数理解析研究所講究録
950 巻 1996 年 22-25

A Note on the Length of M-Programs over Nonsolvable Groups

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Abstract

In a seminal paper, Barrington [Bar89] showed a lovely result that, for all nonsolvable groups G, a Boolean circuit of depth d can be simulated by an M-program of length at most $(4|G|)^d$ working over G. In this tiny note, we improve the upper bound on the length from $(4|G|)^d$ to 4^d.

1. Preliminaries

We assume that the readers are familiar with Boolean circuits. We only note that our circuits consist of NOT-gates, AND-gates with fan-in two, OR-gates with fan-in two, and input gates with each of which a Boolean variable is associated. In this section, we first give the definition of M-programs over groups.

Definition 1.1. Let G be a group and n a positive integer. We define an *monoid-instruction*(M-instruction for short) γ over G to be a three-tuple (i,a,b) where i is a positive integer, and both a and b are elements in G. We define an *monoid-program*(M-program for short) P over G to be a finite sequence $(i_1,a_1,b_1),(i_2,a_2,b_2),\ldots,(i_k,a_k,b_k)$ of M-instructions over G. For this M-program P, we call the number of M-instructions the *length of P* and denote it with $\ell(P)$. Furthermore, we call the maximum value among $i_1,i_2,\ldots,i_k$ the *input size of P* and denote it with $n(P)$.

We suppose the M-program P to compute a Boolean function in the following manner. Let n be the input size of P and let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\{0,1\}^n$ be a vector of Boolean values that is given as an input to P. Then, we define the *value of an M-instruction* $\gamma_j=(i_j,a_j,b_j)$, denoted by $\gamma_j(\vec{x})$, as follows:

$$\gamma_j(\vec{x})=\begin{cases} a_j & \text{if } x_j=0 \\ b_j & \text{if } x_j=1 \end{cases}.$$

We further define the *value $P(\vec{x})$ of the M-program P* by $P(\vec{x})=\gamma_1(\vec{x})\gamma_2(\vec{x})\cdots\gamma_k(\vec{x})$. Then we say that *$P$ computes a Boolean function* $f:\{0,1\}^n\to\{0,1\}$ if, for all $\vec{x}\in\{0,1\}^n$, if $f(\vec{x})=0$, then $P(\vec{x})=e_G$, and otherwise, $P(\vec{x})\neq e_G$, where e_G denotes the identity element of G. ♠

We further assume that the readers are familiar with elementary notions in group theory. Thus, we only give a breif definition for the nonsolvability of groups.

Definition 1.2. Let G be any finite group. For any two elements a, b of G, we define the *commutator* of a and b to be the element represented as $a^{-1}b^{-1}ab$ and denote it by $[a, b]$. We further define the *commutator subgroup* of G to be the subgroup of G generated by all the commutators, and we denote it by $D(G)$. Then, we inductively define $D_i(G)$, for all integers $i \geq 0$, as follows: $D_0(G) = G$, and for all $i \geq 1$, $D_i(G) = D(D_{i-1}(G))$. We say that G is *solvable* if $D_i(G) = \{e_G\}$ for some $i \geq 0$, where e_G denotes the identity element of G. If G is not solvable, we say that it is *nonsolvable*. It is easy to show that $D_{i+1}(G)$ is a subgroup of $D_i(G)$ for all $i \geq 0$. Hence, we see that G is nonsolvable if and only if there exists a subgroup H such that $H \neq \{e_G\}$ and $H = D(H)$. We will use this fact later. ♠

2. An improvement of Barrington's result

To show our result, we use the following lemmas. The first lemma was implicitly used by Barrington in order to show that for all circuits C of depth d, the Boolean function computed by C can be computed by an M-program of length at most 4^d working over the alternating group of degree 5.

Lemma 2.1. Let G be a finite group and let e_G be the identity element of G. Suppose that there exists a subset W of G such that $W \neq \{e_G\}$ and for all elements $w \in W$, there are two elements $a, b \in W$ with $w = [a, b]$. Moreover, let w be an arbitrary element of W. Then, for all Boolean circuits C of depth d, there exists an M-program P_w over G that satisfies the conditions below.

