数理解析研究所講究録
950 巻 1996 年 160-166

Modular Confluence of Conditional Term Rewriting Systems with Extra Variables in Right-Hand Sides

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Abstract

Middeldorp has proved that confluence is a modular property for conditional term rewriting systems with extra variables in the conditions of the rewrite rules. In this paper, we try to show that confluence is also modular for conditional term rewriting systems with extra variables in the right-hand sides of the rewrite rules.

1 Introduction

Term rewriting systems (TRSs, for short) have been studied as a theoretical foundation of algebraic specifications and functional programs. Conditional term rewriting systems (CTRSs) have been proposed as an extension of TRSs. CTRSs have great importance for integrating functional and logic programming paradigms. Extensive surveys for TRSs and CTRSs can be found in [DJ90] and [Kl92].

For CTRSs, there are important properties, such as termination and confluence. Since they are known to be undecidable properties, some sufficient conditions have been proposed. However, if a complicated CTRS is given, then it may be difficult to show the validity of a property by the known sufficient conditions.

So, it is desirable that the validity of a property of a CTRS can be shown by dividing the CTRS into smaller systems. In this paper, we consider the case that the divided systems have no function symbols in common. In this case, the union of CTRSs is called the disjoint union of them. We say a property P of CTRSs is modular when every two CTRSs R_1 and R_2 satisfy the following condition: Both R_1 and R_2 satisfy the property P if and only if their disjoint union $R_1 \oplus R_2$ satisfies P.

Concerning with the confluence property, at first, Toyama has proved the following theorem[To87, KMTV94].

- Confluence is modular for TRSs.

Next, Middeldorp has extended the above result for a class of CTRSs [Mi93]:

- Confluence is modular for CTRSs with extra variables in the conditions of the rewrite rules (2-CTRSs).

In this paper, we try to extend the above Middeldorp's theorem. We try to show the modularity of confluence for CTRSs with extra variables in the right-hand sides of the rewrite rules (3-CTRSs). The class of 3-CTRSs includes the class of 2-CTRSs.

In concluding remarks of [Mi93], Middeldorp has remarked that it is worthwhile to consider whether the results of [Mi93] can be extended to the following subclass of 3-CTRSs.

- Suppose R belongs to the class. The class is characterized by the phrase "if $s \rightarrow_R t$ then $s \rightarrow t$ is a legal unconditional rewrite rule. "

$s \rightarrow t$ is said to be a legal unconditional rewrite rule, if every variable which appears in t also appears in s. Let us call such 3-CTRSs as legal 3-CTRSs. Here, we show a 3-CTRS which is not legal. Suppose a CTRS contains two rules $A \rightarrow B(x) \Leftarrow C(x) = D$ and $C(x) \rightarrow D$. Then the first rule can be considered as the rule $A \rightarrow B(x)$.

3-CTRSs which are not legal yield some problems. The following is one of them: in the proof for 2-CTRSs, given two CTRSs are transformed into two TRSs and the Toyama's theorem for modular confluence of TRSs is applied to the two TRSs. However, in the case of 3-CTRSs which are not legal, CTRSs are transformed to rewrite systems containing rules such as $A \rightarrow B(x)$. We call such systems 'term arbitrary rewriting systems (TARSs)', and try to prove that confluence is modular for TARSs.

The proof is based on the following idea. Assume a TARS contains the rule $A \rightarrow B(x)$ and this TARS is confluent. Then, the rewrite $B(x) \xrightarrow{*} x$ is impossible. If this is possible, then we have both $A \rightarrow B(x) \xrightarrow{*} x$ and $A \rightarrow B(y) \xrightarrow{*} y$, and this fact contradicts to the confluence of the TARS.

In this paper, notions and notations for TRSs and CTRSs are based on [Mi93]. For the space limitation, we cannot show them here, except modulaity (Section 2). We try to prove the modularity of confluence for TARSs in Section 3. However, after the talk at RIMS, it has been revealed that the proof contains several mistakes. See Section 4.

