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Topology of plane trigonometric curves and a duality for strangeness of plane curves derived from real pseudo-line arrangements

by Goo ISHIKAWA

石川剛郎 (いしかわ ごうお): 北海道大学大学院理学研究科

1 Introduction

Consider a parametric plane curve $f : S^1 \longrightarrow \mathbf{R}^2$ which is **generic**, namely an immersion with just transverse self-intersections. Since f is **topologically stable** (= stable topologically under small perturbations), the curve f has the same isotopy type with a curve defined by some leading terms of the Fourier expansion of f. Therefore any isotopy type of generic plane curves can be realized by a trigonometric plane curve. Then, for the classification problem of generic plane curves, it is natural to classify the isotopy types realized by trigonometric curves of a given degree from $1, 2, 3, \ldots$ and so on.

Let $f : S^1 \to \mathbf{R}^2 = \mathbf{C}$ be a plane curve defined by a trigonometric polynomial map $f(z) = \sum_{i=-n}^{n} a_i z^i$ of degree n, where $z = \cos(\theta) + \sqrt{-1}\sin(\theta) \in S^1 \subset \mathbf{C}$, and $a_i \in \mathbf{C}, -n \leq i \leq n$. We simply call f a **Fourier curve of degree** $\leq n$. Non-generic Fourier curves of degree $\leq n$ form a semi-algebraic set Σ in $\mathbf{C}^{2n+1}$. Then the problem is to classify generic Fourier curves $f \in \mathbf{C}^{2n+1} - \Sigma$ up to isotopy.

The general classification theory of parametric plane curves goes back to Gauss (Gauss words or Gauss diagrams [12][6][11]). Whitney [28] gave essential results on the regular homotopy (= deformation through immersions) invariant, Gauss index or Whitney index (= the mapping degree of the Gauss map), of immersed plane curves. Scott Carter [6] showed that the Gauss diagram is the complete invariant for the plane curves as spherical curves. Arnold [2][3] gave three kinds of basic isotopy invariants $J^{\pm}$ and **strangeness** St of generic plane curves. These are first order invariants of Vassiliev type [22]. Also there exist many works about plane curves related to the knot theory.

First remark that for a Fourier curve f of degree n there naturally corresponds a **real rational curve** $f : \mathbf{RP}^1 \longrightarrow \mathbf{RP}^2$ of degree $2n$, if we set $\cos(\theta) = (1-t^2)/(1+t^2)$, $\sin(\theta) = 2t/(1+t^2)$. This reflects the fact that $S^1 = \{x^2 + y^2 = 1\} \subset \mathbf{R}^2 \subset \mathbf{RP}^2$ is rational of degree 2.

Viro [26] studies the classification problem of real rational curves from Vassiliev's point of view. On the other hand this problem has analogous feature to **the 16th problem of Hilbert** [24], especially to the first half of it.

The first half of the 16th problem of Hilbert [24] treats the classification problem of non-singular real algebraic curves of fixed degree m up to isotopy in the real projective plane $\mathbf{RP}^2$. This problem is solved for $m \leq 7$ up to present. (For the case $m \leq 5$ the classification is classically known; Gudkov classified for $m = 6$ and Viro for $m = 7$. Harnack showed that the number ℓ of connected components of a real algebraic curve of degree m is at most $(1/2)(m-1)(m-2)+1$. A non-singular curve of degree m is called (after Petrovskii) an M-curve if it has the maximal number of connected components. For the first non-trivial case, namely $m = 6$, the classification result shows clearly certain symmetry or duality. For instance, there exist three isotopy types of M-curves of degree 6 as in Fig.2. However, as far as I know, the full explanation of this duality is not given yet.

We observe that, in the topological classification problem, the strategy then is first to find restrictions or estimates of topological invariants and second to find realizations or constructions of given topological types.

We are naturally led to the topological classification problem of real rational curves and Fourier curves.

For the basic topological restrictions on the real rational curves, we have easily the following:

Lemma 1.1 *Let f be a generic (= with just transverse self-intersections) real rational curve of degree m. Then*

(1) *The number of self-intersection points $\ell \leq (1/2)(m-1)(m-2)$ (by, for instance, genus formula).*

(2) *The number of intersection points with a generic line is at most m and it is congruent to m modulo 2 (by the theorem of Bezóut).*

Example 1.2 : Consider generic real rational curves of degree 4. Then $\ell \leq 3$. In fact the list of isotopy types are given in Fig.1.

