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On Tractable Slices of Some NP-Complete Functions
NP 完全なブール関数に対する多項式時間スライス関数について

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Abstract

In this paper, we show a number of tractable slices of graph-theoretic NP-complete functions, such as MAXIMUM CLIQUE SIZE, INDEPENDENT SET, CHROMATIC NUMBER, HAMILTONIAN CIRCUIT, and BANDWIDTH. Here, the word tractable means that the slice functions can be computed by some polynomial size circuits. In order to show these results, we consider the recognition problems of several specific graphs, such as Turán graphs.

本論文では、MAXIMUM CLIQUE SIZE, INDEPENDENT SET, CHROMATIC NUMBER, HAMILTONIAN CIRCUIT, BANDWIDTH などのグラフの性質に関する NP 完全関数のスライス関数で、多項式サイズ回路で計算できるものをいくつか示す。これらの結果を得るために、Turán グラフなどの特定のグラフの認識問題を考察する。

1 Introduction

A combinational circuit is a circuit which consists of AND, OR, and NOT gates, and a monotone circuit is a one which consists of AND and OR gates. Despite some considerable effort, it is not known that some explicitly defined family of Boolean functions has superlinear combinational circuit complexity. On the other hand, for monotone circuit model, Razborov gave superpolynomial lower bounds for clique functions [13]. Subsequently, Alon and Boppana improved Razborov's result to exponential [2]. But, Tardos showed that there exist exponential gaps between monotone and combinational complexity [14]. So, in general, we cannot derive strong lower bounds for the combinational circuit complexity using those bounds for the monotone circuit complexity.

Let $X_n = \{x_1, x_2, \ldots, x_n\}$, $f(X_n)$ be a Boolean function with n variables, and k be any integer such that $1 \leq k \leq n$. In [3], Berkowitz introduced the *k-slice function* of f, denoted $k\text{-}sl(f)$, which is defined as follows :

$$k\text{-}sl(f)(X_n) = (f(X_n) \wedge T_n^k(X_n)) \vee T_n^{k+1}(X_n).$$

Here $T_n^k(X_n)$ is the k-th threshold function. Note that $k\text{-}sl(f)$ is a monotone Boolean function from the definition. Also note that $k\text{-}sl(f)$ is 0 (resp. 1) for assignments to X_n in which fewer (resp. more) than k variables are set to 1, and is equal to f for assignments to X_n in which exactly k variables are set to 1.

Berkowitz also showed that, for any slice function $k\text{-}sl(f)$ of f, the combinational circuit complexity $C(k\text{-}sl(f))$ and the monotone circuit complexity $C^m(k\text{-}sl(f))$ are polynomially related. Therefore, if we

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can show a superpolynomial lower bound for $C^m(k\text{-}sl(f))$ for some NP-complete Boolean function f, this would imply P $\neq$ NP. Thus, it is very important to study the circuit complexity of slice functions of NP-complete Boolean functions.

In order to give some insight into this line of research, we show several slices of graph-theoretic NP-complete functions that have polynomial circuit complexity. For some *monotone* NP-complete Boolean functions, such as CLIQUE, HAMILTONIAN CIRCUIT, it is known that their canonical slice functions have polynomial circuit complexity [7]. In this paper, we show a number of slices of monotone and *non-monotone* NP-complete Boolean functions, which have polynomial circuit complexity. In other words, we show several tractable instances of graph-theoretic NP-complete problems.

In order to show these results, we consider the recognition problems for some specific graphs. We first show that any Turán graph is recognizable within $O(n^2)$ time, where n is the number of vertices in G and k is the number of maximal independent sets in G. Then, using this fact, we show that the t-th slice of MAXIMUM CLIQUE SIZE (MAXCLIQUE for short) has polynomial circuit complexity, where

$$t = \binom{n}{2} - \binom{q+1}{2} r - \binom{q}{2} (k-r)$$

is the number of edges included in the Turán graph with n vertices and k classes such that $n = qk + r, 0 \leq r \leq k-1$.

