数理解析研究所講究録
946 巻 1996 年 43-46

F-algebra M of holomorphic functions

Hong Oh Kim
KAIST, Tagjon, KOREA

1. Introduction.

Let U be the unit disc $\{|z| < 1\}$ in $\mathbb{C}$. A function f holomorphic in U is said to belong to the class M if

$$\rho(f) \equiv \int_0^{2\pi} \log(1 + Mf(\theta))\, d\theta < \infty$$

where $Mf(\theta) = \sup_{0\leq r<1} |f(re^{i\theta})|$ and $\log^+ x = \max(\log x, 0), \quad x > 0$. The class M was introduced and studied in [1, 2, 3, 4, 5]. The class M is related to the usual Hardy space $H^p (p > 0)$ and the Nevanlinna class N^+ as

$$\cup_{p>0} H^p \subsetneqq M \subsetneqq N^+$$

The class M with the metric $d(f, g) = \rho(f - g)$ is an F-algebra, i.e, a topological vector space with a complete translation invariant metric in which multiplication is continuous. The class M has many similarities with N^+, but it is not fully studied as N^+. In this report we wish to summarize the works on the class M [1, 2, 3, 4, 5] and some open problems. We refer to [7] for the Hardy space and the Smirnov class.

2. M as a class of functions.

For a real-valued function h in $L^1(\partial U)$, we let

$$f(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} h(e^{it}) dt\right).$$

We have

2.1 Theorem. If $P[h^+] \in \mathrm{Re}\, H^1$, then $f \in M$ where $P[h^+]$ is the Poisson integral of $h^+ = \max(h, 0)$. The converce is false.

2.2 Problem. Find a necessary and sufficient condition on h in order that $f \in M$. That is, characterize those outer functions in M.

Unlike N^+, the inner factor cannot be cancelled in M as in the following theorem.

2.3 Theorem. [1] There exists an f in M whose outer factor F is not in M. It is easy to see that a finite Blaschke factor of $f \in M$ can be cancelled in M but we do not know whether an infinite Blaschke factor of f can be cancelled in M or not.

2.4 For $\alpha > 1$, we define

$$M_\alpha f(e^{i\theta}) = \sup\{|f(z)| : z \in \Gamma_\alpha(e^{i\theta})\}$$

where $\Gamma_\alpha(e^{i\theta})$ is the nontangential region at $e^{i\theta}$ defined as

$$\Gamma_\alpha(e^{i\theta}) = \{z \in U : |e^{i\theta} - z| < \frac{\alpha}{2}(1 - |z|^2)\}$$

In the definition of M, the radial maximal function $Mf(e^{i\theta})$ can be replaced by the nontangential maximal function $M_\alpha f(e^{i\theta})$. Precisely we have

2.5 Theorem. There exists a positive constant C_α such that

$$\int_0^{2\pi} \log\left(1 + Mf(e^{i\theta})\right)\, d\theta \ \le C_\alpha \int_0^{2\pi} \log\left(1 + Mf(e^{i\theta})\right)\, d\theta$$

2.6 Corollary. The class M is invariant under the composition of automorphisms of the unit disc U. More precisely, if $M \in M$ then $f \circ \varphi \in M$ for any $\varphi \in$ Aut (U).

2.7 Problem. Is M invariant under the composition of inner functions? Recall that N^+ is invariant under the composition of inner fuctions.

For the boundary values of functions in M, the following is proved in [5].

2.8 Theorem. [5] A measurable function $g(e^{i\theta})$ on ∂U coincides with the angular boundary value of some function f in M if and only if there exists a sequence of polynomials p_n with properties :

(a)$p_n(e^{i\theta}) \to g(e^{i\theta})$ a.e. on ∂U and

(b)$\overline{\lim_{n\to\infty}} \displaystyle\int_0^{2\pi} \log\left(1 + Mp_n(\theta)\right) d\theta < \infty$.

3. M as an F-space

It is proved in [1] that M with the metric $d(f,g) = \rho(f-g)$ is a separable F-space. The space M has many similarities as N^+ as F-spaces.

3.1 Theorem. M is not locally bounded.

3.2 Theorem. If Λ is a continuous linear functional on M, then there exists a $g \in A^\infty(U)$(i.e., g is analytic in U and C^∞ on $\bar{U}$) such that

$$\Lambda f = \lim_{r \to 1} \int_0^{2\pi} f(re^{i\theta})\overline{g(e^{i\theta})}d\theta, \quad f \in M.$$

Conversely, if $g \in A^\infty(U)$ and if

$$\Lambda f = \lim_{r \nearrow 1} \int_0^{2\pi} f(re^{i\theta})\overline{g(e^{i\theta})}d\theta$$

exists for all $f \in M$, then Λ defines a continnous linear functional on M.

3.3 Problem. Describe $g \in A^\infty(U)$ more precisely in the above theorem.

3.4 Theorem. M is not locally convex.

4. M as an F-algebra

As an F-algebra M, the invertible elements, multiplicative linear functionals, closed maximal ideals and onto algebra endomorphisms of M are determined as we see in the following theorems .

4.1 Theorem. The only invertible elements of M are those outer function f with $\log |f| \in \text{Re } H^1$.

4.2 Theorem. γ is a nontrivial multiplicative linear functional on M if and only if $\gamma(f) = f(\lambda)$, $f \in M$, for some $\lambda \in U$. Therefore, every nontrivial multiplicative linear functional is continuous.

4.3 Theorem. Every closed maximal ideal of M is the kernel of a multiplicative linear functional.

4.4 Theorem. There exists a maximal ideal M which is not the kernel of a multiplicative linear functional.

4.5 Theorem. $\Gamma : M \to M$ is an onto algebra endomorphism if and only if $\Gamma(f) = f \circ \varphi$, $f \in M$, for some automorphism φ of U. In particular, Γ is invertible.

References

[1] B. R. Choe and H. O. Kim, On the boundary behavior of functions holomorphic on the ball, Complex Variables, 20(1992), 53 - 61
[2] H. O. Kim, On an F-algebra of holomorphic functions, Can. J. Math. 40(1988), 718-741
[3] H. O. Kim, On closed maximal ideals of M, 62(1985), 343-346
[4] H. O. Kim and Y. Y. Park, Maximal functions of plurisubharmonic functions, Tsukuba J. Math., 16(1992), 11-18
[5] V. I. Gavrilov and V. S. Zakharyan, Conformal invariance and characteristic property of boundary values of functions of class M, Dokl. of Academy of Sciences of Armenia, 93(1992), 105-109 (Russian)
[6] N. Yanagihara and Y. Nakamura, Sugaku 28(1976), 323-334 (Japanese)
[7] P. L. Duren, Theory of H^p spaces, Pure and Apple. Math. 38, Academic Press, 1970