2 Modularity

Suppose that two CTRSs (F_1, R_1) and (F_2, R_2) are given. If $F_1 \cap F_2 = \emptyset$ then we say R_1 and R_2 are *disjoint*. In this case, the union $(F_1 \cup F_2, R_1 \cup R_2)$ is called the *disjoint union* of R_1 and R_2. And we denote the union as $R_1 \oplus R_2$. A property P for CTRSs is *modular* if the following statement is hold for every two disjoint CTRSs R_1 and R_2. "Both R_1 and R_2 satisfy P if and only if $R_1 \oplus R_2$ satisfies P". For TRSs, the following theorem is known[To87, KMTV94].

Theorem 2.1 Confluence is modular for TRSs. □

For CTRSs, the following two theorems hold[Mi93].

Theorem 2.2 Confluence is modular for join 2-CTRSs. □

Theorem 2.3 Confluence is modular for semi-equational 2-CTRSs. □

In the rest of this subsection, we explain some notions and some usual manners for proofs of the modularity. Consider two disjoint CTRSs (F_1, R_1) and (F_2, R_2). Here, we assume that the elements of F_1 are denoted by capital letters and the elements of F_2 are denoted by small letters. Furthermore, we give the colors to the function symbols. For the elements of F_1, we give "black", and for those of F_2, we give "white". A *black(white) term* is a term consists of elements of $F_1(F_2)$ and variables. We sometimes remark only the black case since such notions are defined similarly for the white case. A *top black term* is a term whose root symbol is black.

We use the following notations: $T_1 = T(F_1, V), T_2 = T(F_2, V), F_\oplus = F_1 \cup F_2$ and $T_\oplus = (F_\oplus, V)$. We often use the symbol $\rightarrow$, instead of $\rightarrow_{R_1 \oplus R_2}$.

For a disjoint union of TRSs, the following proposition is obvious.

Proposition 2.1 $\rightarrow_{R_1\oplus R_2} = \rightarrow_{R_1} \cup \rightarrow_{R_2}$ on $T_\oplus$. $\square$

Suppose $t \equiv C[t_1, \cdots, t_n]$ and $C[, \cdots,] \not\equiv \square$. In the following definition, suppose that the pair of indices (a, b) has the value $(1, 2)$ or $(2, 1)$. If $C[, \cdots,] \in T(F_a \cup \{\square\}, V)$ and the root symbols of $t_1, \cdots, t_n$ are in F_b, then we denote $t \equiv C[\![t_1, \cdots, t_n]\!]$. In this case we say $t_1, \cdots, t_n$ are the *principal subterms* of t. We define the *rank* of a term $t \in T_\oplus$, as follows.

$$rank(t) = \begin{cases} 1 & \text{,if } t \in T_1 \cup T_2, \\ 1 + max\{rank(t_i) | (1 \leq i \leq n)\} & \text{,if } t \equiv C[\![t_1, \cdots, t_n]\!]. \end{cases}$$

The *special subterms* of $t \in T_\oplus$ are the elements of the multiset defined as follows.

$$\begin{aligned} S_1(t) &= [t], \\ S_{j+1}(t) &= \begin{cases} [\,] & \text{,if } rank(t) = 1, \\ S_j(t_1) \cup \cdots \cup S_j(t_n) & \text{,if } t \equiv C[\![t_1, \cdots, t_n]\!], \end{cases} \\ S(t) &= \cup_{j \geq 1} S_j(t). \end{aligned}$$

In the preceding, we have defined the notation $C[\![, \cdots,]\!]$. Now we define some other notations for contexts. $C\langle, \cdots, \rangle$ is a context containing at least zero hole. $C\{, \cdots, \}$ is a context containing at least zero hole, which is not a hole itself. If $t \equiv C\{t_1, \cdots, t_n\} \in T_\oplus$ and $t_1, \cdots, t_n$ are the principal subterms of t, then we denote $t \equiv C\{\{t_1, \cdots, t_n\}\}$. $t \equiv C\langle t_1, \cdots, t_n \rangle \in T_\oplus$ and if one of the following conditions holds, then we denote $t \equiv C\langle\langle t_1, \cdots, t_n \rangle\rangle$.