We also show that the $(e-t)$-th slice functions of INDEPENDENT SET, and t-th slice functions of CHROMATIC NUMBER have polynomial circuit complexity by using the above fact on the recognition of Turán graphs, where $e = n(n-1)/2$. Furthermore, we present tractable slices of HAMILTONIAN PATH and HAMILTONIAN CIRCUIT based on some observation on extremal problems for Hamiltonian path.

A graph $P_n^k = (V, E)$ is a one such that $V = \{1, 2, \ldots, n\}$ and $E = \{(u, v) \mid u \in V, v \in \{u+1, u+2, \ldots, u+k\} \cap V\}$. It is easy to show that any P_n^k is linear-time recognizable. Then, using this fact, we show that the l-th slice of BANDWIDTH has polynomial circuit complexity, where

$$l = nk - k(k+1)/2$$

is the number of edges included in P_n^k.

The remainder of this paper is organized as follows. Section 2 gives basic definitions and notations. Then, we show tractable slices of MAXCLIQUE and other NP-complete problems in Section 3. We also give a tractable slice of HAMILTONIAN CIRCUIT in Section 4, and a tractable slice of BANDWIDTH in Section 5.

2 Preliminaries

Let $X_n = \{x_1, x_2, \ldots, x_n\}$, i.e. the set of n Boolean variables. Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be a Boolean function. The *circuit complexity* of f, denoted $C(f)$, is the number of gates in the smallest circuit computing f, which consists of AND, OR, and NOT gates. On the other hand, the *monotone circuit complexity* of f, denoted $C^m(f)$, is the number of gates in the smallest monotone circuit computing f, which consists of AND and OR gates.

Let $f(X_n)$ be a Boolean function with n variables and k be any integer such that $1 \leq k \leq n$. The *k-slice function* of f, denoted $k\text{-}sl(f)$, is the monotone Boolean function

$$k\text{-}sl(f)(X_n) = (f(X_n) \wedge T_n^k(X_n)) \vee T_n^{k+1}(X_n),$$

where $T_n^k(X_n)$ is the k-th threshold function.

Proposition 2.1 ([3]) *Let f be an arbitrary Boolean function with n variables. Then, for an arbitrary integer k such that $1 \le k \le n$, $C(k - sl(f))$ and $C^m(k - sl(f))$ are polynomially related.*

Graph-theoretic problems are normally encoded using an adjacency matrix to represent an n-vertex graph. Let $X_n^U = \{x_{ij} \mid 1 \le i < j \le n\}$, where each x_{ij} is a Boolean variable. Then, an assignment to the variables in X_n^U represents an undirected graph G in the following way : G contains an edge $\{i,j\}$ iff x_{ij} in X_n^U is 1. Let $e(n) = |X_n^U| = n(n-1)/2$.

On the relationship between the circuit complexity and the time complexity on Turing machines, the following proposition is known.

Proposition 2.2 ([8]) *Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. If f is computable within $O(T(n))$ time on a deterministic Turing machine, then $C(f) = O(T(n)\log T(n))$.*

For details of the circuit complexity theory, see [7], and for elementary concepts from graph theory, see [10].

3 The Recognition of Turán Graphs and Its Applications

In this section, we first show an efficient algorithm for the recognition of Turán Graph. Thus by using this result, we show that tractable slices of MAXCLIQUE and other NP-complete functions.

