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SINGULARITIES FOR PROJECTIONS OF CONTOUR LINES OF SURFACES ONTO PLANES

YASUHIRO KUROKAWA (黒川 康宏)

Department of Mathematics, Faculty of Science
Hokkaido University, Sapporo 060 JAPAN

ABSTRACT. We study the visions for contour lines of surfaces when one looks at it from a distant view in some direction. The study of such a landscape (i.e. so-called "topography") is reduced to the study of a certain divergent diagram of the smooth mappings $\mathbb{R} \leftarrow M \rightarrow \mathbb{R}^2$, where M is a smooth surface. We give a generic semi-local classification of such divergent diagrams.

1. FORMULATIONS AND RESULTS

In this paper we give a generic semi-local classification for the singularities of orthogonal projections of contour lines of surfaces onto planes.

Let M be a surface in $\mathbb{R}^3 = \{(x, y, z)\}$ and let E_d be a hyperplane in $\mathbb{R}^3$ with the normal direction d such that $E_d \cap M = \phi$. We denote by $Emb(M, \mathbb{R}^3)$ the space of all embeddings $M \rightarrow \mathbb{R}^3$ endowed with the Whitney C^∞-topology. Let $\pi_d : \mathbb{R}^3 \rightarrow E_d$ be a orthogonal projection along the direction d. Then consider a level set of the height function $z : i(M) \rightarrow \mathbb{R}$, that is $i(M) \cap \{z\text{=constant}\}$ for $i \in Emb(M, \mathbb{R}^3)$. We call the set *a contour line* on $i(M)$. If one looks at a contour line on $i(M)$ from a distant view in some direction d, then one will get $\pi_d(i(M) \cap \{z = c\})$ as the viewing image. We study such a landscape as one of the problems in the vision theory. That is, our subject is a semi-local classification of singularities for one parameter families $\{\pi_d(i(M) \cap \{z = c\})\}_{c \in \mathbb{R}}$ which is called *a topography* of $i(M)$ with respect to a direction d.

Let us formulate our theorems. Throughout this paper we shall suppose that all mappings, map germs and manifolds are of class C^∞ unless otherwise stated. Now,

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without loss of generality we can suppose that $d = (0, \cos\xi, \sin\xi) \in S^2 \cap \{x = 0\}$ where $0 < \xi \leqq \frac{\pi}{2}$. For a direction d, by the transformation of $\frac{\pi}{2} - \xi$ rotation around x-axis in $\mathbb{R}^3$, we choose a new coordinate (x', y', z'). Then the direction d becomes $(0,0,1)$ and the height function z is expressed by $-y' \sin(\frac{\pi}{2} - \xi) + z' \cos(\frac{\pi}{2} - \xi)$ in the new coordinate. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection defined by $\pi(u, v, w) = (u, v)$. We call the following divergent diagram of mappings *a topographic diagram* of $i(M)$, which is denoted by (μ_i, g) :

$$\mathbb{R} \xleftarrow{\mu_i} M \xrightarrow{g} \mathbb{R}^2,$$

where $\mu_i = -i_2 \sin\theta + i_3 \cos\theta \, (0 \leqq \theta < \frac{\pi}{2}), g = \pi \circ i$.

Since our concern is to describe the discriminant set of $\pi \circ i$ (the outline of $i(M)$) and the bifurcation of $\pi \circ i(\mu_i^{-1}(c))$ along the parameter $c \in \mathbb{R}$ in semi-local situation, we introduce the following definitions. Let $i \in Emb(M, \mathbb{R}^3)$ and let $\{p_1, \dots, p_r\}$ be a subset of M whose elements are all distinct points in M such that $\pi \circ i(p_1) = \cdots = \pi \circ i(p_r)$, where r is a positive integer. Then the multigerm of a topographic diagram at $\{p_1, \dots, p_r\}$ which is denoted by ${}_rT_i$

