b^p for $1 < p < \infty$ by Proposition 3.1. Let q be the conjugate exponent of $p \in (0, \infty)$. Using the duality $(b^p)^* = b^q$ ([9]) under the standard integral pairing, one can check that the adjoint operator of $Q_0 : \ell^p \to b^p$ can be identified with $T_0 : b^q \to \ell^q$ defined by $T_0 u = \left(z_{mn}^{n/q} u(z_m)\right)$.

Let $\{z_m\}$ be a sequence in H. Let $k \geq 0$ be an integer and $1 \leq p < \infty$. Associated with the sequence $\{z_m\}$ is the operator T_k taking a b^p-function u into the sequence $T_k u$ of complex numbers defined by

$$T_k u = \left(z_{mn}^{n/p+k} D^k u(z_m)\right) \tag{4.1}$$

where D denotes the differentiation with respect to the last component. We say that $\{z_m\}$ is a *b^p-interpolating sequence of order k if $T_k(b^p) = \ell^p$.*

Separation is necessary for b^p-interpolation.

Proposition 4.1. *Every b^p-interpolating sequence of order k is separated.*

On the other hand, separation ensures the boundedness of the operator T_k.

Proposition 4.2. *Let $1 \leq p < \infty$ and $k \geq 0$ be an integer. Suppose $\{z_m\}$ is a δ-separated sequence. Let T_k be the associated operator as in* (4.1)*. Then, for $1 \leq p < \infty$, $T_k : b^p \to \ell^p$ is bounded.*

Instead of the lattice density condition for representation, we need the sufficient separation condition for interpolation.

Theorem 4.3. *Let $1 \leq p < \infty$ and $k \geq 0$ be an integer. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ-separated sequence with $\delta > \delta_0$ and let $T_k : b^p \to \ell^p$ be the associated linear operator as in* (4.1)*. Then there is a bounded linear operator $S_k : \ell^p \to b^p$ such that $T_k S_k$ is the identity on ℓ^p. In particular, $\{z_m\}$ is a b^p-interpolating sequence of order k.*

5. The Limiting Case $p \to \infty$

When one tries to describe the dual of b^1, one may expect that the dual of b^1 would be the Bergman projections of L^∞-functions. However, the Bergman integral is not even defined on L^∞, simply because the kernel $R(z,\cdot)$ is not integrable. Overcoming this difficulty, Ramey and Yi [9] have shown that the dual of b^1 is identified with the "modified" Bergman projections of L^∞. They consider the integral operator

$$\tilde{R}f(z) = \int_H f(w)\tilde{R}(z,w)\,dw,$$

where

$$\tilde{R}(z,w) = R(z,w) - R(z_0,w)$$

is a kernel which is an integrable function of w for each fixed z, and prove the duality $(b^1)^* = \tilde{R}(L^\infty)$. Ramey and Yi [9] also give an intrinsic description of the space $\tilde{R}(L^\infty)$ by means of the growth restriction of derivatives. To be more precise, let u be a harmonic function on H. We shall say $u \in \tilde{\mathcal{B}}$, the harmonic Bloch space, if $u(z_0) = 0$ and if

$$||u||_{\tilde{\mathcal{B}}} = \sup_{w\in H} w_n|\nabla u(w)| < \infty.$$

It then turns out that $\tilde{R}(L^\infty) = \tilde{\mathcal{B}}$. We also say that $u \in \tilde{\mathcal{B}}_0$, the harmonic little Bloch space, if $u \in \tilde{\mathcal{B}}$ satisfies the additional boundary vanishing condition

$$\lim w_n|\nabla u(w)| = 0$$

where the limit is taken as $w \to \partial H \cup \{\infty\}$. It is not hard to verify that $\tilde{\mathcal{B}}$ is a Banach space and $\tilde{\mathcal{B}}_0$ is a closed subspace of $\tilde{\mathcal{B}}$. Also, $\tilde{\mathcal{B}}_0$ is identified with the predual of b^1 in [11].

More generally, for an integer $k \geq 0$, consider the modified kernel

$$\tilde{R}_k(z,w) = R_k(z,w) - R_k(z_0,w).$$

Then $\tilde{R}_k(z,w)$ has the following reproducing property for harmonic Bloch functions: If $u \in \tilde{\mathcal{B}}$, then

$$u(z) = \int_H u(w)\tilde{R}_k(z,w)\,dw \tag{5.1}$$

for all $z \in H$. The associated integral operator $\tilde{R}_k$ defined by the formula

$$\tilde{R}_k f(z) = \int_H f(w)\tilde{R}_k(z,w)\,dw$$

takes L^∞ onto $\tilde{\mathcal{B}}$ boundedly. A consideration of approximating Riemann sum of the reproducing formula (5.1) leads us to a similar definition of representing sequences for the spaces $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}_0$.