3.1 The Recognition of Turán Graphs

By $T_k(n)$, we denote the Turán graph, which is the complete k-partite graph with n vertices such that each class of it has exactly $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ vertices (see Figure 1). Let $t_k(n)$ be the number of edges

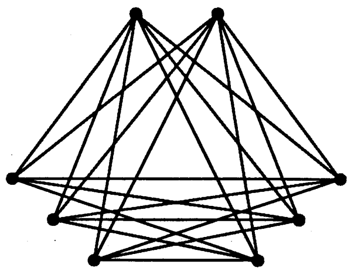

Figure 1: The Turán Graph $T_3(8)$

included in $T_k(n)$. Note that

$$t_k(n) = \binom{n}{2} - \binom{q+1}{2} r - \binom{q}{2} (k-r)$$

where $n = qk + r, 0 \le r \le k-1$. The Turán's theorem is as in follows:

Theorem 3.1 ([4]) *Let G be a simple graph with n vertices and $t_k(n)$ edges. G has no (k+1)-clique iff G is isomorphic to $T_k(n)$.*

Using this fact, we show that the recognition problem of Turán graphs can be solved in $O(n^2)$ time.

Theorem 3.2 *Given any graph G with n vertices and k be an integer such that $1 \le k \le n$. Then, it is decidable whether G is isomorphic to $T_k(n)$ within $O(n^2)$ time.*

Proof. Let $G = (V, E)$ be an input graph. Note that G is isomorphic to $T_k(n)$ iff $\overline{G}$ consits of k complete graphs with sizes $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$. So, we check whether $\overline{G}$ consits of k complete graphs with sizes $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$.

First, construct $\overline{G}$ in $O(n^2)$ time. Then, decompose $\overline{G}$ into the connected components of it. It is known that we can find a list of vertices of each connected component of any input graph $H = (V', E')$ in $O(\max\{|V'|, |E'|\})$ [1]. In this case, the number of edges is $O(n^2)$. So, we can find a list of vertices of each connected component of $\overline{G}$ in $O(n^2)$. For each component of $\overline{G}$, check whether the cardinality of it is $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$. If some components do not satisfy this condition, G is not a Turán graph. This check can be solved in $O(kn)$ time. □

Let $TU_n^k(X_n^U)$ be an $e(n)$-variable Boolean function whose value is 1 iff X_n^U represents a graph G which is isomorphic to $T_k(n)$. By Proposition 2.2, we obtain the following corollary.

Corollary 3.3 $C(TU_n^k) = O(n^2 \log n)$.

3.2 A Tractable Slices of MAXCLIQUE

The well known NP-complete problem *MAXIMUM CLIQUE SIZE* (*MAXCLIQUE* for short) is as follows [9]:

> INSTANCE: A graph G and a positive integer k.
> QUESTION: Does the largest complete subgraph in G contain exactly k vertices ?

It is known that the central slice (i.e. $e(n)/2$-slice) of the clique function is NP-complete [7]. Let $CL_n^{n/2}(X_n^U)$ be the $e(n)$-variable Boolean function whose value is 1 iff X_n^U represents a graph contains $n/2$-clique. And let $e = e(n)$. The following property is known.

Proposition 3.4 ([7]) *$e/2$-$sl(CL_n^{n/2})$ is NP-complete.*

We can prove a similar theorem in the case of MAXCLIQUE. Let $MC_n^{n/2}(X_n^U)$ be the $e(n)$-variable Boolean function whose value is 1 iff X_n^U represents a graph such that the size of maximum clique of it is exactly $n/2$.

Theorem 3.5 *$e/2$-$sl(MC_n^{n/2})$ is NP-complete.*

Here, we show a tractable slice of MAXCLIQUE using Corollary 3.3.

Theorem 3.6 *t-$sl(MC_n^k)$ has polynomial circuit complexity.*

Proof. Let G be an arbitrary graph with n vertices and t edges. Then the size of maximum clique of G is k iff G is isomorphic to $T_k(n)$. Thus, we can use a circuit C_1 for TU_n^k to decide whether the size of maximum clique of G is k by combining the outputs of C_1 and a circuit C_2 which checks that the number of edges in G is exactly t using an AND gate (see Figure 2). Since the function computed by the circuit C_2 is $T_n^k \wedge \overline{T_n^{k+1}}$, we can construct the combinational circuit C_2 using $O(n)$ gates and then the theorem follows. □

By Proposition 2.1, we obtain the following corollary.

