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On a global solution and a blow-up solution to one-phase Stefan problem

Toyohiko AIKI 愛木 豊彦
Department of Mathematics, Faculty of Education, Gifu University
Gifu 501-11, Japan

Hitoshi IMAI 今井 仁司
Faculty of Engineering, Tokushima University
Tokushima 770, Japan

1 Introduction

Let us consider the following one-phase Stefan problem DP (resp. NP) with homogeneous Dirichlet (resp. Neumann) boundary condition in one-dimensional space: The problem is to find a curve $x = \ell(t) > 0$ on $[0,T]$, $(0 < T < \infty)$, and a function $u = u(t,x)$ on $Q_\ell(T) := \{(t,x); 0 < t < T, 0 < x < \ell(t)\}$ satisfying that

$$u_t = u_{xx} + u^{1+\alpha} \quad \text{in } Q_\ell(T), \tag{1.1}$$

$$u(0,x) = u_0(x) \quad \text{for } 0 \le x \le \ell_0, \tag{1.2}$$

$$u(t,0) = 0 \quad \text{for } 0 < t < T, \tag{1.3}$$

$$(\text{resp. } \frac{\partial}{\partial x}u(t,0) = 0 \quad \text{for } 0 < t < T,) \tag{1.4}$$

$$u(t,\ell(t)) = 0 \quad \text{for } 0 < t < T, \tag{1.5}$$

$$\frac{d}{dt}\ell(t) = -u_x(t,\ell(t)) \quad \text{for } 0 < t < T, \tag{1.6}$$

$$\ell(0) = \ell_0 \tag{1.7}$$

where α and ℓ_0 are given positive constants and u_0 is a given initial function on $[0, \ell_0]$.

For the above problems DP and NP Fasano-Primicerio (cf. [6]) established the local existence in time and the uniqueness for solutions to the above DP and NP in the classical formulation (which means that u_t and u_{xx} are continuous functions). Besides, for solutions of DP and NP in the distribution sense (which means that u_t and u_{xx} belong to L^2-class) the existence, the comparison and the behavior were studied by Aiki-Kenmochi [1, 5, 10].

It is well known that there are blow-up solutions of the usual initial boundary value problem for semilinear equation (1) in bounded domain, accordingly, by using comparison principle it is clear that our problems DP and NP have blow-up solutions. Here, we note the the following global existence result of a solution: Let $[0, T^*)$ be the maximal interval of existence of the solution to DP and NP, we see (cf. [5]) that the following cases (a) or (b) must occur:

(a) $T^* = +\infty$;
(b) $T^* < +\infty$ and $|u|_{L^\infty(Q_\ell(t))} \to +\infty$ as $t \uparrow T^*$.

However, from the above result we can get no information for the behavior of free boundary ℓ at blow-up time. In the present paper we shall show the behavior of blow-up solutions to DP and NP at finite blow-up time.

The first result concerned with the problem NP is stated as follows (Theorem 2.2): Under the suitable assumptions for the initial data u_0 if the blow-up time T^* is finite, then u blows up at only one point $x = 0$ and $\ell(t)$ converges to some number L as $t \uparrow T^*$ where $\{u, \ell\}$ is a solution to NP. Our proof for this result is essentially due to papers by Friedman-McLeod [7] and Fujita-Chen [9]. In case homogeneous Neumann boundary condition and $u_{0,x} \le 0$, the maximum point for u is always the point $x = 0$. But, in case Dirichlet boundary condition the maximum point may move. By this difficulty we can not estimate the blow-up point. So, we have obtained the following result (Theorem 2.3): We assume that u_0 satisfies some conditions. If T^* is finite, then the cases (A) or (B) must occur:

(A) u blows up at only one point and the free boundary ℓ converges.

(B) $\ell(t)$ tends to infinity as $t \uparrow T^*$ and u is locally bounded with respect to $x \ge 0$.

By the numerical experiments it seems that the case (B) does not occur (section 4). But we can not prove it, theoretically.

