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Finite Element Analysis of Axisymmetric Flow Problems and Its Application

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1 Introduction

In many fields of science and engineering we are required to solve partial differential equations. Typical examples are found in the problems of the structural dynamics, the fluid dynamics, and the magnet electric dynamics. In these fields it is not sufficient to give a qualitative analysis such as the existence, the uniqueness, or some properties of the solution, but we need quantitative analysis, for example, to find drag coefficients of airfoils and critical exterior forces which structures can support. Nowadays we have computers and we can use them for the solution of these problems. The finite element method (FEM) is one of the most powerful numerical techniques to solve partial differential equations (PDEs).

The appearance of the first computer in the world ENIAC was in 1945 and the idea of FEM is found in the paper of Courant [1] written in 1943. We can say that it is by the combination of computers and FEM that PDEs have been solved in the practical sense.

Finite element schemes are not derived directly from PDEs. At first we derive variational formulations corresponding to PDEs. Discretizing them, we obtain finite element schemes. It gives a clear contrast with the finite difference method (FDM) where schemes are derived by approximating PDEs directly. One advantage of FEM is that the derivation is natural from the mathematical view point, which allows us to use various mathematical tools in establishing error estimates. Thus FEM is an excellent application of the variational procedure to the practical purpose for solving PDEs.

In this paper we discuss on the finite element analysis of axisymmetric flow problems and its application to the computation of drag coefficients.

Throughout this article we consider a regular family [2] of triangulations of Ω. We denote by h the maximum element size of each triangulation.

2 A variational formulation of axisymmetric flow problems

Here we derive a variational formulation of the Navier-Stokes equations in the cylindrical domain. For a variational formulation in the Cartesian coordinate we refer to [3].

Let Ω be the meridian of an axisymmetric domain in $\mathbf{R}^3$. We denote the point in Ω by $x = (x_1, x_2)$, where x_1 is the distance from the axis and x_2 is the axis coordinate. The stationary axisymmetric Navier-Stokes equations are written in the cylindrical coordinates as

$$(u \cdot \mathrm{grad})u + \frac{1}{Re}Lu + \mathrm{grad}\ p = f, \tag{1}$$

$$\mathrm{div}_1\ u = 0, \tag{2}$$

where $u = (u_1, u_2)^T$ is the velocity, p is the pressure, $f = (f_1, f_2)^T$ is an external force, Re is the Reynolds number, and

$$\mathrm{div}_1 u = \frac{1}{x_1}\mathrm{div}(x_1 u), \quad \Delta_1 \equiv \mathrm{div}_1 \cdot \mathrm{grad} = \frac{\partial^2}{\partial x_1^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1} + \frac{\partial^2}{\partial x_2^2}, \quad L = \begin{bmatrix} -\Delta_1 + \frac{1}{x_1^2} & 0 \\ 0 & -\Delta_1 \end{bmatrix}.$$

We note that the differential operators acting each component of the velocity are different each other and that they have singularities on the axis $x_1 = 0$. The boundary of Ω is divided into three parts,

$$\partial\Omega = \Gamma_0 + \Gamma_1 + \Gamma_2 \quad (\Gamma_0 \equiv \partial\Omega \cap \{x_1 = 0\})$$

and we impose the boundary conditions

$$u = g^1 \quad (x \in \Gamma_1), \qquad \sigma n = g^2 \quad (x \in \Gamma_2) \tag{3}$$

where g^1 is a given velocity, g^2 is a given surface force, n is a unit outer normal to $\partial\Omega$ and $\sigma = [\sigma_{ij}]$ is the stress tensor defined by

$$\sigma_{ij} = -p\delta_{ij} + \frac{2}{Re}D_{ij}(u), \quad D_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4}$$

We introduce function spaces

$$X^{\ell,2}_{1/2}(\Omega) = \{v \in \mathcal{D}'(\Omega);\ x_1^{\frac{1}{2}-\ell+|\beta|}D^\beta v \in L^2(\Omega),\ 0 \le |\beta| \le \ell\},$$

$$W^{\ell,2}_{1/2}(\Omega) = \{v \in \mathcal{D}'(\Omega);\ x_1^{\frac{1}{2}}D^\beta v \in L^2(\Omega),\ 0 \le |\beta| \le \ell\}$$

and define

$$V(g^1) = \{v \in X^{1,2}_{1/2}(\Omega) \times W^{1,2}_{1/2}(\Omega);\ v = g^1\ (x \in \Gamma_1)\ \},$$

$$V = V(0), \quad Q = L^2_{1/2}(\Omega) \equiv W^{0,2}_{1/2}(\Omega).$$

We derive a variational formulation corresponding to (1)-(3): Find $(u,p) \in V(g^1) \times Q$ such that

$$a_1(u,u,v) + a(u,v) + b(v,p) = < F, v > \quad (\forall v \in V), \tag{5}$$

$$b(u,q) = 0 \quad (\forall q \in Q), \tag{6}$$

where

$$a(u,v) = \frac{2}{Re}\int_\Omega \{\sum_{i,j=1}^{2} D_{ij}(u)D_{ij}(v) + \frac{u_1 v_1}{x_1^2}\} x_1 dx, \quad b(v,q) = -\int_\Omega q \ \mathrm{div}_1 v \ x_1 dx, \tag{7}$$

$$a_1(w,u,v) = \int_\Omega \sum_{i=1}^{2} (w \cdot \mathrm{grad}\ u_i) v_i \ x_1 dx, \quad < F, v >= \int_\Omega f \cdot v \ x_1 dx + \int_{\Gamma_2} g^2 \cdot v \ x_1 ds.$$