- $C\langle, \cdots, \rangle$ is not a hole and $t_1, \cdots, t_n$ are the principal subterms of t.
- $C\langle, \cdots, \rangle$ is a hole and $t \in \{t_1, \cdots, t_n\}$.

For 2-CTRSs R_1 and R_2, the following proposition holds[Mi93].

Proposition 2.2 If $s \rightarrow_{R_1 \oplus R_2} t$, then $rank(s) \geq rank(t)$. $\square$

An *inner reduction* $s \rightarrow^i t$ is a reduction which rewrites a subterm of a principal subterm of t. An *outer reduction* $s \rightarrow^o t$ is a reduction which is not a inner one.

We inductively define a *destructive reduction* as follows. $s \rightarrow t$ is destructive at level 1 if the root symbols of s and t have the different colors. $s \rightarrow t$ is destructive at level $n+1$ if $s \equiv C[\![s_1, \cdots, s_j, \cdots, s_n]\!] \rightarrow^i C[s_1, \cdots, t_j, \cdots, s_n] \equiv t$ and $s_j \rightarrow t_j$ is destructive at level n. t is *root preserved* if the root symbols of t and s have the same color for all s such that $t \xrightarrow{*} s$. t is *preserved* if t is root preserved and all principal subterms of t are preserved. t is *inner preserved* if all principal subterms of t are preserved. We say a conditional rewrite rule $l \rightarrow r \Leftarrow C$ is a *collapsing rule* if $r \in V$. We write $s \rightarrow_c t$, if the followings hold:

- $s \equiv C[s']$ and $t \equiv C[t']$ by a context $C[\,]$,
- $s' \xrightarrow{*} t'$,
- root symbols of s' and t' have the different colors.

This relation $\rightarrow_c$ is called *collapsing reduction.* In the above case, we call s' a *collapsing redex.*

3 Confluence is modular for TARSs

We use two join 3-CTRSs whose names are R_1 and R_2.

Here, we define legality of a 3-CTRS.

Definition 3.1 Suppose a 3-CTRS R is given. R is legal if $Var(s) \supseteq Var(t)$ for every rewrite $s \rightarrow_R t$. □

R' in the next example is confluent and not legal.

Example 3.1

$$R' = \{ \begin{array}{ll} A \rightarrow D(x) & \Leftarrow B(x) \downarrow C, \\ B(x) \rightarrow C, & \\ D(x) \rightarrow E & \} \end{array}$$

□

We regard that R' essentially contains a rewrite rule $A \rightarrow D(x)$. The term A can rewrite to every instance of $D(x)$. In spite of this fact, the rule $D(x) \rightarrow E$ ensures the confluence of R'.

Now, we define (unconditional) rewrite systems containing rules such as $A \rightarrow D(x)$.

Definition 3.2 $l \rightarrow r(l, r \in T)$ is called an *arbitrary rewrite rule (A-rule)* if $l \notin V$ and $Var(l) \not\supseteq Var(r)$. A set of rewrite rules and A-rules is called a *term arbitrary rewriting system (TARS).* We call an element of $Var(r) - Var(l)$ as an *arbitrary variable (A-variable)* of the A-rule $l \rightarrow r$. □

We define the relation $\rightarrow_1$ and $\rightarrow_2$ which are restricted relations of $\rightarrow_{R_1}$ and $\rightarrow_{R_2}$, respectively.

Definition 3.3 We write $s \rightarrow_1 t$ if one of the following conditions holds.

