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Global structure of Brezis-Nirenberg type equations on the unit ball

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1 Introduction

We consider the existence and uniqueness of radial solutions of

$$\begin{cases} \Delta u + \lambda u + u^{(n+2)/(n-2)} = 0, \quad \text{in } B = \{x \in \mathbf{R}^n : |x| < 1\}, \\ u > 0, \quad \text{in } B, \\ \kappa \dfrac{\partial u}{\partial \nu} + u = 0, \quad \text{on } \partial B, \end{cases} \tag{1}$$

where ν is the outward unit normal vector on ∂B, $\lambda < \lambda_*^2$ (λ_*^2 is the first eigenvalue of $-\Delta$ with 0-Dirichlet condition on B) and $\kappa \geq 0$.

In the case $\kappa = 0$, it is well-known that any solution of (1) is radially symmetric by Gidas-Ni-Nirenberg [5]. Moreover, Brezis-Nirenberg [2] proved that (1) with $n = 3$ has a solution if and only if $\lambda \in (\pi^2/4, \pi^2)$ and that (1) with $n \geq 4$ does if and only if $\lambda \in (0, \lambda_*^2)$. Later, Kwong-Li [4] and Zhang [15] proved that the solution obtained by [2] is unique.

Even though the 0-Dirichlet problem has no positive solution for the case $\lambda < 0$, the homogeneous Neumann problem has a positive one. There is also a nonradial solution which has a peak on the boundary at least if λ is near $-\infty$ by Ni-Takagi [7] [9].

As for the third boundary conditions, X.-J. Wang [11] treated more general problems than (1) under the "least energy" condition. Recently, X.-B. Pan [10] treated the asymptotic behavior of solutions to (1) as $\lambda \to -\infty$ in analogous to [7].

Though there seems to be no results similar to Gidas-Ni-Nirenberg [5] for the third boundary conditions for small $|\lambda|$, we restrict our attention only to *radial solutions.*

We consider the initial value problem

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + \lambda u + u_+^{(n+2)/(n-2)} = 0, \quad 0 < r < 1, \\ \\ u(0) = \alpha, \ u_r(0) = 0 \end{cases} \tag{2}$$

and seek a suitable number $\alpha > 0$ satisfying $u(r) > 0$ on $(0, 1)$ and

$$\kappa u_r(1) + u(1) = 0, \tag{3}$$

where $u_+ = \max\{u, 0\}$. Note that (2) has a solution (denoted by $u(r; \lambda, \alpha)$) for any $\alpha > 0$ and λ.

The main purpose of this article is to make clear the range of λ in which (1) has a unique solution and find out the relation between λ and α.

Hereafter we restrict ourselves to the case $\underline{n = 3}$.

2 Results

To state our theorems, we introduce four numbers. Let $\lambda_\kappa \in (0, \pi]$ satisfy $\tan \lambda_\kappa = \kappa\lambda_\kappa/(\kappa - 1)$ if $\kappa \neq 1$ and $\lambda_\kappa = \pi/2$ if $\kappa = 1$. Define λ_2 by $\tan(\lambda_2 - \pi/2) = \kappa\lambda_2/(\kappa - 1)$ if $0 < \kappa \leq 1$. As we see in Theorem 2, λ_2 is a blow-up point. Set $\lambda_3 \tanh \lambda_3 = (\kappa - 1)/\kappa$ if $\kappa \geq 1$. For $\kappa \in [0,1)$, we define $\lambda_4 > 0$ by $\tanh \lambda_4 = \kappa\lambda_4$ if $0 < \kappa < 1$ and $\lambda_4 = \infty$ if $\kappa = 0$.

Note that λ_κ^2 is the first eigenvalue of $-\Delta$ with the boundary condition $\kappa \partial u/\partial\nu + u = 0$ on ∂B. Our methods are based on Yotsutani [12] and Yanagida-Yotsutani [13], [14]

Theorem 1 *Let $n = 3$.*

Case (I): $0 \leq \kappa \leq 1$. *If $\lambda_2^2 < \lambda < \lambda_\kappa^2$, then (1) has a unique radial solution.*

Case (II): $1 < \kappa$. *If $-\lambda_3^2 < \lambda < \lambda_\kappa^2$, then (1) has a unique radial solution.*

Remark. If $-\lambda_4^2 \leq \lambda \leq \lambda_2^2$, then (1) has no radial solution. Moreover, for such λ the inequality $\kappa u_r(1;\lambda,\alpha) + u(1;\lambda,\alpha) > 0$ holds for any $\alpha > 0$. For $\lambda < -\lambda_4$, there may be at least two solutions, while λ_4 may not be sharp.

By this theorem, there is a one-to-one mapping from λ to α, that is, α is a function of λ. So we can draw the graph of $\alpha = \alpha(\lambda;\kappa)$.

