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ファラデー水面波のパターン選択

Pattern Formation in Parametrically Excited Surface Waves

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Abstract: Pattern formation in parametrically excited gravity-capillary waves is studied by a weakly nonlinear three-mode system. Coefficients of nonlinear interaction are obtained by the average Lagrangian method. Stability and bifurcations are investigated in detail. It is shown that, in capillary waves, supercritical bifurcation of a square pattern is stable to line-type disturbances with an arbitrary intersecting angle and to *internal* stability, patterns of line and rhombus are stable in the confined subcritical branch and there is no stable hexagonal pattern. On the other hand, a stable pattern is line in deep gravity waves. The presented theory are compared with the experiments of capillary wave and interfacial wave close to the critical point.

Pattern formation in fluids (e.g. Bénard convection) is an interesting physical phenomenon where nonlinearity plays a dominant role. Recent experimental investigations have revealed that various patterns of line [2,5,7], square [1,2,8,5], hexagon [6,8,7], triangle [8], rhombus [6] and quasi-periodicity [7] may be selected in parametrically excited (Faraday) surface waves with various fluids and forcing conditions. Tufillaro et al. [1] showed capillary waves of n-butyl alcohol driven by sinusoidal forcing prefer square pattern at the onset of wave excitation. On the other hand, Fauve et al. [2] observed that the patterns on the liquid-vapor interface of CO_2 change from squares to straight lines as the temperature approaches the critical value. Theoretical analyses were also given on this problem by somewhat different methods and approximations [4]. In this paper, we derive a three-mode system of Faraday surface waves, which can describe line, square, hexagonal and rhombic patterns simultaneously, by using the average Lagrangian method. Coefficients of nonlinear interaction between lines intersecting at an arbitrary angle are calculated. One of them takes values significantly different between in capillary and gravity waves on infinitely deep fluid. The stability of various modes is examined to disturbances of the third line mode with an arbitrary intersecting angle in addition to the stability in the two-mode system. It is shown that a square pattern is stable in the supercritical bifurcation in capillary waves and a branch of line is stable in gravity waves. Other patterns are unstable or stable only in the subcritical branches.

The evolution of modes of standing surface waves with external acceleration $4a_0\omega^2 \cos 2\omega t$ is well described by the following Hamiltonian system derived by the average Lagrangian method:

$$\left[\frac{d}{d\tau} + \alpha\right](p_i, q_i) = \left(-\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i}\right) H(p_1, q_1, \cdots, p_n, q_n), \qquad i = 1, \cdots, n \tag{1}$$

where p_i, q_i are ith amplitudes of in- and out of phase, α is a linear damping coefficient, $\tau = \epsilon^2\omega t$ is a slow time-variable and H is a Hamiltonian function. The expression of H is given by

$$H = \sum_i [\frac{1}{2}\beta_i r_i^2 + \frac{1}{2}A_0(p_i^2 - q_i^2) + \frac{1}{4}A_i r_i^4] + \sum_{i<j} [\frac{1}{2}C_{ij} r_i^2 r_j^2 + \frac{1}{2}D_{ij} M_{ij}^2],$$

$$r_i^2 = p_i^2 + q_i^2, \qquad M_{ij} = p_i q_j - p_j q_i,$$

$$\beta_i = (1 - \omega_i^2/\omega^2)/(2\epsilon),$$

$$A_0 = a_0 \kappa_1 \tanh \kappa_1 d/\epsilon^2,$$

where κ_i and ω_i are the wavenumber and angular frequency of the ith mode, d the depth of the fluid, ϵ a small expansion parameter. The surface displacement is expressed as

$$\eta(t, \boldsymbol{x}) = \sum_i \eta_i(t)\psi_i(\boldsymbol{x}),$$

where ψ_i is an eigenmode, $\boldsymbol{x} = (x, y)$ and the amplitude η_i is given by

$$\eta_i(t) = \epsilon(\kappa_i \tanh \kappa_i d)^{-1}(p_i \cos \omega t + q_i \sin \omega t) + O(\epsilon^2)$$

In the following, we consider cases of degenerate three modes. We put $\beta_1 = \beta_2 = \beta_3 \equiv \beta$ and use convenient notation $C_{ij} = C_k$, $D_{ij} = D_k$ for (i,j,k) =(1,2,3), (2,3,1) and (3,1,2).