(1) P_w is of length at most 4^d and is of the same input size as C.

(2) For all inputs $\vec{x} \in \{0,1\}^n$ where n is the input size of both C and P_w, $P_w(\vec{x}) = e_G$ if $C(\vec{x}) = 0$, and $P_w(\vec{x}) = w$ otherwise.

Proof. We show this lemma by an induction on the depth of a given circuit C. When the depth of C is 1 (that is, the Boolean function computed by C is either an identity function or its negation), it is obvious that an M-program consisting of single M-instruction computes the same function. Thus we have the lemma in this case.

Now assume, for some $d \geq 1$, that we have the lemma for all Boolean circuits of depth at most d and all elements $w \in W$. Suppose further that C is of depth $d + 1$, it is of input size n, and g is the output gate of C. We below consider three cases according to the type of the gate g.

Suppose g is a NOT-gate. Let h be a unique gate that gives an input value to g and let C_h denote the subcircuit of C whose output gate is h. Then, by inductive hypothesis, there exists an M-program Q_w that satisfies the following conditions.

(3) Q_w is of length at most 4^d and is of input size at most n.

(4) For all inputs $\vec{x} \in \{0,1\}^n$, $Q_w(\vec{x}) = e_G$ if $C_h(\vec{x}) = 0$, and $Q_w(\vec{x}) = w$ otherwise.

From this Q_w, we construct an M-program $Q_{w^{-1}}$ such that:

(5) $Q_{w^{-1}}$ is of length at most 4^d and is of input size atmost n, and

(6) for all inputs $\vec{x} \in \{0,1\}^n$, $Q_{w^{-1}}(\vec{x}) = e_G$ if $C_h(\vec{x}) = 0$, and $Q_{w^{-1}}(\vec{x}) = w^{-1}$ otherwise.

To construct $Q_{w^{-1}}$, we may first replace each M-instruction (i_j, a_j, b_j) by $(i_j, a_j^{-1}, b_j^{-1})$ and may further reverse the sequence of those M-instructions. Finally, we define P_w to be an

M-program obtained from $Q_{w^{-1}}$ by replacing its first M-instruction, say (i_1, c_1, d_1), with (i_1, wc_1, wd_1). Then, we can easily see that P_w satisfies the conditions (1) and (2) above.

Suppose next that g is an AND-gate (with fan-in two). Let h_1 and h_2 are gates of C that give input values to g, and let C_1 and C_2 denote the subcircuits of C whose output gates are h_1 and h_2 respectively. Furthermore, let a and b be elements of W such that $w = [a, b]$. Then, by inductive hypothesis, we have two M-programs Q_a and Q_b such that:

(5) both Q_a qand Q_b are of length at most 4^d and they are of input size at most n, and

(6-1) for all inputs $\vec{x} \in \{0,1\}^n$, $Q_a(\vec{x}) = e_G$ if $C_1(\vec{x}) = 0$, and $Q_a(\vec{x}) = a$ otherwise, and

(6-2) for all inputs $\vec{x} \in \{0,1\}^n$, $Q_b(\vec{x}) = e_G$ if $C_2(\vec{x}) = 0$, and $Q_b(\vec{x}) = b$ otherwise.

Then, we define P_w by $P_w = Q_{a^{-1}}, Q_{b^{-1}}, Q_a, Q_b$, where $Q_{a^{-1}}$ and $Q_{b^{-1}}$ denote M-programs obtained from Q_a and Q_b, respectively, by using the same method as mentioned in the above paragraph. It is not difficult to see that P_w satisfies the conditions (1) and (2) above. Thus we have the lemma in this case.

Suppose g is an OR-gate. In this case, we can obtain a desired M-program by using De Morgan's Law and the technique mentioned above. We leave the detail to the reader. ♠

From this lemma, we may show that any finite nonsolvable group has a subset W satisfing the conditions mentioned above. We below show this. Then, we can immeidiately obtain our result mentioned in the abstract section.

The following lemma is obtained by a simple calculation.

Lemma 2.2. Let G be any finite group and let a, b, c be any elements in G. Then, we have the following equations.