Besides, we know the following result concerned with global existence for solutions of Cauchy problem for equation (1.1) (cf. Fujita [8] and Levine [11]): In case $\alpha > 2$ there are global solutions in time for sufficiently small initial data; in case $0 < \alpha \le 2$ the non-trivial solution always blows up at some finite time. Also, if a domain is bounded then for $\alpha > 0$ the problem has a global solutions. Hence, it follows from comparison principle that for $\alpha > 2$ $NP(DP)$ has a global solution in time for a small initial function u_0. One of purposes of this paper is to establish the global existence result for NP (Theorem 3.1) with $\alpha > 1$.

In the final section we carried out the following numerical computations for the problem DP. We put $u_{0\lambda} = \lambda v_0$ for $\lambda > 0$ and some non-negative function v_0. Let $\{u_\lambda, \ell_\lambda\}$ be a solution to DP with initial data $u_0 = u_{0\lambda}$. Then, in our computations for small $\lambda > 0$ $T^* = \infty$ and the free boundary $\ell_\lambda(t)$ converges to some number $\ell_{\infty\lambda}$ as $t \uparrow T^*$. Also, for large $\lambda > 0$ we obtain that $T^* < \infty$ and $\ell_\lambda(t)$ converges to $\ell_{\infty,\lambda}$ as $t \uparrow T^*$. So, we shall show the relationship between λ and $\ell_{\infty\lambda}$.

2 Blow-up points

In this section we discuss blow-up points for the solution to NP and DP. For the proof of our results we shall use the storing maximum principle. Hence, we deal with the classical solution.

Here, we give assumptions (H1) $\sim$ (H4) for initial data ℓ_0 and u_0.

(H1) $\ell_0 > 0$ and $u_0 \in C^2((0, \ell_0)) \cap C^1([0, \ell_0])$ and $u_0(x) > 0$ for $x \in (0, \ell_0)$,

(H2) $u_{0,xx}(x) + u_0^{1+\alpha}(x) \ge 0$ for $x \in (0, \ell_0)$,

(H3) $u_0(\ell_0) = 0$, $u_{0,x} < 0$ for $x \in (0, \ell_0)$ and $u_{0,x}(0) = 0$,

(H4) $u_0(0) = u_0(\ell_0) = 0$, $u_{0,x} > 0$ on $[0, x_0)$ and $u_{0,x} < 0$ on $(x_0, \ell_0]$ for some $x_0 \in (0, \ell_0)$.

We begin with the precise definition of a solution to DP and NP. Let $C^{1,0}(\overline{Q_\ell(T)})$ be the set of functions which are continuous on $\overline{Q_\ell(T)}$ with their $x-$derivatives.

Definition A couple $\{u, \ell\}$ of functions $u = u(t, x)$ and $x = \ell(t)$ is said to be a solution of DP (resp. NP) on a compact interval $[0, T]$, $0 < T < +\infty$, if the following conditions (S1) and (S2) are satisfied:

(S1) $\ell \in C^1([0, T])$, and $u \in C^{1,0}(\overline{Q_\ell(T)})$, u_{xx} and u_t are continuous in $Q_\ell(T)$;
(S2) (1.1) $\sim$ (1.3) and (1.5) $\sim$ (1.7) (resp. (1.1) $\sim$ (1.2) and (1.4) $\sim$ (1.7)) hold in the classical sense.

Also, we call a couple $\{u, \ell\}$ is a solution of DP(resp. NP) on an interval $[0, T')$, $0 < T' \leq \infty$, if it is a solution of DP(resp. NP) on $[0, T]$ in the above sense for any $0 < T < T'$.

First, we recall the theorem concerned with local existence of solutions to the above DP and NP.

Theorem 2.1. *(cf. [6, Theorem 1]) We assume that $u_0 \in C^1([0, \ell_0])$, $u_0 \geq 0$ on $[0, \ell_0]$, $u_0(\ell_0) = 0$ and $u_0(0) = 0$(resp. $(u_{0,x}(0) = 0)$. Then there exists a positive number T_0 depending only on $|u_0|_{C^1([0,\ell_0])}$, ℓ_0 and α such that problem DP(resp. NP) has a unique solution $\{u, \ell\}$ on $[0, T_0]$.*

For the problems DP and NP, we say that $[0, T)$, $0 < T \leq +\infty$, is the maximal interval of existence of the solution, if the problem has a solution on time-interval $[0, T']$, for every T' with $0 < T' < T$ and the solution can not be extended in time beyond T.