Let

$$C = \{(\lambda, \alpha(\lambda)) \mid \alpha \text{ satisfies } (2) - (3)\},$$

$$D_C = \{(\lambda, y) \mid y > \alpha(\lambda)\},$$

$$D_S = \{(\lambda, y) \mid y < \alpha(\lambda)\}.$$

Note that for $(\lambda, \alpha) \in D_C$, $\kappa u_r(1;\lambda,\alpha) + u(1;\lambda,\alpha) < 0$ or $u(r;\lambda,\alpha)$ has a zero in $(0,1)$. Similarly, for $(\lambda,\alpha) \in D_S$, $\kappa u_r(1;\lambda,\alpha) + u(1;\lambda,\alpha) > 0$.

Theorem 2 *$\alpha(\lambda)$ is a continuous function of λ satisfying $\alpha(\lambda) \to 0$ as $\lambda \to \lambda_\kappa^2 - 0$ and $\alpha(\lambda) \to \infty$ as $\lambda \to \lambda_2^2 + 0$. More precisely, $\alpha(\lambda)$ satisfies*

$$\lim_{\lambda \to \lambda_2^2 + 0} (\lambda - \lambda_2^2)\alpha(\lambda)^2 = \frac{2\sqrt{3}\pi\lambda_2^2\left\{(1-\kappa)\sin\lambda_2 + \kappa\lambda_2\right\}}{\sin\lambda_2}.$$

Remark. As a matter of fact, the curve C must be a C^1 curve. As we see from the standard bifurcation theory, $(\lambda_\kappa^2, 0)$ is a bifurcation point.

The blow-up rate of $\alpha(\lambda)$ as $\lambda \to \lambda_2^2 + 0$ is known by Brezis-Peletier [3] for $\kappa = 0$. We show the graph of $\alpha(\lambda)$ in Section 5.

3 Reduction to a Matukuma-type equation

Our idea for the proof of Theorem 1 is to reduce (2)-(3) to an exterior Neumann problem of a Matukuma-type equation. For a solution φ to

$$\begin{cases} \varphi_{rr} + \dfrac{2}{r}\varphi_r + \lambda\varphi = 0, \quad 0 < r < 1, \\ \varphi(0) = 1, \ \varphi_r(0) = 0, \end{cases} \tag{4}$$

let $u = v\varphi$. Note that

$$\varphi = \begin{cases} \dfrac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r} & \text{if } \lambda > 0, \\ \dfrac{\sinh(\sqrt{-\lambda}r)}{\sqrt{-\lambda}r} & \text{if } \lambda < 0. \end{cases}$$

If u is a solution to (2)-(3), then v satisfies

$$\begin{cases} v_{rr} + (\dfrac{2}{r} + 2\dfrac{\varphi_r}{\varphi})\varphi_r + \varphi^4 v^5 = 0, \quad 0 < r < 1, \\ v(0) = 1, \\ \dfrac{\kappa\varphi(1)}{\varphi(1) + \kappa\varphi_r(1)} v_r(1) + v(1) = 0. \end{cases} \tag{5}$$

Next, let $g(r) = r^2\varphi^2$. Then v satisfies

$$\begin{cases} \dfrac{1}{g^2}(gv_r)_r + \varphi^4 v^5 = 0, \quad 0 < r < 1, \\ v(0) = 1, \\ \dfrac{\kappa\varphi(1)}{\varphi(1) + \kappa\varphi_r(1)} v_r(1) + v(1) = 0. \end{cases} \tag{6}$$

Finally, let

$$h(r) = g(r)\Big(\int_r^1 \frac{ds}{g(s)} + \frac{\kappa\varphi(1)}{g(1)(\varphi(1) + \kappa\varphi_r(1))}\Big),$$

$$w(\tau) := \frac{g(r)}{h(r)} v(r),$$

and

$$\tau := \exp\Big(\int_r^1 \frac{ds}{h(s)}\Big).$$

Then $w(\tau)$ satisfies the exterior Neumann problem

$$\begin{cases} \dfrac{1}{\tau^2}(\tau^2 w_\tau)_\tau + K(\tau)v^5 = 0, \quad \tau > 1, \\ w_\tau(1) = 0, \\ \lim_{\tau\to\infty} \tau w(\tau) > 0, \end{cases} \tag{7}$$

where

$$K(\tau) := \frac{1}{\tau^2}\frac{h(r)^6}{g(r)^4}\varphi(r)^4.$$

We can apply the modified version of [13] to obtain Theorem 1.