Various patterns are described by superposition of line modes $\psi^\theta(\boldsymbol{x}) = \sqrt{2}\cos \boldsymbol{k}\cdot\boldsymbol{x}, \theta = \tan^{-1}(k_y/k_x)$ with different directions of wavevector $\boldsymbol{k}$. Alternatively, an expression of orthonormal eigenmodes $(m,n) = \psi_{mn}(\boldsymbol{x}) = [(2-\delta_{m0})(2-\delta_{n0})]^{1/2}\cos k_x x \cos k_y y$ in a square container of length l is invoked, where $(k_x, k_y) = (m/l, n/l)$, in order to use the previous result of two-mode competitions. Line and rhombic modes may be expressed as $(m,0)$ [or $(0,m)$] and $(m_1, n_1) = (m/2, \sqrt{3}m/2)$. Square and hexagon are mixed states of $(m,0)$, $(0,m)$ and $(m,0)$, (m_1,n_1), respectively.

Fixed points of the dynamical system (1) can be classified as single-mode (denoted by S_i), double-mode (D_{ij}) and triple-mode states (T), respectively. A double-mode state is subdivided into $M_{ij} = 0$ (denoted by $\mathrm{D}^{\mathrm{a}}_{ij}$) and $M_{ij} \neq 0$ ($\mathrm{D}^{\mathrm{b}}_{ij}$). Similarly, A triple-mode has substates T^{a} with all vanishing M_{ij}, T^{b} where one of M_{ij} vanishes and T^{c} with all nonzero M_{ij}.

Linear analysis shows that the stability boundary of S_i lies on the hyperbola

$$A_0 = A_0^{\mathrm{S}i,j} \equiv \left[\alpha^2 + \left(\frac{A_i - C_{ij} - D_{ij}}{A_i + C_{ij} + D_{ij}}\beta\right)^2\right]^{1/2},$$

which denotes a bifurcation to $\mathrm{D}^{\mathrm{b}}_{ij}$. The stability curves of $\mathrm{D}^{\mathrm{a}}_{ij}$ are given by

$$A_0 = A_0^{\mathrm{M}ij\mathrm{a},k} \equiv \left[\alpha^2 + \left(\frac{E_{ijk}}{F_{ijk}}\beta\right)^2\right]^{1/2}, \tag{2}$$

$$E_{ijk} = A_iA_j - A_iC_i - A_jC_j + C_iC_k + C_jC_k - C_k^2 - A_iD_i + C_kD_i - A_jD_j + C_kD_j$$

$$F_{ijk} = A_iA_j + A_iC_i + A_jC_j - C_iC_k - C_jC_k - C_k^2 + A_iD_i - C_kD_i + A_jD_j - C_kD_j$$

and

$$A_0 = A_0^{\mathrm{M}ij\mathrm{a,b}} \equiv \left[\alpha^2 + \left(\frac{-D_k(A_i + A_j - 2C_k)}{D_k(A_i + A_j - 2C_k) + 2(A_iA_j - C_k^2)}\beta\right)^2\right]^{1/2} \tag{3}$$

Eq. (2) shows the bifurcation to the T state, and eq. (3) is where $\mathrm{D}^{\mathrm{b}}_{ij}$ coincides with $\mathrm{D}^{\mathrm{a}}_{ij}$.