Theorem 2.2. *(cf. [2]) Assume that (H1) $\sim$ (H3) hold. Let $\{u, \ell\}$ be a solution of NP and $[0, T^*)$ be the maximal interval of existence of solutions to NP. If $T^* < \infty$, then $\ell(t) \uparrow L < +\infty$ as $t \uparrow T^*$, $u(t, 0) \to +\infty$ as $t \uparrow T^*$, and for any $x \in (0, L)$ there exists a positive number $M(x)$ such that*

$$|u(t, x)| \leq M(x) \quad \textit{for any } t \textit{ with } (t, x) \in Q_\ell(T).$$

In [9] Fujita and Chen studied the following initial boundary value problem.

$$u_t = u_{xx} + u^{1+\alpha} \quad \text{in } (0, T) \times (0, 1),$$

$$u_x(t, 0) = 0 \quad \text{for } t \in (0, T],$$

$$u(t, 1) = 0 \quad \text{for } t \in (0, T],$$

$$u(0, x) = u_0(x) \quad \text{for } x \in [0, 1].$$

They showed that under the similar assumptions for u_0 to (H1) $\sim$ (H3) if the solution u blows up then blow-up point is one and only one point $x = 0$. In the proof of Theorem 2.2 we done with help of the idea in [9].

Theorem 2.3. *(cf. [3]) Assume that (H1),(H2) and (H4) hold. Let $[0, T^*)$ be the maximal interval of existence of the solution $\{u, \ell\}$ to DP. If T^* is finite, then either the*

following cases (A) or (B) always happens:
(A) $\ell(t) \to \ell_\infty$ *as* $t \uparrow T^*$ *where* ℓ_∞ *is some positive number, there exists one and only one point* $x^* \in (0, \ell_\infty)$ *such that* $u(t,x) \to +\infty$ *as* $t \uparrow T^*$ *and for* $x \in (0, \ell_\infty)$ *with* $x \neq x^*$ *there is a positive constant* $M_1(x)$ *such that* $|u(t,x)| \leq M_1(x)$ *for* t *with* $(t,x) \in Q_\ell(T^*)$;
(B) $\ell(t) \to +\infty$ *as* $t \uparrow T^*$ *and for any* $x > 0$ *there is a positive number* $M_2(x)$ *satisfying that* $|u(t,\xi)| \leq M_2(x)$ *for* $(t,\xi) \in Q_\ell(T^*) \cap \{\xi < x\}$.

The proof of Theorem 2.3 is done in the following way.

First, we shall show continuation of solutions of our problem DP. Precisely speaking, for $0 < T_0 < \infty$ if a solution $\{u, \ell\}$ to SP on $[0, T_0)$ satisfies that

$$|u(t,x)| \leq K \quad \text{for } (t,x) \in \overline{Q_\ell(T_0)} \cap \{t < T_0\}$$

where K is a positive constant, then the solution $\{u, \ell\}$ is extended in time beyond T_0.

Next, we assume that (B) in the assertion of Theorem 3 does not hold, that is, (A1) or (A2) is valid:

(A1) There is a number $x_1 \in (0, \infty)$ satisfying that for some sequence $\{(t_n, \xi_n)\} \subset Q_\ell(T^*)$ $t_n \uparrow T^*$, $\xi_n \to x_1$ and $u(t_n, \xi_n) \to \infty$ as $n \to \infty$;

(A2) there exists a positive constant L_0 such that $\ell(t) \leq L_0$ for any $t \in [0, T^*)$.

Therefore, it is sufficient to show that (A1) is a sufficient condition for (A) in the statement of Theorem 3 and (A2) is, too.

Under the condition (A1) by the similar argument to those of Friedman-McLeod [7] we infer that the set of blow-up points consists of only one point $x_1 \in (0, \infty)$. Clearly, $u(t, x_1) \to +\infty$ as $t \uparrow T^*$ and for any $x_2 > x_1$ there is a positive constant $M_3(x_2)$ such that

$$|u(t,x)| \leq M_3(x_2) \quad \text{for any } (t,x) \in Q_\ell(T^*) \cap \{x > x_2\}.$$

By using the above estimate and a comparison theorem for solutions to one-phase Stefan problems we conclude that (A) holds. Similarly, we can prove that (A2) implies (A).