Now we turn to the classification problem of trigonometric curves. Remark that when we seek topological restrictions on generic plane curves, we may assume more generic conditions upon the curves because of its topological stability of the real locus.

Lemma 1.3 *Let $f : S^1 \longrightarrow \mathbf{C}$ be a generic Fourier curve of degree n. Then, for f,*

(1) *The number of self-intersection points $\ell \leq (2n-1)(n-1)$.*

(2) $|$index of Gauss map$| := i \leq n$, $i \equiv \ell + 1$ (mod. 2).

(1) is clear. For (2), we use the formula $df/d\theta = \sum_{i=-n}^{n} \sqrt{-1} i a_i z^i$. (The differential defines on $\mathbf{C}^{2n+1}$ just a diagonal linear action with eigenvalues $-\sqrt{-1}n, \ldots, \sqrt{-1}n$.)

A generic Fourier curve of degree n is called M-curve if $\ell = (2n-1)(n-1)$. Similarly this definition is applied also to generic real rational curves.

Then the first result of this note is the following:

Proposition 1.4 *Let f be a generic Fourier curve of degree n. If f is an M-curve, then f has no inflection points.*

This gives a severe topological restriction on the isotopy types of M-curves.

Example 1.5 : There does not exist a Fourier curve of degree 2 with the topological type as in Fig.3. In fact assume it does exist. Then it is an M-curve, so it does not have inflection points. However, on the other hand, we see $i = 0$, so the number of inflection points is at least 2. This leads a contradiction.

The list of isotopy types of generic Fourier curves of degree 2 is the same with the remaining types appeared in the list of isotopy types of generic real rational curves of degree 4.

Next let us turn to classify Fourier M-curves of degree 3. We observe the following result:

Proposition 1.6 *There exists a set of* 14 *isotopy types I of generic Fourier M-curves of degree* 3 *and an involution $\sigma : I \longrightarrow I$ such that* $\mathrm{St}(i) + \mathrm{St}(\sigma(i)) = 2$.

These 14 isotopy types are shown in Fig.4 with the adjacency and the values of Arnold's strangeness St. Notice that this result is obtained by the experiments using **ParametricPlot** of Mathematica. I wish to find an alternative theoretic proof of Proposition 1.6. I conjecture that the isotopy types of generic Fourier M-curves of degree 3 are exhausted by these 14 types. To confirm it, we need to exclude the isotopy type shown in Fig.5. But I have no proof about it yet.

We observe that these 14 isotopy types can obtained as perturbations of the bouquet or blossom $b_5 : f = z^3 - z^{-2}$ with five stalks or petals (Fig. 6). (We may call it "sakura", which means the cherry blossom.)

In general we set $b_{2n-1} : f = z^n - z^{-(n-1)}$, a non-generic Fourier curve of degree n (the bouquet with $(2n-1)$-stalks or the blossom with $(2n-1)$-petals). This curve provides the unique topological type of real algebraic curves in $\mathbf{RP}^2$ of degree $2n$ with one real ordinary $(2n-1)$-multiple point. (This remark is found during a conversation with T. Fukui.) We get necessarily an M-Fourier curve of degree n if we perturb b_{2n-1} into an generic Fourier curve of degree n. The existence of the duality, in the case $n = 3$, is clear from the Fig.4. However we observe that this duality comes from more general connection with the topology of Fourier curves and the topology of (pseudo-)line arrangements [13] on the real plane $\mathbf{R}^2$.

A **pseudo-line arrangement** $\mathcal{A}$ of s strings in $\mathbf{R}^2$ is a proper (say C^∞) embedding $F : \coprod_s \mathbf{R} \to \mathbf{R}^2$ with the following properties: If we denote by F_i the restriction of F to the i-th $\mathbf{R}$ and by $L_1, \ldots, L_s$ the images $F_1(\mathbf{R}), \ldots, F_s(\mathbf{R})$, then L_i and L_j intersect at only one point in $\mathbf{R}^2$. A point $p \in \mathbf{R}^2$ is called an m-**multiple point** (with respect to F) if $F^{-1}(p)$ consists of m-points in $\coprod_s \mathbf{R}$. A pseudo-line arrangement is called **simple** if there does not exist m-multiple points with $m \geq 3$.

Two mappings $F : M \to N$ and $G : M' \to N'$ between topological spaces are called **homeomorphic** if there exist homeomorphisms $\sigma : M \to M'$ and $\tau : N \to N'$ such that $G \circ \sigma = \tau \circ F$.