The explicit expressions of the nonlinear coefficients A_i, C_i, D_i derived by using Umeki's previous result [3] will be published separately [9]. They can be calculated by two coefficients

$C(\theta)$ and $D(\theta)$ which describe interaction between two line modes intersecting at an angle θ. In capillary waves on infinitely deep fluid, they reduce to

$$C_c(\theta) = \frac{1}{16} + \frac{3}{8}\cos^2\theta + \frac{1}{2}(\cos^3\frac{\theta}{2} + \sin^3\frac{\theta}{2}) - \frac{\cos\theta/2}{2(1-2\cos^3\theta/2)}(1-\cos\frac{\theta}{2} - \frac{1}{2}\cos^2\frac{\theta}{2})^2$$

$$-\frac{\sin\theta/2}{2(1-2\sin^3\theta/2)}(1-\sin\frac{\theta}{2} - \frac{1}{2}\sin^2\frac{\theta}{2})^2,$$

$$D_c(\theta) = -\frac{1}{4}(3+\cos^2\theta).$$

and gravity waves on infinitely deep fluid lead to

$$C_g(\theta) = -\frac{1}{4} + \frac{1}{2}(\cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2}) + \frac{1}{2}(\cos^3\frac{\theta}{2} + \sin^3\frac{\theta}{2})$$

$$-\frac{\sin\theta/2}{2-\sin\theta/2}(1-\sin\frac{\theta}{2} - \frac{1}{2}\sin^2\frac{\theta}{2})^2 - \frac{\cos\theta/2}{2-\cos\theta/2}(1-\cos\frac{\theta}{2} - \frac{1}{2}\cos^2\frac{\theta}{2})^2,$$

$$D_g(\theta) = D_c(\theta).$$

In the case of infinitely deep fluid, A_1 of the line mode can be calculated as

$$A_1 = \frac{1}{2}\lim_{\theta\to 0} C(\theta). \tag{4}$$

It is shown that the experimental data of Tufillaro et al. leads to values of nonlinear coefficients very close to those for capillary waves on infinitely deep fluid.

In order to analyze the stability of various patterns, the following three cases are worth studying. First, a system of three symmetric line modes, which intersect at the angle of $\pi/3$, is considered. It can describe patterns of line by S, rhombus by D^a and hexagon by T^a. Nonlinear coefficients are symmetric, i.e. $A_1 = A_2 = A_3$ given by (4), $C_1 = C_2 = C_3 = C(\pi/3)$ and $D_1 = D_2 = D_3 = D(\pi/3)$.

Second, we consider the linear stability of line and square patterns to line-type disturbance with an arbitrary intersecting angle θ. We take modes as $1 = \psi^0 = (m,0)$, $2 = \psi^{\pi/2} = (0,m)$ and $3 = \psi^\theta$. The coefficients are given by $A_1 = A_2 = A_3$ in (4), $C_3 = C(\pi/2)$, $D_3 = D(\pi/2)$, $C_2 = C(\theta)$, $D_2 = D(\theta)$, $C_1 = C(\pi/2-\theta)$ and $D_1 = D(\pi/2-\theta)$.

Third, modes are taken as $1 = (m,0)$, $3 = (m_3, n_3)$ and $2 = \psi^\theta$. Nonlinear coefficients are $A_1 = A_2$ given by (4), $C_3 = C(\theta)$, $D_3 = D(\theta)$, $A_3 = [A_1 + C(2\varphi)]/2$, where $\varphi = \tan^{-1} m_3/n_3$,

$C_2 = C(\varphi)$, $D_2 = D(\varphi)$, $C_1 = [C(\theta-\varphi)+C(\theta+\varphi)]/2$, and $D_1 = [D(\theta-\varphi)+D(\theta+\varphi)]/2$. If $\theta = \pi/2$ and $\varphi = \pi/3$ are substituted in the third case, line (corresponding to S_1, S_2), square (D^a_{12}), hexagon (D^a_{31}) and rhombic patterns (S_3) can be treated with simultaneously in the three-mode system.