3 Global existence

The purpose of this section is to show that in case $\alpha > 1$ there is a global solution $\{u, \ell\}$ of $NP(u_0, \ell_0)$ for a sufficiently small initial function u_0, which satisfies that the set $\{\ell(t)\}_{t \geq 0}$ is bounded and $|u(t)|_{L^\infty(0,\ell(t))}$ decays in exponential order. We can get no results concerned with the global existence to our problem in case $0 < \alpha \leq 1$.

In this section we consider a solution to NP in the distribution sense. Here, we give a definition of a solution. We say that a pair $\{u, \ell\}$ is a solution of $NP(u_0, \ell_0)$ on $[0, T]$, $0 < T < \infty$, if the following properties are fulfilled:

(S1) $u \in W^{1,2}(0, T; L^2(0, \ell(t))) \cap L^\infty(0, T; W^{1,2}(0, \ell(t)))$, and $\ell \in W^{1,2}(0, T)$ with $0 < \ell$ on $[0, T]$.

(S2) (1.1) holds in the sense of $\mathcal{D}'(Q_\ell(T))$ and (1.2) $\sim$ (1.7) are satisfied.

Also, we call a couple $\{u, \ell\}$ is a solution of NP on an interval $[0, T')$, $0 < T' \leq \infty$, if it is a solution of NP on $[0, T]$ in the above sense for any $0 < T < T'$.

Theorem 3.1. *(cf. [4]) We assume that $\alpha > 1$, $\ell_0 > 0$, $u_0 \in W^{1,2}(0,\ell_0)$ with $u_0 \geq 0$ and $u_0(\ell_0) = 0$. Then, there exist positive numbers $\delta > 0$ and $p_0 > 1$ such that if $\int_0^{\ell_0} u_0^{1+\alpha} dx \leq \delta$, $\int_0^{\ell_0} u_0^2 dx \leq \delta$, $\int_0^{\ell_0} u_0^{p_0} dx \leq 1$ and $\int_0^{\ell_0} u_{0,x}^2 dx \leq 1$, then the problem NP with initial condition u_0 and ℓ_0 has a solution $\{u,\ell\}$ on $[0,\infty)$ satisfying that*

$$\int_0^{\ell(t)} u(t,x)dx + \ell(t) \leq C \quad \text{for } t > 0,$$

$$\frac{d}{dt}|u_x(t)|^2_{L^2(0,\ell(t))} \leq 0 \quad \text{for a.e. } t > 0,$$

$$|u(t)|_{L^\infty(0,\ell(t))} \leq C\exp(-\mu t) \quad \text{for } t > 0$$

where C and μ are some positive constants.

Here, we note Gagliardo-Nirenberg inequalities.

Lemma 3.1. *(1) For $p \geq 2$ and $\alpha \geq 0$ we put $q = 2(p+\alpha)/p$ and $r \in (0,q)$. Then we have*

$$\int_0^d u^{p+\alpha} dx \leq \left(\frac{q+2}{2}\right)^{\frac{2(q-r)}{r+2}} |(u^{\frac{p}{2}})_x|^{\frac{2(q-r)}{r+2}}_{L^2(0,d)} \left(\int_0^d u^{\frac{pr}{2}} dx\right)^{\frac{q+2}{r+2}}$$

$$\text{for } u \in W^{1,2}(0,d) \text{ with } u(d) = 0.$$

(2) For $q \geq 1$ we have

$$|u|_{L^\infty(0,d)} \leq \left(\frac{q+2}{2}\right)^{\frac{2}{q+2}} |u_x|^{\frac{2}{q+2}}_{L^2(0,d)} |u|^{\frac{q}{q+2}}_{L^q(0,d)} \quad \text{for } u \in W^{1,2}(0,d) \text{ with } u(d) = 0.$$

In order to prove Theorem 3.1 we prepare the following energy estimates.