(1) $c^{-1}[a,b]c = [c^{-1}ac, c^{-1}bc]$. (2) $[ab, c] = b^{-1}[a,c]b[b,c]$. (3) $[a, bc] = [a,c]c^{-1}[a,b]c$. ♠

By using the above equations repeatedly, we can easily obtain the following lemma. We leave the detailed proof to the interested reader.

Lemma 2.3. Let G be any finite group, let V be a subset of G such that $V = \bigcup_{g \in G} g^{-1}Vg$, and let $a_1, \ldots, a_k, b_1, \ldots, b_m$ be any elements of V. Then, the commutator $[a_1 \cdots a_k, b_1 \cdots b_m]$ is represented as a product of commutators of elements in V. ♠

Lemma 2.4. For all finite nonsolvable groups G, there exists a subset W of G such that $W \neq \{e_G\}$ and for all $w \in W$, there are two elements $a, b \in W$ with $w = [a, b]$, where e_G denotes the identity element of G.

Proof. Let H be a subgroup of G satisfying that $H \neq \{e_G\}$ and $H = D(H)$. Such a subgroup surely exists since G is nonsolvable. Furthermore, let S be a subset of H that generates H, and let us define U by $U = \bigcup_{g \in G} g^{-1}Sg$. Then, we inductively define a subset V_i of G, for all integers $i \geq 0$, as follows.

$$V_0 = U, \quad V_i = \{[a,b] \ : \ a, b \in V_{i-1}\} \quad (i \geq 1).$$

We below observe that for each $i \geq 0$, (i) $V_i = \bigcup_{g \in G} g^{-1}V_ig$, and (ii) V_i generates H, by induction on i. From the definition of $U = V_0$, it is obvious that V_0 satisfies (i). Moreover,

V_0 generates H since it includes all elements in S. Assume V_i satisfies (i) and (ii). Since $H = D(H)$, each element h in H is represented as a product, say $[h_{1,1}, h_{1,2}][h_{2,1}, h_{2,2}] \cdots [h_{k,1}, h_{k,2}]$, of commutators of elements of H. Moreover, since V_i generates H, each $h_{i,j}$ is represented as a product of elements in V_i. Hence, the element h is represented as a product of elements of the form $[a_1 \cdots a_k, b_1 \ldots b_m]$ where each a_i and each b_i are elements in V_i. Then, from Lemma 2.3 and the inductive hypothesis that V_i satisfies (i) above, we have that h is represented as a product of elements in V_{i+1}. Thus V_{i+1} generates H. From Lemma 2.2(1) and the inductive hypothesis, it follows that V_{i+1} satisfies the condition (i) above.

Since each V_i is a subset of G which is finite, there exists two integers $i, j \geq 0$ such that $i < j$ and $V_i = V_j$. Then, we define a desired set W by $W = \bigcup_{k=i}^{j-1} V_k$. Since $H \neq \{e_G\}$ and each V_i generates H, we have $W \neq \{e_G\}$. Moreover, from the definitions of each V_i and W, we see that for all $w \in W$, there are two elements a, b with $w = [a, b]$. Thus we have the lemma. ♠

Combining Lemma 2.4 with Lemma 2.1, we immediately obtain the following theorem.

Thoerem 2.5. Let G be any finite nonsolvable group and C any circuit of depth d. Then, the Boolean function computed by C is computed by an M-program over G of length at most 4^d. ♠

3. Concluding Remarks

In [CL94], Cai and Lipton imporved Barrington's result on the alternating group of degree 5. They showed that any circuit of depth d can be simulated by an M-program over the group of length at most $2^{\lambda d}$ where $\lambda = 1.81\ldots$. However, it is unknown whether their result holds for all nonsolvable groups. They further showed a lower bound on the length of M-programs over groups: for any group G and any M-program P over G, if P computes the conjunction of n Boolean variables , then it must be of length at least $\Omega(n \log\log n)$. Hence, any M-program over any group simulating a circuit of depth d must have length asymptotically greater than 2^d.

In [Cle90], Cleve showed that for any constant $\varepsilon > 0$, a circuit of depth d can be simulated by a bounded-width branching program of length $2^{(1+\varepsilon)d}$. It would be interesting to ask whether the same result holds for M-programs over groups.

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