Remark 1. If u_1 belongs to $X^{1,2}_{1/2}(\Omega)$, then $u_1 = 0 \quad (x \in \Gamma_0)$.

3 A finite element approximation

Let ϕ_i and ψ_j be shape functions at the nodal point P_i and P_j $(\in \bar{\Omega})$ of velocity and pressure satisfying the inf-sup conditions in the Cartesian coordinates ([3], e.g. P2/P1 elements, P1+bubble /P1 elements). We assume that on each element $\{\phi_i\} \supset \mathcal{P}_k$ and $\{\psi_j\} \supset \mathcal{P}_{k-1}$, where $\mathcal{P}_k$ is the polynomial space of degree k (k=2 when P2/P1 elements and k=1 when P1+bubble/P1 elements).

Lemma 1.

$$\phi_i \in X^{1,2}_{1/2}(\Omega) \quad (P_i \notin \Gamma_0), \quad \phi_i \in W^{1,2}_{1/2}(\Omega) \quad (\forall i), \quad \psi_j \in L^2_{1/2}(\Omega) \quad (\forall j).$$

We define W_h by the linear combination of

$$(\phi_i, 0)^T \quad (P_i \notin \Gamma_0), \quad (0, \phi_i)^T \quad (\forall i), \tag{8}$$

Q_h by the linear combination of ψ_j $(\forall j)$, and $V_h(g^1)$ and V_h by

$$V_h(g^1) = \{v_h \in W_h;\ v_h(P_i) = g^1(P_i) \quad (P_i \in \Gamma_1)\}, \qquad V_h = V_h(0).$$

The mixed finite element formulation of (5) and (6) is: Find $(u_h, p_h) \in V_h(g^1) \times Q_h$ such that

$$a_1(u_h,u_h,v_h) + a(u_h,v_h) + b(v_h,p_h) = < F, v_h > \quad (\forall v_h \in V_h), \tag{9}$$

$$b(u_h,q_h) = 0 \qquad (\forall q_h \in Q_h). \tag{10}$$

The core of the analysis of the mixed finite element method is found in the corresponding Stokes problem: Find $(u_h, p_h) \in V_h(g^1) \times Q_h$ such that

$$a(u_h,v_h) + b(v_h,p_h) = < F, v_h > \quad (\forall v_h \in V_h), \tag{11}$$

$$b(u_h, q_h) = 0 \qquad (\forall q_h \in Q_h). \tag{12}$$

As in the Cartesian coordinates, we can show that the inf-sup condition is satisfied for the choice of basis functions ϕ_i and ψ_j.

Lemma 2. *There exists a positive constant β independent of h such that*

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{||q_h||_Q ||v_h||_V} \geq \beta.$$

After preparing an approximation theory in the weighted Sobolev spaces $X^{\ell,2}_{1/2}(\Omega)$ and $W^{\ell,2}_{1/2}(\Omega)$, we have the following error estimate for the Stokes problem.

Theorem 1.[4] (i) *There exists a unique solution (u_h, p_h) of (11) and (12).*
(ii) *Suppose that the exact solution (u, p) belongs to $(W^{k+1,2}_{1/2}(\Omega))^2 \times W^{k,2}_{1/2}(\Omega)$, $k \geq 1$. Then we have*

$$||u - u_h||_V + ||p - p_h||_Q \leq ch^k\{|u|_{(W^{k+1,2}_{1/2}(\Omega))^2} + |p|_{W^{k,2}_{1/2}(\Omega)}\},$$

where c is a positive constant independent of h and (u, p).

4 Drag coefficients of axisymmetric bodies

In this section we apply the finite element method for axisymmetric flow problems to the computation of drag coefficients of axisymmetric bodies.

Let G be an axisymmetric body in a velocity field. Let U be the representative velocity, ρ be the density of the fluid, and A be the area of the cross section of G to the direction U. The drag coefficient of G is defined by

$$C_D = -\frac{4\pi}{\rho U^2 A} \int_C \sum_{j=1}^{2} \sigma_{2j}(u, p) n_j x_1 ds, \tag{13}$$

where σ is the stress tensor defined by (4) and C is an intersection curve of the surface of G and the meridian $(x_1 > 0)$.

The following proposition is proved easily by using the Gauss-Green theorem.