Lemma 3.2. *We suppose that $\alpha > 0$, $\ell_0 > 0$ and $u_0 \in W^{1,2}(0,\ell_0)$ such that $u_0 \geq 0$ on $[0,\ell_0]$ and $u_0(\ell_0) = 0$. Let $\{u,\ell\}$ be a solution of NP on $[0,T]$, $0 < T < \infty$. Then we have:*

(1) For each $p > 1$

$$\frac{d}{dt}\int_0^{\ell(t)} u^p(t,x)dx$$
$$\leq \left\{-\frac{4(p-1)}{p} + p(2+\frac{\alpha}{p})^2(\int_0^{\ell(t)} u^{1+\alpha}(t,x)dx)^{\frac{\alpha}{1+\alpha}} \ell(t)^{2-\frac{1}{1+\alpha}}\right\} |(u^{\frac{p}{2}}(t))_x|^2_{L^2(0,\ell(t))}$$

for any $t \in (0,T]$.

(2) $\frac{d}{dt}\{\int_0^{\ell(t)} u(t,x)dx + \ell(t)\} = \int_0^{\ell(t)} u(t,x)^{1+\alpha} dx$ for $t \in (0,T]$.

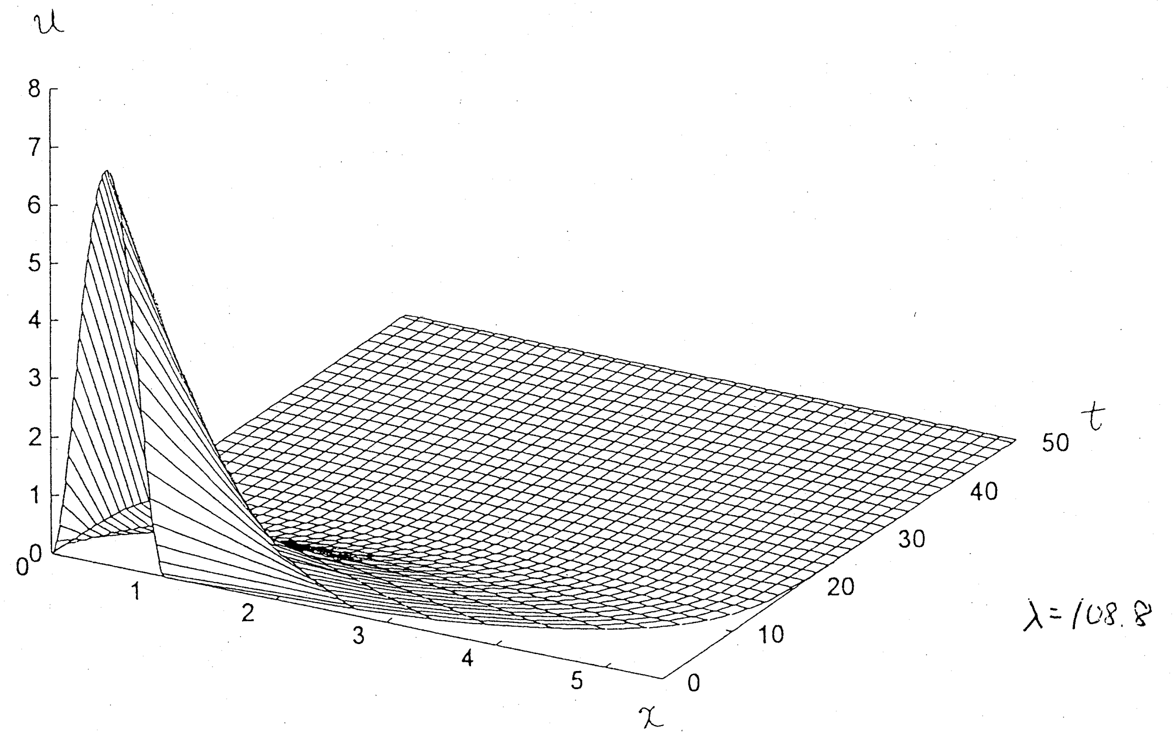
(3) For a.e. $t \in [0,T]$

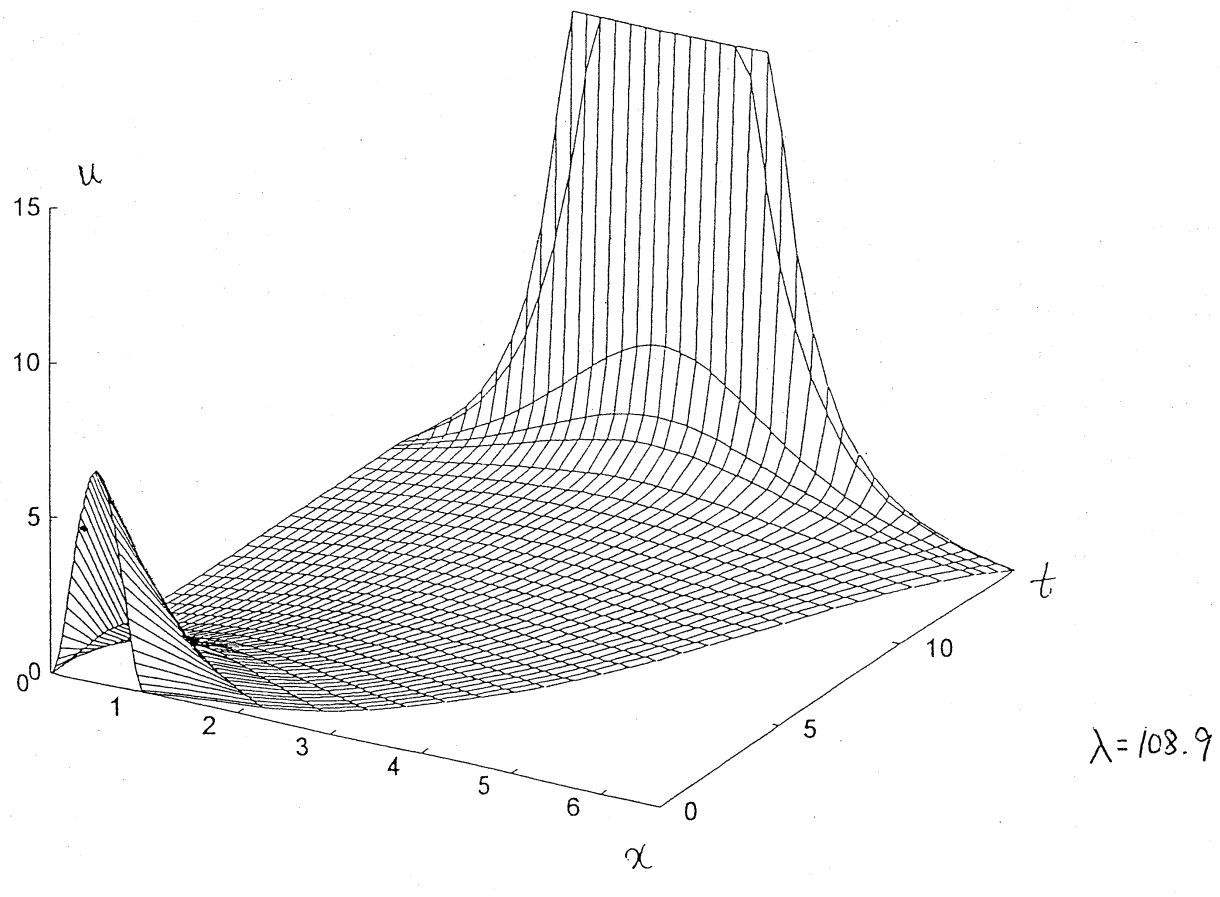
$$|u_t(t)|^2_{L^2(0,\ell(t))} + \frac{1}{2}|\ell'(t)|^3 + \frac{1}{2}\frac{d}{dt}|u_x(t)|^2_{L^2(0,\ell(t))} = \frac{1}{2+\alpha}\frac{d}{dt}|u(t)|^{2+\alpha}_{L^{2+\alpha}(0,\ell(t))}.$$

Applying Gagliardo-Nirenberg inequalities to the above inequalities we can prove Theorem 3.1.