$$\begin{tikzcd}
(\mathbb{R},0) & (\mathbb{R}^2,0) \arrow[l, "\mu_1"'] \arrow[r, "g_1"] & (\mathbb{R}^2,0) \\
(\mathbb{R},0) & (\mathbb{R}^2,0) \arrow[l, "\mu_2"'] \arrow[ur, "g_2"] & \\
\vdots & \vdots & \\
(\mathbb{R},0) & (\mathbb{R}^2,0) \arrow[l, "\mu_r"'] \arrow[uuur, "g_r"'] &
\end{tikzcd},$$

where μ_k, g_k are germs of $\mu_i, \pi \circ i$ at p_k respectively $(k = 1, \dots, r)$, is called *a topographic multigerm of* i. Let ${}_rT_i$ and ${}_rT_{i'}$ be topographic multigerms. Then ${}_rT_i$ and ${}_rT_{i'}$ are said to be *equivalent* if there exist diffeomorphism germs $\lambda_k : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, $\psi_k : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, where $k = 1, \dots, r$, and $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\lambda_k \circ \mu_k = \mu'_k \circ \psi_k, \phi \circ g_k = g'_k \circ \psi_k$.

We shall state a genericity theorem for topographic multigerms.

Theorem A. *There exists a residual subset (hence dense)* ${}_rO$ *in* $Emb(M, \mathbb{R}^3)$ *such that for any* $i \in {}_rO$ *the topographic multigerms* ${}_rT_i \, (1 \leq r \leq 3)$ *is one of the following types:*

In the case $r = 1$.

(I) μ_1 *is a submersion and* g_1 *is regular.*

(II) μ_1 *is a Morse type and* g_1 *is regular.*

(III) μ_1 *is a submersion,* g_1 *is a fold,* μ_1 *restricted to the singular set of* g_1 *is regular and* $(\mu_1, g_1) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is regular.*

(IV) μ_1 *is a submersion,* g_1 *is a fold,* μ_1 *restricted to the singular set of* g_1 *is a Morse type and* $(\mu_1, g_1) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is regular.*

(V) μ_1 *is a submersion,* g_1 *is a fold,* $(\mu_1, g_1) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is a Whitney's umbrella such that the line of double points of which is transversal at* 0 *to the direction* $\{0\} \times \mathbb{R}^2$ *in* $\mathbb{R}^3$.

(VI) μ_1 *is a submersion,* g_1 *is a cusp and* $(\mu_1, g_1) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is regular.*

In the case $r = 2$.

$(I, I)_0$:

$$(\mu_1, g_1), (\mu_2, g_2) \text{ are both of type (I)}$$
$$\text{and } (\mu_1 \circ g_1{}^{-1}, \mu_2 \circ g_2{}^{-1}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \text{ is regular.}$$

$(I, I)_1$:

$$(\mu_1, g_1), (\mu_2, g_2) \text{ are both of type (I)}$$
$$\text{and } (\mu_1 \circ g_1{}^{-1}, \mu_2 \circ g_2{}^{-1}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \text{ is a fold.}$$

$(I, I)_2$:

$$(\mu_1, g_1), (\mu_2, g_2) \text{ are both of type (I)}$$
$$\text{and } (\mu_1 \circ g_1{}^{-1}, \mu_2 \circ g_2{}^{-1}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \text{ is a cusp.}$$

(II, I) :

$$(\mu_1, g_1) \text{ is of type (II), } (\mu_2, g_2) \text{ is of type (I)}$$
$$\text{and } (\mu_1 \circ g_1{}^{-1}, \mu_2 \circ g_2{}^{-1}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \text{ is a fold.}$$

$(III, I)^0$:

$$(\mu_1, g_1) \text{ is of type (III), } (\mu_2, g_2) \text{ is of type (I)}$$
$$\text{and the discriminant set of } g_1 \text{ and } g_2(\mu_2^{-1}(0)) \text{ are transversal.}$$

$(III, I)^1$:

$$(\mu_1, g_1) \text{ is of type (III), } (\mu_2, g_2) \text{ is of type (I)}$$
$$\text{and the discriminant set of } g_1 \text{ and } g_2(\mu_2^{-1}(0)) \text{ have two point contact.}$$

(IV, I) :

(μ_1, g_1) is of type (IV), (μ_2, g_2) is of type (I)

and the discriminant set of g_1 and $g_2(\mu_2^{-1}(0))$ are transversal.