Proposition 1. *Let (u_1, u_2, p) be any functions in $X^{2,2}_{1/2}(\Omega) \times W^{2,2}_{1/2}(\Omega) \times W^{1,2}_{1/2}(\Omega)$. Then for any function $v \in X^{1,2}_{1/2}(\Omega) \times W^{1,2}_{1/2}(\Omega)$ we have*

$$\begin{aligned} & a_1(u, u, v) + a(u, v) + b(v, p) \\ = & \int_\Omega \{(u \cdot \mathrm{grad})u + \frac{1}{Re} Lu + \mathrm{grad} p + \frac{1}{Re} \mathrm{grad}(\mathrm{div}_1 u)\} v x_1 dx + \int_{\partial\Omega} [\sigma(u, p)] n \ v x_1 ds. \end{aligned}$$

Suppose f is given in $(L^2_{1/2}(\Omega))^2$. Let (u, p) be a solution of (1) and (2) subject to a boundary condition. If (u, p) has the regularity assumed in Proposition 1, we have

$$\int_{\partial\Omega} [\sigma(u, p)] n \ v x_1 ds = a_1(u, u, v) + a(u, v) + b(v, p) - \langle f, v \rangle, \tag{14}$$

for any function $v \in X^{1,2}_{1/2}(\Omega) \times W^{1,2}_{1/2}(\Omega)$, where

$$\langle f, v\rangle = \int_\Omega f \cdot v \; x_1 dx.$$

Since (14) is meaningful for the function $(u_1, u_2, p) \in X^{1,2}_{1/2}(\Omega) \times W^{1,2}_{1/2}(\Omega) \times L^2_{1/2}(\Omega)$, (14) is valid for the (weak) solution (u, p) of (1) and (2). Let $v^* = (0, v_2^*)$ be a fixed function such that

$$v_2^* \in W^{1,2}_{1/2}(\Omega), \quad v_2^* = 1 \text{ on } C, \quad v_2^* x_1 \sum_{j=1}^{2} \sigma_{2j}(u,p) n_j = 0 \text{ on } \partial\Omega \backslash C. \tag{15}$$

From (13) and (14) we obtain

$$C_D = -\frac{4\pi}{\rho U^2 A}\{a_1(u,u,v^*) + a(u,v^*) + b(v^*,p) - \langle f, v^*\rangle\}. \tag{16}$$

We employ (16) for the computation of the drag coefficient.

Let $(u_{1h}, u_{2h}, p_h) \in X^{1,2}_{1/2}(\Omega) \times W^{1,2}_{1/2}(\Omega) \times L^2_{1/2}(\Omega)$ be a corresponding finite element solution. Let v_h^* be an interpolation of v^* in the finite element space. We define an approximate drag coefficient C_D^h by

$$C_D^h = -\frac{4\pi}{\rho U^2 A}\{a_1(u_h,u_h,v_h^*) + a(u_h,v_h^*) + b(v_h^*,p_h) - \langle f, v_h^*\rangle\}. \tag{17}$$

Remark 2. (i) The drag coefficient C_D^h applied to the problem in the Cartesian coordinates coincides with that obtained from the consistent flux method [5],[6],[7]. In those papers, however, the function v^* is not used. As will be shown soon the use of v^* enables us to derive an error estimate.
(ii) The direct boundary integration (13) for finite element solutions produces poor results. Similar facts are pointed out also in the literatures cited just above. In (13) the boundary values of grad$u$ and p appear. In general, the estimate at the boundary of a function is harder than that in the interior (we need more regularity). That is the reason why we employ (17) for the computation of drag coefficients, where only interior integrals appear and there are no boundary terms.
(iii) In the real computation of C_D^h we replace the function v_h^* by a more convenient function, which makes the computation simpler but does not change the value [8].

We have the following error estimates.

Proposition 2. *There exists a positive constant*

$$c_1 = c_1(||u||_V, ||u_h||_V, ||v^*||_V, ||f||_{(L^2_{1/2}(\Omega))^2}, \rho, U, A, Re)$$

such that

$$|C_D - C_D^h| \le c_1\{||u-u_h||_V + ||p-p_h||_Q + ||v^* - v_h^*||_V\}. \tag{18}$$

Theorem 2.[8] *Suppose that there exist positive constants c_2 and α independent of h such that*

$$||u - u_h||_V, \quad ||p - p_h||_Q, \quad ||v^* - v_h^*||_V \leq c_2 h^\alpha. \tag{19}$$

Then we have

$$|C_D - C_D^h| \leq c_3 h^\alpha, \tag{20}$$

where c_3 is a positive constant independent of h.

Remark 3. In the case of low Reynolds numbers the same error estimate as Theorem 1 can be obtained for (9) and (10). Therefore we get $\alpha = 2$ and $\alpha = 1$ in (20) for P2/P1 and P1+bubble/P1 elements, respectively.

5 Concluding remarks

We have presented a finite element analysis of axisymmetric flow problems and its application to the computation of drag coefficients. We can also apply to these problems a stabilized finite element method, which does not require the inf-sup condition [4], e.g., the choice of P1/P1 element is possible. For the precise computation of drag coefficients of the sphere we refer to [8].

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