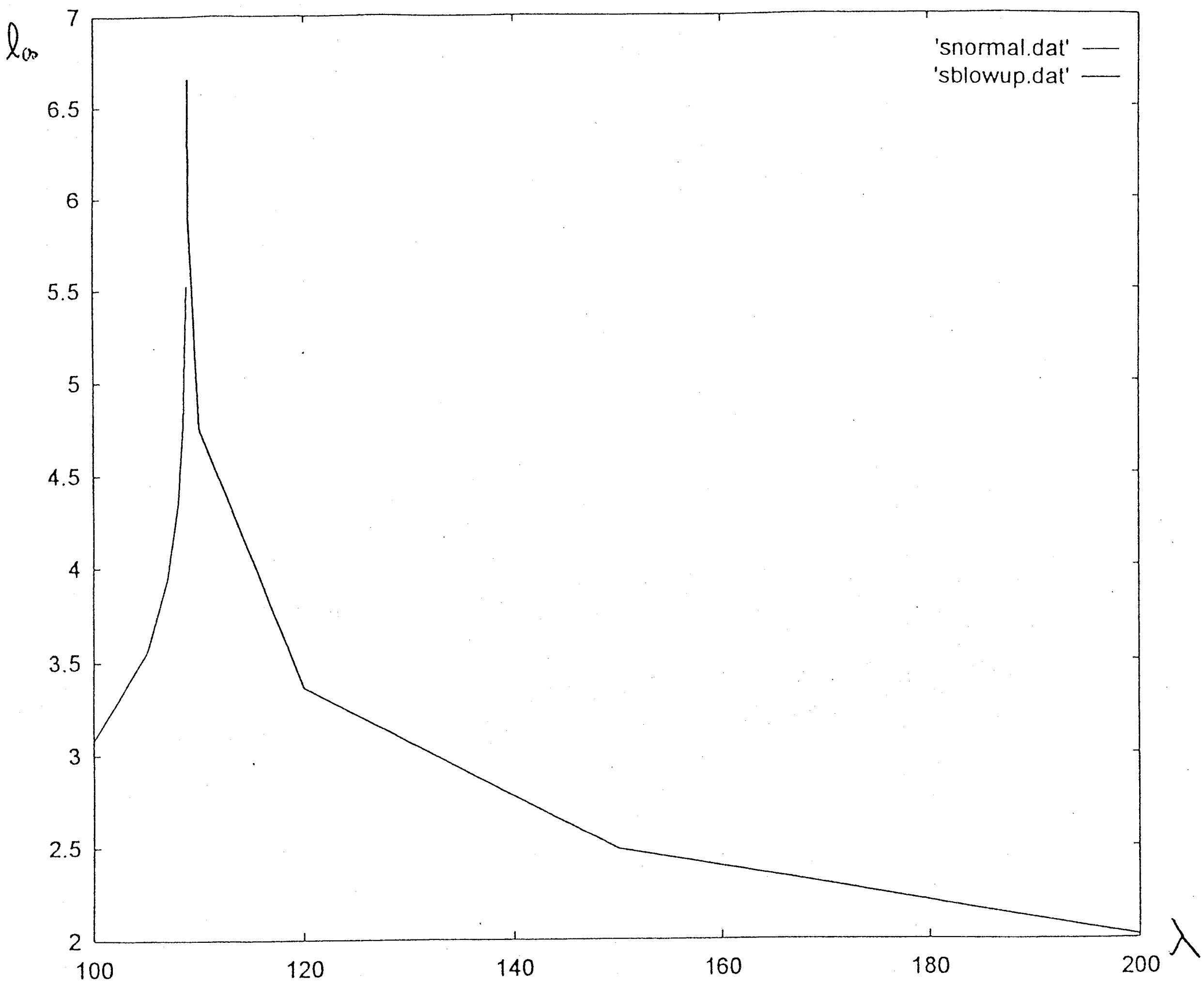
4 Numerical experiments

We carried out numerical computations to DP with $\alpha = 1$, $\ell_0 = 1$ and $u_0(x) = \lambda v_0(x)$, $\lambda > 0$, where $v_0(x) = x^2(x-1)^2$.





The follwing figure discribes the relationship between λ and ℓ_∞.



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