Eigenvalues of the Jacobi matrix of (1) for the various patterns are computed while changing θ of the mode ψ^θ. The second case is used for the line and square patterns, and the third case with $\varphi = \pi/3$ for the rhombus and hexagon. For the rhombus, instability in the direction of two superposed line modes, which can be obtained by the first case, is also considered. This instability is called *internal.* Figure 1 shows stable regions of various modes in the parameter space (β, A_0) for capillary waves (a) and gravity waves (b). In capillary waves, the square pattern is stable on the supercritical branch $\beta > 0$, the line and rhombic patterns are stable only on the confined subcritical branches $\beta < 0$ and the hexagon is unstable everywhere. In contrast with it, the line mode is the only stable pattern in gravity waves.

Computed bifurcation diagrams of the fixed points of the third case with $\theta = \pi/2$ and $\varphi = \pi/3$ in capillary waves are shown in detail in figure 2. It is shown that a square pattern is the only supercritically stable branch. The structure in the subcritical branch, which includes T^c and D^b_{ij} states, is more complicated than that in the supercritical branch.

This result explains well the observation of square patterns in the capillary-wave experiment [1]. In addition, the transition from square to line patterns found in the liquid-vapor interface of CO_2 near the critical point [2] matches with the change of selected patterns in capillary and gravity waves in the present analysis, since the surface tension decreases as T approaches T_c. A quantitative measurement of dependence of density difference and the surface tension on $T - T_c$ will be necessary in order to confirm this conjecture.

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Figure captions

Figure 1. Stable regions of various patterns in the parameter space (β, A_0) in capillary waves (a) and gravity waves (b).

Figure 2. Bifurcations of the three-mode system including patterns of line, square, rhombus and hexagon for $\beta < 0$ (a) and $\beta > 0$ (b). Stable branches are denoted by thick curves.