- $s \equiv C[l\sigma], t \equiv C[r\sigma]$ by an unconditional rewrite rule $l \rightarrow r \in R_1$, a substitution σ, and a context $C[\,]$.
- $s \equiv C[l\sigma], t \equiv C[r\sigma]$ and $s_i\sigma \downarrow_1^o t_i\sigma$ for each $i(1 \leq i \leq n)$ by a conditional rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, \cdots, s_n \downarrow t_n \in R_1$, a substitution σ, and a context $C[\,]$.

$\rightarrow_2$ is defined similarly. □

The following TARS R_1^m serves the rewrite relation equivalent to $\rightarrow_1$.

Definition 3.4 The following TARS R_1^m defined by the CTRS R_1 is called the *monochrome TARS* of R_1.

$$R_1^m = \{s \to t | s, t \in T_1, s \to_1 t\}$$

□

If R_1 is a 2-CTRS, then R_1^m is a TRS. Also, if R_1 is a legal 3-CTRS, then R_1^m is a TRS . This is because $\to_1 \subseteq \to_{R_1}$. However, if R_1 is a 3-CTRS which is not legal, then R_1^m is not a TRS. In the proof of the theorem for 2-CTRSs (Theorem 2.2), the confluence of the union $R_1^m \oplus R_2^m$ is proved by using the Toyama's theorem for modular confluence of TRSs (Theorem 2.1). In the case of 3-CTRSs, we cannot show the confluence of $R_1^m \oplus R_2^m$ unless we extend the Toyama's theorem to the theorem for TARSs.

In [KMTV94], induction on the rank of a term is used to prove the Toyama's theorem. For TRSs, this way is available since the rank of a term does not increase (Proposition 2.2). However, in the case of TARSs, the rank of a term can increase. To conquer this problem, we modify the notion of rank.

The new rank for a term is given in the reduction sequence which the term belongs to. An intuitive explanation of the new rank is as follows: in the rewrite step by an A-rule, every subterm at the A-variable position of the rewritten term has rank 0, and such a subterm (and its all subterms) remains to have rank 0 in the reduction sequence.

To explain the formal definition of new rank, we define some notions here. The confluence of a TARS (T, R) is shown by the confluence of every term t in T. We call the term t, which should be shown its confluence, as the *primary term*. We give it the ordinary rank which has already defined. Contrary, we give the new rank to a term which appears in the diagram of the confluence proof of t. We call the diagram as the *proof diagram* of a primary term t and call the terms in the proof diagram, except t, as the *secondary terms*. By these words, we define the notion of embedded subterms.

Definition 3.5 Suppose a primary term s is given. We define the embedded subterms of a secondary term $t \in T_\oplus$, s.t. $s \xrightarrow{*} u \to t$, as follows. We call a subterm of t as a *embedded subterm* of t, if one of the following conditions holds.

- A subterm at the position of an A-variable of r, if $u \to t$ is done by an A-rule $l \to r$.
- A subterm of a embedded subterm of u.

□

Here, we give the new definitions of principal subterms and rank.

Definition 3.6 Consider a top black term $t \equiv C[t_1, \cdots, t_n]$, where t_i's are the maximal top white subterms of t. Suppose $t_{i_1}, \cdots, t_{i_m}$ ($\{i_1, \cdots, i_m\} \subseteq \{1, \cdots, n\}$) are not the embedded subterms of t and the others are the embedded ones. In this case we say $t_{i_1}, \cdots, t_{i_m}$ are the principal subterms of t and write $t \equiv C'[\![t_{i_1}, \cdots, t_{i_m}]\!]$. Note that the context $C'[\,]$ contains the top white and embedded subterms of s. □

Definition 3.7 The rank of a secondary term $t \in T_\oplus$ is defined as follows. We give the rank 0 to every embedded subterm of t. The total rank of the term t is given by the same way as a primary term. □

By these new definitions, we can prove that rank of a term cannot increase. Furthermore, the modifications are harmless to the other propositions which are used in the proof for TRSs. This fact is due to Proposition 3.2 which we show afterward. For the proof of Proposition 3.2, we explore a property of confluent TARSs by observing the following examples. For the confluence of a TARS, it must contain no A-rule which is also a collapsing one. See R''.