(V, I) :

(μ_1, g_1) is of type (V), (μ_2, g_2) is of type (I),

the discriminant set of g_1 and $g_2(\mu_2^{-1}(0))$ are transversal.

(VI, I) :

(μ_1, g_1) is of type (VI), (μ_2, g_2) is of type (I)

and the tangent cone of the discriminant set of g_1 and $g_2(\mu_2^{-1}(0))$ are transversal.

(III, III) :

$(\mu_1, g_1), (\mu_2, g_2)$ are both of type (III)

and the discriminant sets of g_1, g_2 are transversal.

In the case $r = 3$.

(I, I, I)$_{1,1}$: $(\mu_j, g_j; \mu_k, g_k)$ is of type (I, I)$_1$ for $1 \leqq j < k \leqq 3$.

(III, I, I)$_1^{0,0}$: $(\mu_1, g_1; \mu_2, g_2)$ is of type (III, I)^0 and $(\mu_2, g_2; \mu_3, g_3)$ is of type (I, I)$_1$.

(III, III, I)^{0,0} : $(\mu_1, g_1; \mu_2, g_2)$ is of type (III, III) and $(\mu_j, g_j; \mu_3, g_3)$ is of type (III, I)^0 for $j = 1, 2$.

Besides the following nine types:

(I, I, I)$_{0,0}$, (II, I, I)$_0^{0,0}$, (III, I, I)$_0^{0,0}$, (IV, I, I)$_0^{0,0}$, (V, I, I)$_0^{0,0}$, (VI, I, I)$_0^{0,0}$: $(\mu_1, g_1; \mu_k, g_k)$ *is of type* (I, I)$_0$, (ν, I), ν=II,...,VI *for* $k = 2, 3$ *and* $(\mu_2, g_2; \mu_3, g_3)$ *is of type* (I, I)$_0$.

(I, I, I)$_{1,0}$, (I, I, I)$_{2,0}$, (III, I, I)$_0^{1,0}$: $(\mu_1, g_1; \mu_2, g_2)$ *is of type* (I, I)$_1$, (I, I)$_2$, (III, I)^1 *and* $(\mu_2, g_2; \mu_3, g_3)$ *is of type* (I, I)$_0$.

Remark 1.1. In the case $r = 2$, the generic condition of (I, I)$_1$ (resp. (I, I)$_2$) means that $g_1({\mu_1}^{-1}(0))$ and $g_2({\mu_2}^{-1}(0))$ have second (resp. third) order contact.

Remark 1.2. In the case $r \geqq 4$ all of generic types are essentially same as the case $r = 3$. That is, we can add only type (I) to the each type in the case $r = 3$ such that $(\mu_j, g_j; \mu_r, g_r)$ for $j = 1, \ldots, r-1$ is not of type (*, I)$_1$, (*, I)$_2$ in the list of the case $r = 2$. In the same sense for the case $r = 3$, the generic types except three types (I, I, I)$_{1,1}$, (III, I, I)$_1^{0,0}$, (III, III, I)^{0,0} are essentially same as in the case $r = 2$.

Next we shall give a normal form for each type stated as above. Denote by $\mathcal{E}_{x_1,\dots,x_n}$ the ring of all smooth function germs on $\mathbb{R}^n$ at 0 with a coordinate $(x_1, \dots, x_n)$ and denote by $\mathcal{M}_{x_1,\dots,x_n}$ the unique maximal ideal of $\mathcal{E}_{x_1,\dots,x_n}$.