Corollary 3.7 *MC_n^t has polynomial monotone circuit complexity.*

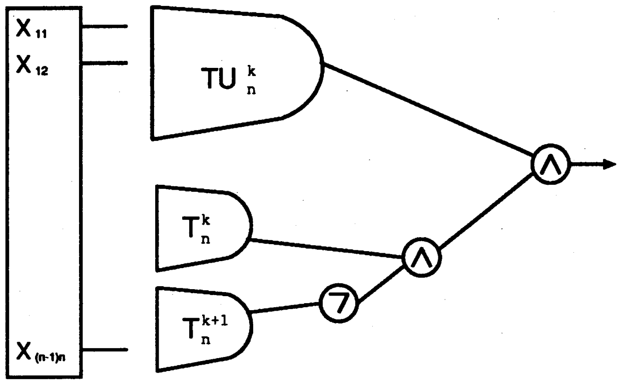

Figure 2: A circuit for $t\text{-}sl(MC_n^k)$

3.3 Tractable Slices of NP-Complete Functions

First, we show a tractable slice of *CHROMATIC NUMBER*. Let $CN_n^k(X_n^U)$ be the $e(n)$-variable Boolean function whose value is 1 iff X_n^U represents a graph whose chromatic number is exactly k.

Theorem 3.8
(i) $C(t\text{-}sl(CN_n^{\lceil n/k \rceil})) = C(t\text{-}sl(TU_n^t))$.
(ii) $C^m(t\text{-}sl(CN_n^{\lceil n/k \rceil})) = C(t\text{-}sl(TU_n^t))$.

Proof. Let G be an arbitrary graph with n vertices and t edges. Then the chromatic number of G is $\lceil n/l \rceil$ iff G is isomorphic to $T_k(n)$. □

Corollary 3.9
(i) *$t\text{-}sl(CN_n^{\lceil n/k \rceil})$ has polynomial circuit complexity.*
(ii) *$t\text{-}sl(CN_n^{\lceil n/k \rceil})$ has polynomial monotone circuit complexity.*

Here we consider a tractable slice of *MAXIMUM INDEPENDENT SET*. Let $MIS_n^k(X_n^U)$ be the $e(n)$-variable Boolean function whose value is 1 iff X_n^U represents a graph such that the size of maximum independent set of it is exactly k.

Theorem 3.10 *Let k be an integer such that $1 \leq k \leq n$. And let $e = e(n)$ and $t = t_k(n)$. $C((e-t)\text{-}sl(MIS_n^{\lceil n/k \rceil})) = C(t\text{-}sl(TU_n^t)) + O(e(n))$.*

Proof. Let G be an arbitrary graph with n vertices and $(e-t)$ edges. Then the size of maximum independent set of G is $\lceil n/k \rceil$ iff $\overline{G}$ is isomorphic to $T_k(n)$. Thus, we can use the circuit for $t\text{-}sl(TU_n^k)$ by inverting all the inputs. In order to do this, we need $e(n)$ NOT gates. □

Corollary 3.11
(i) *$(e-t)\text{-}sl(MIS_n^{\lceil n/k \rceil})$ has polynomial circuit complexity.*
(ii) *$(e-t)\text{-}sl(MIS_n^{\lceil n/k \rceil})$ has polynomial monotone circuit complexity.*

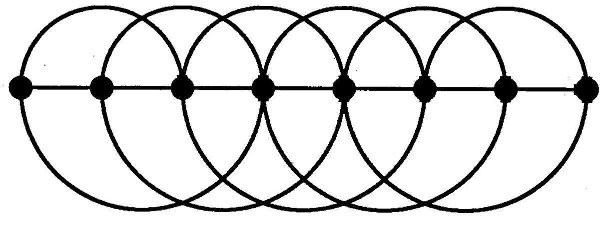

Figure 3: The graph P_8^3.