F and G are called **isotopic** if there exist a homeomorphism $\sigma : M \to M'$ and a one-parameter family of homeomorphisms $\tau_\lambda : N \to N'$ ($\lambda \in [0,1]$) such that $\tau_0 = 1_N$ and that $G \circ \sigma = \tau_1 \circ F$.

F and G are called **strictly isotropic** if there exist one-parameter families of homeomorphisms $\sigma_\lambda : M \to M'$ and $\tau_\lambda : N \to N'$ ($\lambda \in [0,1]$) such that $\sigma_0 = 1_M, \tau_0 = 1_N$ and that $G \circ \sigma_1 = \tau_1 \circ F$.

Any pseudo-line arrangement F of s strings is strictly isotopic to an arrangement F' with the following property: there exists a permutation ρ of $\{1, \ldots, s\}$ such that each string $F_{\rho(i)}$ coincides with the straight line $\ell_i : \mathbf{R} \to \mathbf{R}^2 = \mathbf{C}$ defined by $\ell_i(t) = te^{\sqrt{-1}(i-1)\pi/s}$ outside of a compact subset of $\mathbf{R}$.

We call an arrangement F' **normarised** if it has the above property.

Any pseudo-line arrangement F of odd $(2n-1)$ strings is isotopic to an arrangement F' with the following properties: $F_1, \ldots, F_s$ coincide with the straight lines

$$\ell_1, -\ell_2, \ell_3, \ldots, -\ell_{2n-2}, \ell_{2n-1}.$$

(In fact we only need a permutation of the components of $\coprod_s \mathbf{R}$ and reversing some of $\mathbf{R}$'s for a normalised representative.) Fig. 6.

We call a pseudo-line arrangement F' of odd strings **admissible** if it has the above property.

We are going to construct two (isotopy types of) closed curves $S^1 \to \mathbf{R}^2$ from an admissible pseudo-line arrangement F of $(2n-1)$ strings as follows: Take a circle C_r on $\mathbf{R}^2$ containing all multiple points of F in the inside and intersecting each string just twice. First start at the point $F_1(0)$ and move along F_1 until we hit C_r. Then move along the arc counterclockwise till we hit $L_2 = F_2(\mathbf{R})$. Change to L_2 and draw along F_2 until we hit C_r. Then draw along the arc counterclockwise. Continuing this process, we get a (piecewise C^∞) closed curve $S^1 \to \mathbf{R}^2$. By smoothing this, we have an immersion $c(F) : S^1 \to \mathbf{R}^2$ of Gauss index n. If we move clockwise instead of counterclockwise and in order $F_1, F_s, F_{s-1}, \ldots$, then we get another plane curve $c'(F) : S^1 \to \mathbf{R}^2$ of Gauss index $-n$. We call $c(F)$ and $c'(F)$ **closures** of F. (Fig.8). If F is simple, then both $c(F)$ and $c'(F)$ are generic and have $(2n-1)(n-1)$ double points.

If F and F' are admissible pseudo-line arrangements and they are isotopic, then the set of isotopy classes $\{c(F), c'(F)\}$ coincides with that of $\{c(F'), c'(F')\}$. Thus, to an isotopy class $\mathcal{A}$ of pseudo-line arrangement of odd strings, there corresponds a set $\{c(\mathcal{A}), c'(\mathcal{A})\}$ of isotopy classes of closed curves $S^1 \to \mathbf{R}^2$ as $\{c(F), c'(F')\}$ for a admissible representative F

of $\mathcal{A}$. We call also $\{c(\mathcal{A}), c'(\mathcal{A})\}$ the **closures** of $\mathcal{A}$. Remark that there is no preference to choose one of the closures $\{c(\mathcal{A}), c'(\mathcal{A})\}$ of an isotopy class $\mathcal{A}$.

Theorem 1.7 *Let $\mathcal{A}, \mathcal{A}'$ be isotopy types of simple $(2n-1)$ pseudo-line arrangements and $\{c(\mathcal{A}), c'(\mathcal{A})\}, \{c(\mathcal{A}'), c'(\mathcal{A}')\}$ closures of $\mathcal{A}, \mathcal{A}'$ respectively. Then we have*

(1) *If $c(\mathcal{A})$ or $c'(\mathcal{A})$is isotopic (resp. homeomorphic) to $c(\mathcal{A}')$ or $c'(\mathcal{A}')$, then $\mathcal{A}$ and $\mathcal{A}'$ are isotopic (resp. homeomorphic).*

(2) $\mathrm{St}(c(\mathcal{A})) + \mathrm{St}(c'(\mathcal{A})) = n-1$.