Theorem B. *The topographic multigerms of each type are equivalent to one of the following multigerms* ${}_rT_i = (\mu_1, g_1; \cdots; \mu_r, g_r)$*:*

In the case $r = 1$.

(I)

$$\mu_1 = y_1, \quad g_1 = (x_1, y_1).$$

(II)

$$\mu_1 = x_1^2 \pm y_1^2, \quad g_1 = (x_1, y_1).$$

(III)

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, y_1^2).$$

(IV)

$$\mu_1 = x_1^2 + y_1, \quad g_1 = (x_1, y_1^2).$$

(V)

$$\mu_1 = x_1 + x_1 y_1 + y_1^3, \quad g_1 = (x_1, y_1^2).$$

(VI)

$$\mu_1 = y_1 + \alpha \circ g_1, \quad g_1 = (x_1, y_1^3 + x_1 y_1),$$

where $\alpha \in \mathcal{M}_{u,v}$.

In the case $r = 2$.

$(I, I)_0$

$$\mu_1 = y_1, \quad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2, \quad g_2 = (x_2, y_2).$$

$(I, I)_1$

$$\mu_1 = y_1, \quad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2^2 + y_2, \quad g_2 = (x_2, y_2).$$

$(I, I)_2$

$$\mu_1 = y_1, \quad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2^3 + x_2 y_2 + y_2, \quad g_2 = (x_2, y_2).$$

(II, I)

$$\mu_1 = x_1^2 \pm y_1^2, \quad g_1 = (x_1, y_1);$$

$$\mu_2 = x_2, \quad g_2 = (x_2, y_2).$$

$(III, I)^0$

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

$$\textit{where } \theta \in \mathcal{M}_{x_2, y_2} \textit{ with } \frac{\partial \theta}{\partial x_2}(0) = 0.$$

$(III, I)^1$

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = y_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

$$\textit{where } \theta \in \mathcal{M}^2_{x_2, y_2} \textit{ with } \frac{\partial^2 \theta}{\partial x_2^2}(0) \neq 0.$$

(IV, I)

$$\mu_1 = x_1^2 + y_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

$$\textit{where } \theta \in \mathcal{M}_{x_2, y_2} \textit{ with } \frac{\partial \theta}{\partial x_2}(0) = 0.$$

(V, I)

$$\mu_1 = x_1 + x_1 y_1 + y_1^3, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

$$\textit{where } \theta \in \mathcal{M}_{x_2, y_2} \textit{ with } \frac{\partial \theta}{\partial x_2}(0) = 0.$$

(VI, I)

$$\mu_1 = y_1 + \alpha \circ g_1, \quad g_1 = (x_1, y_1^3 + x_1 y_1);$$

$$\mu_2 = x_2 + \theta(x_2, y_2), \quad g_2 = (x_2, y_2),$$

$$\textit{where } \alpha \in \mathcal{M}_{u,v}, \theta \in \mathcal{M}_{x_2, y_2} \textit{ with } \frac{\partial \theta}{\partial x_2}(0) = 0.$$

(III, III)

$$\mu_1 = y_1 + \alpha_1 \circ g_1, \quad g_1 = (x_1, y_1^2);$$

$$\mu_2 = x_2 + \alpha_2 \circ g_2, \quad g_2 = (x_2^2, y_2),$$

where $\alpha_1, \alpha_2 \in \mathcal{M}_{u,v}$ *with* $\frac{\partial \alpha_1}{\partial u}(0) \neq 0, \frac{\partial \alpha_2}{\partial v}(0) \neq 0$.

Remark 1.3. The normal forms of type $(\mathrm{I, I, I})_{1,1}$, $(\mathrm{III, I, I})_1^{0,0}$, $(\mathrm{III, III, I})^{0,0}$ are the following:

$(\mathrm{I, I, I})_{1,1}$: $\mu_1, g_1; \mu_2, g_2$ have the same form as the normal form of type $(\mathrm{I, I})_1$,

$$\mu_3 = y_3 + a{x_3}^2 + \theta(x_3, y_3), \quad g_3 = (x_3, y_3)$$

where $a \in \mathbb{R} - \{0, 1\}$, $\theta \in \mathcal{M}^2_{x_3,y_3}$ with $\frac{\partial^2 \theta}{\partial {x_3}^2}(0) = 0$.