$$R'' = \{A \to x, \cdots\}$$

In R'', the term A can rewrite to two different variables x and y. So R'' is not confluent while it contains any other rules. The situation is similar in the case such as the following R'''.

$$R''' = \{A \to B(x), B(x) \to x, \cdots\}$$

Definition 3.8 Suppose a term $s \equiv C[s']$, where $C[\ ] \not\equiv \Box$. If $s \xrightarrow{*} s'$, then we say a subterm s' of s is *exposed.* □

By the above examples, we obtain the following proposition.

Proposition 3.1 If a TARS R is confluent, R satisfies the followings.

1. Every A-rule is not a collapsing one.

2. If a term has embedded subterms, then they are not exposed.

□

Note that 1. of this proposition ensures the termination of collapsing reduction for TARSs. By 2. of this proposition, we can prove the following Proposition 3.2.

Proposition 3.2 Suppose a top black term $t \equiv C[\![t_1, \cdots, t_n]\!]$ is given. Then the top black context $C[, \cdots ,]$ is confluent with respect to $\to_{R_1 \oplus R_2}$. □

(Proof) The top black context $C[, \cdots ,]$ may contain top white subterms, but they are the embedded subterms of t. Using 2. of Proposition 3.1, every embedded subterm s of t cannot be exposed. So, we may regard s as a variable, even if s is top white. Thus, the confluence of $C[, \cdots ,]$ is proved by the confluence of R_1. □

This is an extension of a proposition for TRSs, which is stated as "monochrome outer reduction is confluent". Using this, we have the following theorem.

Theorem 3.1 Confluence is modular for TARSs. □

This is proved by the following modification of the proof for TRSs. We write the propositions, the lemma and the theorem in [KMTV94] by the italic letters.

- use 1. of Proposition 3.1 to show the termination of collapsing reduction (*Proposition 2.5*),

- replace *Proposition 3.1* to our Proposition 3.2,

- use our Proposition 3.2 in the base steps of induction of *Lemma 3.4* and *Theorem 4.4*.

Note that our Proposition 3.2 is also used in the inductive step of *Lemma 3.4*, by the replacement.

4 Conclusion

In this paper, we have tried to prove the modularity of confluence for TARSs, in order to prove that for 3-CTRSs.

However, after the talk at RIMS, it has been revealed that the proof of this paper contains several mistakes (by reviewers of the conference RTA96).

The following is one of them: $R = \{A \to B(y), B(y) \to C, B(C) \to C\}$ is a confluent TARS. However, in the reduction $B(C) \to C$, the embedded subterm C of $B(C)$ is exposed. This is contradict to 2. of Proposition 3.1.

Reviewers have also remarked that the definitions of some notions are imprecise.

Now, we are considering the modification of the proof.

References

[DJ90] Dershowitz, N. and Jouannaud, J.-P.: "Rewrite Systems", Handbook of Theoretical Computer Science B: Formal Methods and Semantics (van Leuwene, J., ed.), North-Holland, pp.243-320 (1990).

[Kl92] Klop, J.W.: "Term Rewriting Systems", Handbook of Logic in Computer Science, Vol.II (Abramsky, S., Gabbay, D. and Maibaum, T., eds.), Oxford University Press, pp.1-116 (1992).

[KMTV94] Klop, J.W., Middeldorp, A., Toyama, Y. and de Vrijer, R.: "Modularity of Confluence: A Simplified Proof", Information Processing Letters, Vol.49, pp.101-109 (1994).

[Mi93] Middeldorp, A.: "Modular Properties of Conditional Term Rewriting Systems ", Information and Computation, Vol.104, No.1, pp.110-158 (1993).

[To87] Toyama, Y.: "On the Church-Rosser Property for the Direct Sum of Term Rewriting Systems", Journal of Association for Computing Machinery, Vol.34, No.1, pp.128-143 (1987).