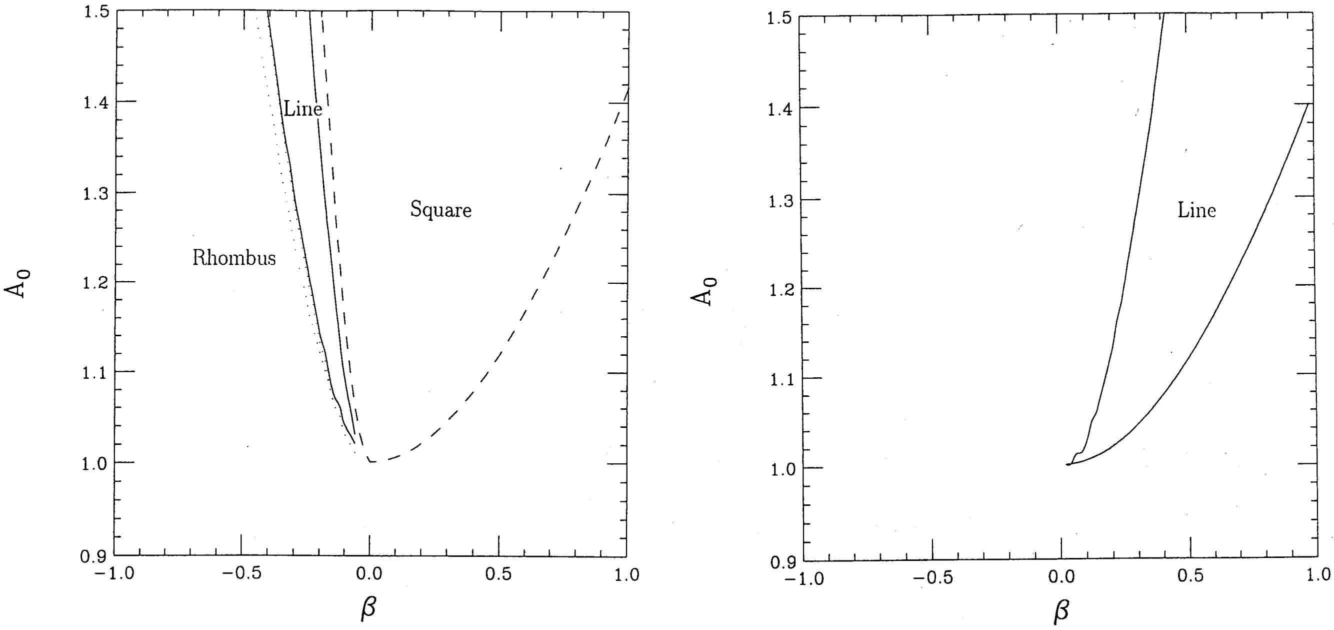

Figure 1. (a) (b)

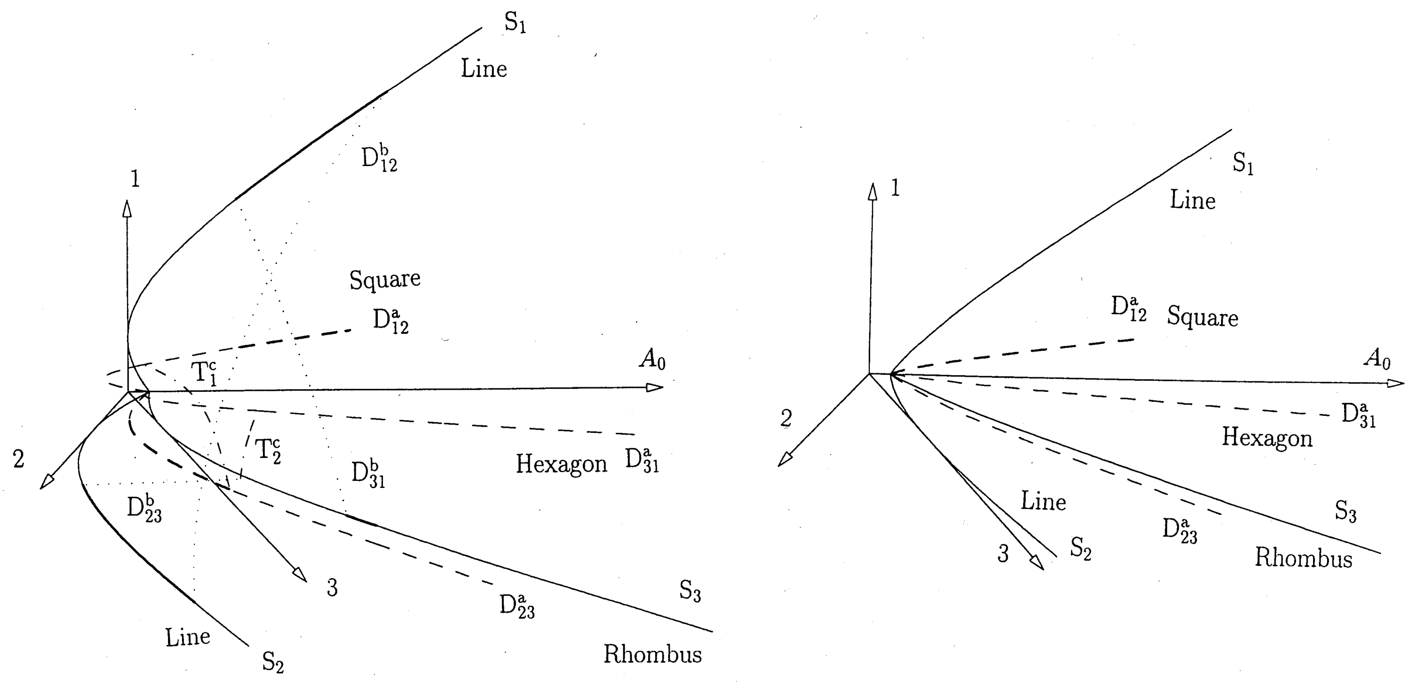

Figure 2. (a) (b)