4 A Tractable Slice of HAMILTONIAN CIRCUIT

In this section, we present tractable slices of HAMILTONIAN PATH and HAMILTONIAN CIRCUIT using the extremal problems for HAMILTONIAN PATH and HAMILTONIAN CIRCUIT. Let $HP_n^k(X_n^U)$ ($HC_n^k(X_n^U)$ resp.) be the $e(n)$-variable Boolean function whose value is 1 iff X_n^U represents a graph which has Hamiltoian path (Hamiltonian circuit resp.).

Let K_n be the complete graph with size k and E^1 be the empty graph with only one vertex. The following proposition is known.

Proposition 4.1 ([4]) (i)*Let G be a simple graph with n vertices and $p = e(n) - (n-3)$. G has no Hamiltonian path iff G is isomorphic to $K^{n-1} \cup E^1$.*
(ii)*Let H be a simple graph with n vertices and $c = e(n) - (n-2)$. H has no Hamiltonian cycle iff G isomorphic to a graph which is obtained by adding an edge to $K^{n-1} \cup E^1$.*

Theorem 4.2 (i) *p-$sl(HP_n)$ has polynomial circuit complexity, where $p = e(n) - (n-3)$.*
(ii) *c-$sl(HC_n)$ has polynomial circuit complexity, where $c = e(n) - (n-2)$.*

Proof. Omitted.

5 A Tractable Slice of BANDWIDTH

The well known NP-complete problem *BANDWIDTH* is as follows [9]:

> INSTANCE: A graph $G = (V, E)$ and a positive integer $k \le |V|$.
> QUESTION: Is there a one-to-one function $f : V \to \{1, 2, \ldots, |V|\}$ such that, for all $\{u, v\} \in E$, $|f(u) - f(v)| \le k$?

A graph $P_n^k = (V, E)$ is a one such that $V = \{1, 2, \ldots, n\}$ and $E = \{(u, v) \mid u \in V, v \in \{u+1, u+2, \ldots, u+k\} \cap V\}$. For example, the graph P_8^3 is shown in Figure 3. It is known that an arbitrary graph G with n vertices has bandwidth k iff G is a subgraph of P_n^k [6].

Proposition 5.1 *Let k and n be integers such that $1 \le k \le n$. Then, the graph P_n^k is recognizable in $O(n+e)$ time, where n is the number of vertices and e is the number of edges.*

Proof. It is known that interval graphs can be recognized in $O(n+e)$ time [5] and the isomorphism problem for interval graphs can be solved in $O(n+e)$ time [12]. For given positive integer n and k, we can construct P_n^k in $O(n+e)$ time. Clearly, P_n^k is a interval graph, thus, P_n^k is recognizable in $O(n+e)$ time. □

Let $PR_n^k(X_n^U)$ be an $e(n)$-variable Boolean function whose value is 1 iff X_n^U represents the graph G which is isomorphic to P_n^k. From Proposition 2.2, we obtain the following.

Proposition 5.2 $C(PR_n^k) = O((n+e)\log n)$.

Let $l = nk - k(k+1)/2$ be the number of edges in P_n^k, and $BW_n^k(X_n^U)$ be an $e(n)$-variable Boolean function whose value is 1 iff X_n^U represents the graph G has bandwidth k. We obtain the following theorem.

Theorem 5.3
(i) $C(l\text{-}sl(BW_n^k)) = C(l\text{-}sl(PR_n^k))$.
(ii) $C^m(l\text{-}sl(BW_n^k)) = C^m(l\text{-}sl(PR_n^k))$.

Proof. Let G be an arbitrary graph with n vertices and l edges. Then G has bandwidth k iff G is isomorphic to P_n^k. Thus, we can use a circuit for PR_n^k to decide whether G has bandwidth k. □

Corollary 5.4
(i) *l-$sl(BW_n^k)$ has polynomial circuit complexity.*
(ii) *l-$sl(BW_n^k)$ has polynomial monotone circuit complexity.*

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