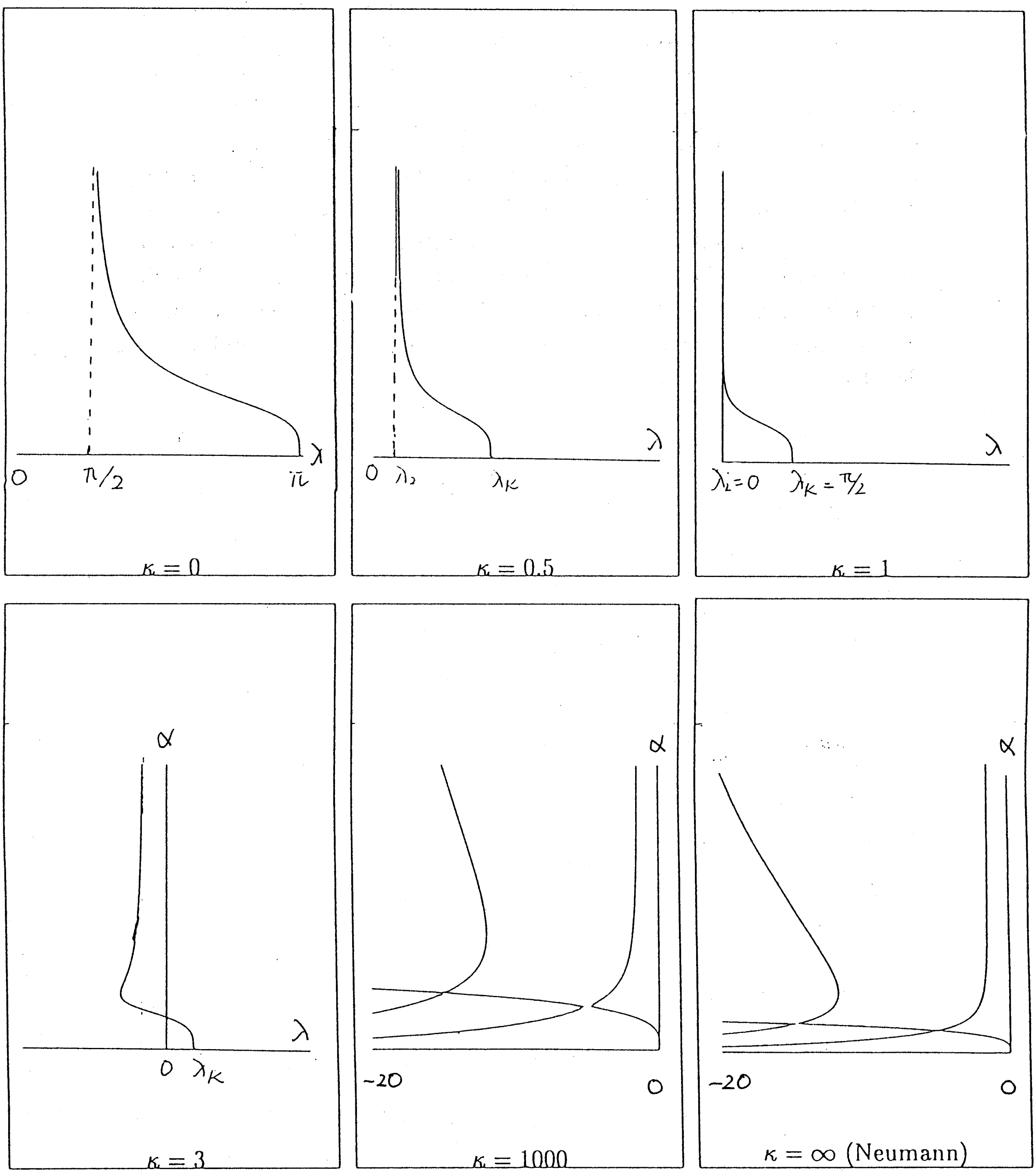
As for Theorem 2, we may follow the argument in [14]. To prove the blow-up rate, we use the argument as in [3].

4 Concluding remarks

So far, we do not pay much attention to the Neumann problem ($\kappa = \infty$). However, there are many results on the radial solutions as well as non-radial ones (least-energy solutions). See for instance, Adimurthi-Yadava [1], Ni-Takagi [9] [10] or Ni-Pan-Takagi [7]. According to their results, nonconstant radial solutions bifurcate from the constant solution at $(-\mu_j/4, (\mu_j/4)^{1/4})$ where μ_j the eigenvalues of $-\Delta$ subject to the homogeneous Neumann problem $(0 = \mu_0 < \mu_1 < \mu_2 < \ldots)$. Moreover, the properties of the bifurcation branch are known by [1], [9] etc. In view of the graphs in the case where $\lambda < 0$and $\kappa > 0$ is sufficiently large, our results seems to be a " homotopy bridge" connecting the Dirichlet problem and the Neumann one. We may regard the graph ($\kappa = 1000$) as an imperfect bifurcation, though we do not have any rigorous proofs. See for instance, Chapter 3 of Golubitsky and Schaeffer [6].

5 Graphs

These computations are due to Mr. H. Morishita of Hyogo University.



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