$(\mathrm{III, I, I})_1^{0,0}$:

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, {y_1}^2);$$

$$\mu_2 = x_2 + \alpha(x_2, y_2), \quad g_2 = (x_2, y_2);$$

$$\mu_3 = x_3 + \beta(x_3, y_3), \quad g_3 = (x_3, y_3),$$

where $\alpha \in \mathcal{M}_{x_2,y_2}, \beta \in \mathcal{M}_{x_3,y_3}$ with $\frac{\partial \alpha}{\partial x_2}(0) = \frac{\partial \beta}{\partial x_3}(0) = 0$, $\frac{\partial \alpha}{\partial y_2}(0) = \frac{\partial \beta}{\partial y_3}(0)$ and $\frac{\partial^2 \alpha}{\partial {y_2}^2}(0) - \frac{\partial \alpha}{\partial y_2}(0)\frac{\partial^2 \alpha}{\partial x_2 \partial y_2}(0) \neq \frac{\partial^2 \beta}{\partial {y_3}^2}(0) - \frac{\partial \beta}{\partial y_3}(0)\frac{\partial^2 \beta}{\partial x_3 \partial y_3}(0)$.

$(\mathrm{III, III, I})^{0,0}$: $\mu_1, g_1; \mu_2, g_2$ have the same form as the normal form of type (III, III),

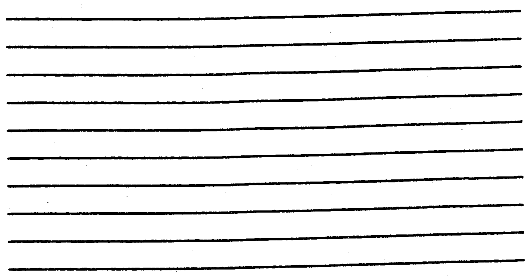
$$\mu_3 = x_3 + y_3 + \theta(x_3, y_3), \quad g_3 = (x_3, y_3) \text{ where } \theta \in \mathcal{M}^2_{x_3,y_3}.$$

Remark 1.4. The divergent diagram $(\mathbb{R}, 0) \leftarrow (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ have been studied by Arnol'd [1], Carneiro [5], Dufour [10] from the viewpoint of envelope, stability theory. Also, the normal forms of each type stated in Theorem A for $r = 1$ has been obtained by Arnol'd [1], Dufour [10].(The classification of Arnol'd is not C^∞ case but formal case.) For $r = 2$, essentially types $(I, I)_k$ $(k = 0, 1, 2)$, (II, I) have been studied by Dufour [7, 8, 9] from the viewpoint of "bi-stability" and its normal forms have been obtained. On the other hand, singularities for certain visual images have been studied by several authors [4, 6, 13, 16, 17]. In particular, Dufour and Tueno have investigated in [13] the pattern of illuminance due to a point source of light which coincide with our Theorem A in the case $r = 1$.