Let $\{z_m\}$ be a sequence in H and $k \geq 0$ be an integer. For $(\lambda_m) \in \ell^\infty$, let

$$\tilde{Q}_k(\lambda_m)(z) = \sum \lambda_m z_{mn}^n \tilde{R}_k(z,z_m) \qquad (z \in H). \tag{5.2}$$

We say that $\{z_m\}$ is a *$\widetilde{\mathcal{B}}$-representing sequence of order k* if $\widetilde{Q}_k(\ell^\infty) = \widetilde{\mathcal{B}}$. We also say that $\{z_m\}$ is a *$\widetilde{\mathcal{B}}_0$-representing sequence of order k* if $\widetilde{Q}_k(c_0) = \widetilde{\mathcal{B}}_0$.

As in the case of b^p-representation, separation implies boundedness of the operator $\widetilde{Q}_k$.

Proposition 5.1. *Let $k \geq 0$ be an integer and suppose $\{z_m\}$ is a δ-separated sequence. Let $\widetilde{Q}_k$ be the associated operator as in* (5.2). *Then, $\widetilde{Q}_k : \ell^\infty \to \widetilde{\mathcal{B}}$ is bounded. In addition, $\widetilde{Q}_k$ maps c_0 into $\widetilde{\mathcal{B}}_0$.*

The following is the limiting version of the b^p-representation theorem (Theorem 3.2).

Theorem 5.2. *Let $k \geq 0$ be an integer. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ-lattice with $\delta < \delta_0$ and let $\widetilde{Q}_k : \ell^\infty \to \widetilde{\mathcal{B}}$ be the associated linear operator as in* (5.2). *Then there exists a bounded linear operator $\widetilde{P}_k : \widetilde{\mathcal{B}} \to \ell^\infty$ such that $\widetilde{Q}_k\widetilde{P}_k$ is the identity on $\widetilde{\mathcal{B}}$. Moreover, $\widetilde{P}_k$ maps $\widetilde{\mathcal{B}}_0$ into c_0. In particular, $\{z_m\}$ is a both $\widetilde{\mathcal{B}}$-representing and $\widetilde{\mathcal{B}}_0$-representing sequence of order k.*

Let $k \geq 1$ be an integer and let $\{z_m\}$ be a sequence in H. For $u \in \widetilde{\mathcal{B}}$, let $\widetilde{T}_k u$ denote the sequence of complex numbers defined by

$$\widetilde{T}_k u = \left(z_{mn}^k D^k u(z_m)\right). \tag{5.3}$$

We say that $\{z_m\}$ is a *$\widetilde{\mathcal{B}}$-interpolating sequence of order k* if $\widetilde{T}_k(\widetilde{\mathcal{B}}) = \ell^\infty$. We also say that $\{z_m\}$ is a *$\widetilde{\mathcal{B}}_0$-interpolating sequence of order k* if $\widetilde{T}_k(\widetilde{\mathcal{B}}_0) = c_0$.

Note that $\widetilde{T}_k : \widetilde{\mathcal{B}} \to \ell^\infty$ is clearly bounded. Also, if $\{z_m\}$ is separated, then $z_m \to \partial H \cup \{\infty\}$ and therefore T_k maps $\widetilde{\mathcal{B}}_0$ into c_0. As in the case of b^p-interpolation, separation turns out to be necessary for $\widetilde{\mathcal{B}}$-interpolation or $\widetilde{\mathcal{B}}_0$-interpolation.

Proposition 5.3. *Every $\widetilde{\mathcal{B}}$-interpolating sequence of order k is separated. Also, every $\widetilde{\mathcal{B}}_0$-interpolating sequence of order k is separated.*

The following theorem shows that "sufficient separation" is also sufficient for $\widetilde{\mathcal{B}}$-interpolation or $\widetilde{\mathcal{B}}_0$-interpolation.

Theorem 5.4. *Let $k \geq 1$ be an integer. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ-separated sequence with $\delta > \delta_0$ and let $\widetilde{T}_k : \widetilde{\mathcal{B}} \to \ell^\infty$ be the associated linear operator as in* (5.3). *Then there exists a bounded linear operator $\widetilde{S}_k : \ell^\infty \to \widetilde{\mathcal{B}}$ such that $\widetilde{T}_k\widetilde{S}_k$ is the identity on ℓ^∞. Moreover, $\widetilde{S}_k$ maps c_0 into $\widetilde{\mathcal{B}}_0$. In particular, $\{z_m\}$ is a both $\widetilde{\mathcal{B}}$-interpolating and $\widetilde{\mathcal{B}}_0$-interpolating sequence of order k.*

6. Remarks

In the holomorphic case representation and interpolation properties of Bergman functions have been studied by several authors on various domains. For representation theorems, see [6], [8]. For interpolation theorems, see [1], [10] for Bergman functions and [2], [4] for Bloch functions.

In the harmonic case, representation theorems for harmonic Bergman functions on the ball are proved in [6]. Theorem 3.2 should be compared with Theorem 3 of Coifman and Rochberg [6]. While their theorem has the advantage of being valid for $p < 1$, it contains the restriction $k \geq 1$ for $1 < p < \infty$.

Proofs of the results stated above can be found in [5] which will appear elsewhere. In [5] our argument takes a more constructive idea of [6] rather than duality argument of [8]. In [5] one can find some other related results and applications.

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