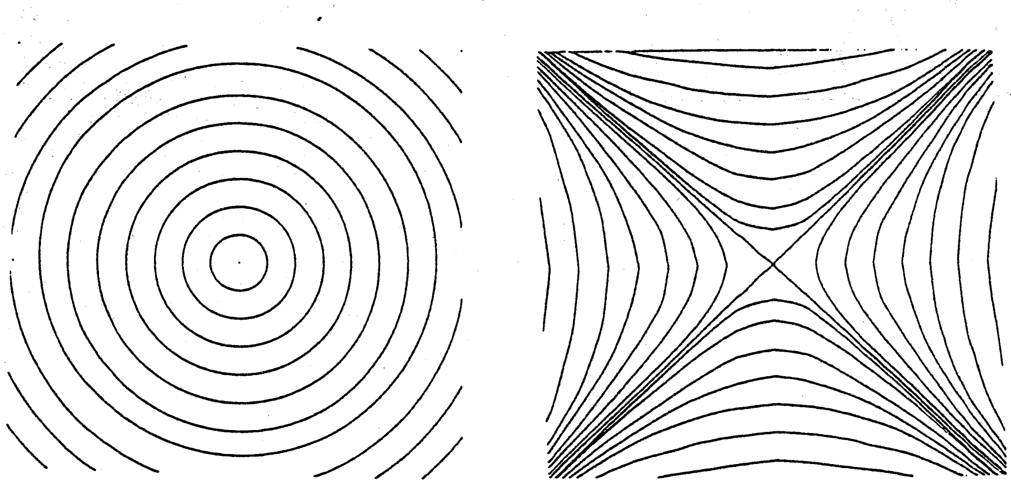
2. Geometric description of the normal forms

In order to understand our classification of topographies geometrically, let us describe the level curves $\{g_k({\mu_k}^{-1}(c))\}$ and the discriminant set of $g_k, 1 \leqq k \leqq r\,(r = 1, 2)$ for each type.

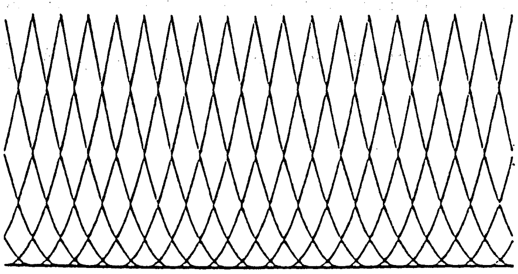
(I)



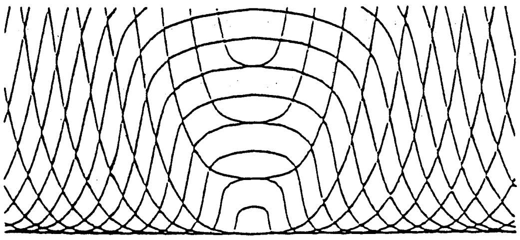
(II)



(III)



(IV)



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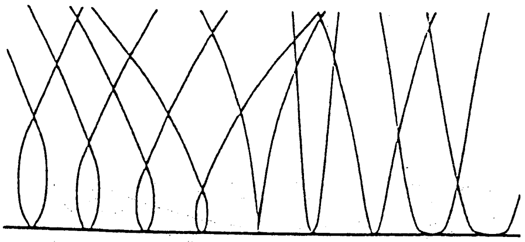
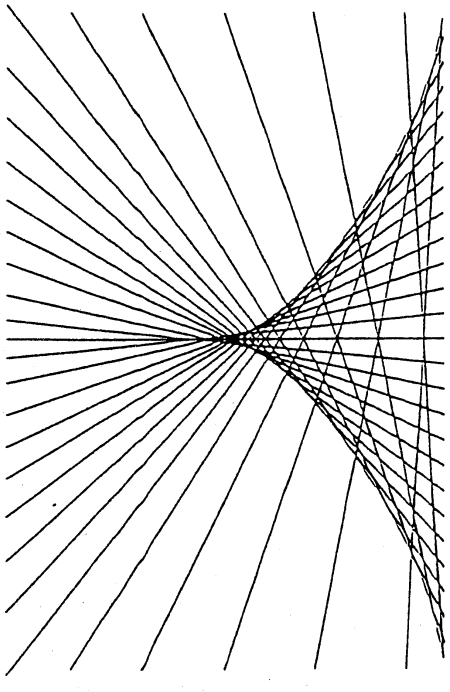
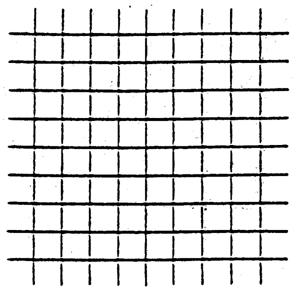


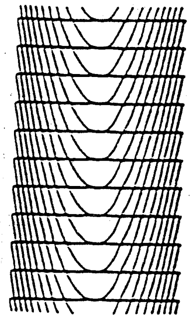
(VI)



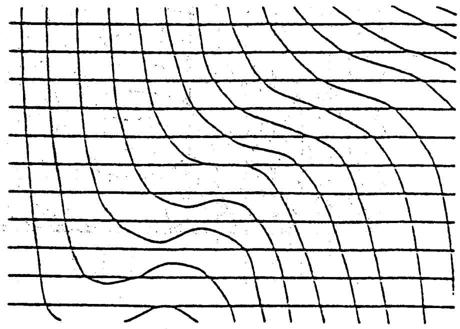
$(\mathrm{I.\ I})_0$



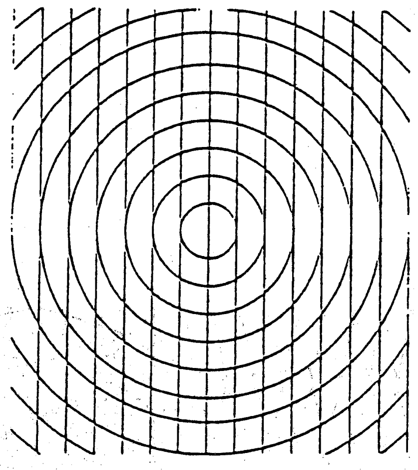
$(\mathrm{I.\ I})_1$

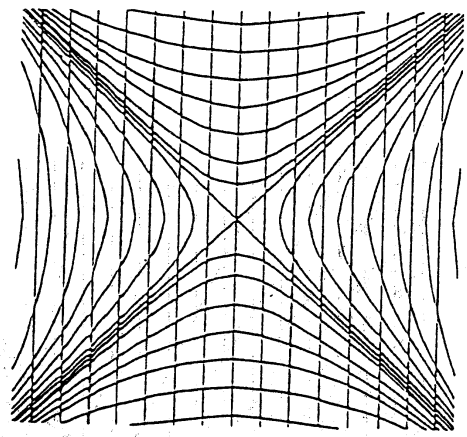


$(\mathrm{I.\ I})_2$

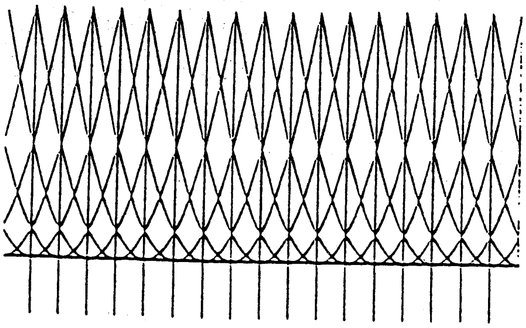


$(\mathrm{II,\ I})$

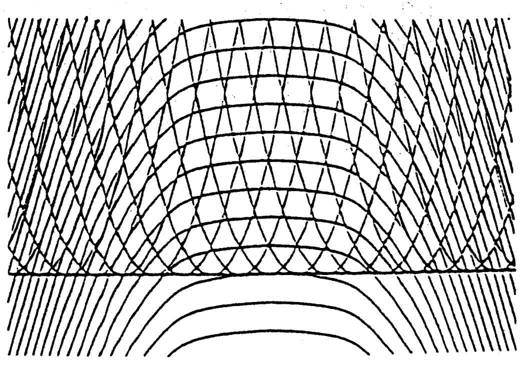




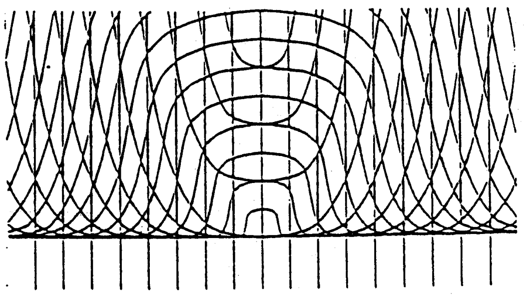
$(\mathrm{III.\ I})^0$



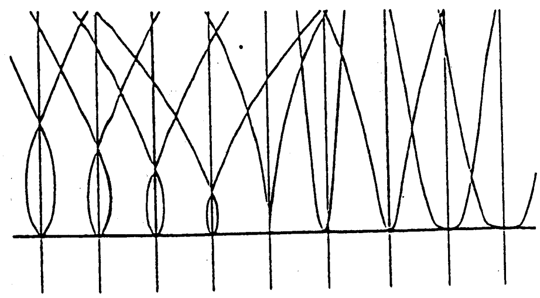
$(\mathrm{III.\ I})^1$



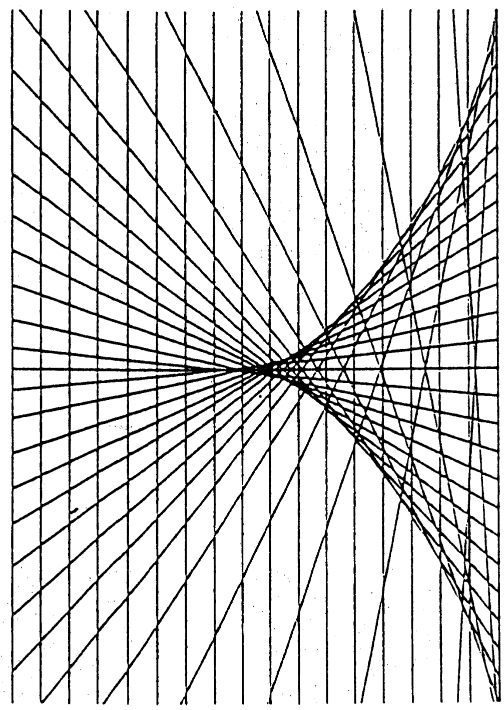
(IV. I)



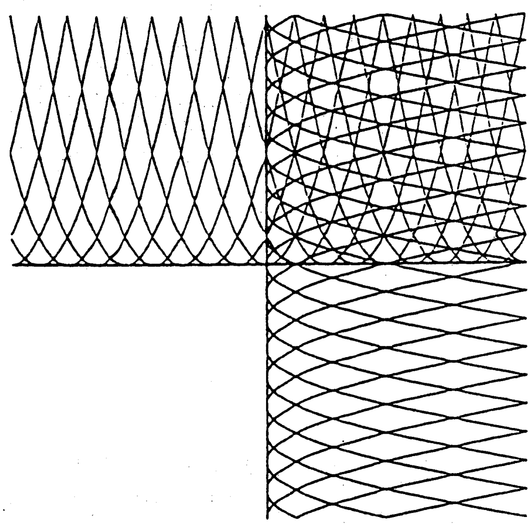
(V. I)



(VI, I)



(III. III)



Remark 2.1. The normal forms in Theorem B depend on arbitrary functions with some conditions, that is so-called "functional moduli" appear in the normal forms. For the type (VI) the uniqueness of the functional moduli have been studied and the complete invariant has been detected ([11], [15]). For other types which appear functional moduli, however we can not obtain the uniqueness result of the functional moduli in this paper. We remark that the topographies have "d-web"(a configuration of d foliations) structure. It is known that the only one functional moduli appear in the local normal form of 3-web which consists of 3 curvilinear foliations in $\mathbb{R}^2$ ([12]). So we observe that it is natural the only one functional moduli appear in our normal forms of the types which have 3-web structures. Also we remark that two functional moduli which appear in our normal form of types (III, III), (VI, I) deeply connect with the 4-web structure which the topographies of type (III, III), (VI, I) have. We can not obtain the result that whether the two functional moduli are reduced to only one or not in this paper.

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