(3) $J^+(c(\mathcal{A})) = J^+(c'(\mathcal{A})) = (n-1)(n-2)$.

(4) $J^-(c(\mathcal{A})) = J^-(c'(\mathcal{A})) = -(n-1)(n+1)$.

From (the proof of) Theorem 1.7, we have the following:

Proposition 1.8 *If n is even, then $c(\mathcal{A})$ and $c'(\mathcal{A})$ are not isotopic (even after the reversing the parameter S^1). In general, if $c(\mathcal{A})$ and $c'(\mathcal{A})$ are isotopic, then, for any addmisible representative F of $\mathcal{A}$ and the admissible arrangement G obtained from F by reversing all orientations of $\coprod_{2n-1} \mathbf{R}$ and rotating by π/s, F and G are strictly isotopic.*

The **line arrangements** are of course important examples of pseudo-line arrangements. In particular, by Theorem 1.7, there exists a duality on the set of isotopy types of plane curves $S^1 \to \mathbf{R}^2$ obtained as a closure from a line arrangement. Remark that this set of isotopy types is contained in the set of isotopy types of curves $S^1 \to \mathbf{R}^2$ without inflection points.

A pseudo-line arrangement is called **stretchable** if it is isotopic to a line arrangement [13]. The pseudo-line arrangement shown in bottom left of Fig.8 is an example of non-stretchable simple pseudo-line arrangement of 9 strings due to Ringel and Grünbaum. As stated in [13], R.J. Canham, E. Halsey showed in 1971 that all simple pseudo-line arrangements of s strings $(s \le 7)$ are stretchable. Grünbaum [13] conjectured that all simple pseudo-line arrangements of s strings $(s \le 8)$ are stretchable.

The numbers n_s of homeomorphism types of (pseudo-)line arrangements of s-strings $(s \le 7)$ are given, for instance in [19]: $n_s = 1$, for $s \le 4$, $n_5 = 6, n_6 = 43, n_7 = 922$.

The number of isotopy types of line arrangements of 5 lines is equal to 7 (Fig.9). The adjacency of these isotopy types is represented by the graph of Fig.10. By Theorem 1.7, we have 14 isotopy types of plane curves with $\ell = 10, i = 3$. and a duality among these isotopy types, which coincides with the duality found in Proposition 1.6. The graph of adjacency (Fig.11) is the double covering of the graph of adjacency for line arrangements of 5 lines.

The Gauss diagrams of the curves No.1 and No.14 are drawn in Fig.12. These are connected by successive bifurcations of death-barth triangles as shown in Fig.13.

A pseudo-line arrangement $F : \coprod_s \mathbf{R} \to \mathbf{R}^2$ is **convex** (resp. **concave**) if there exists a closed disk $D_r \subset \mathbf{R}^2$ containing all multiple points of F such that on $F^{-1}(D_r)$ the determinant $|F'(t)F''(t)| > 0$ (resp. < 0.)

If F is convex, then the closure $a(F) : S^1 \to \mathbf{R}^2$ is a convex curve. If F is concave, then $a'(F)$ is concave. The isotopy type of a simple line arrangement are realised by a simple convex (resp. concave) pseudo-line arrangement. This fact causes the duality on a class of isotopy types of plane curves without inflection points.

Miscellaneous Remarks and Questions:

(1) We remark that the example (Fig.8) of simple non-stretchable pseudo-line arrangement is realised by a convex pseudo-line arrangement: Fig.14. I do not know any example for isotopy types of simple pseudo-line arrangements which can not be realised by convex (or concave) pseudo-line arrangements.

Related to the existence of non-stretchable pseudo-line arrangements we observe the following result from the singularity theory:

Proposition 1.9 *The deformation $\tilde{F} : \{\coprod_9(\mathbf{R}, 0)\} \times \mathbf{R}^N \to \mathbf{R}^2$ by all line arrangements of the parametrization $F : \coprod_9(\mathbf{R}, 0) \to \mathbf{R}^2, 0$ of straight 9-lines (Fig.15) is not topologically versal.*

(A deformation of a map-(multi-)germ is topologically versal if it contains all topological types appearing as perturbations of the original germ. For the rigorous definition, see [8] [9] for instance.)

The proof is a easy consequence of Puppus's theorem (Fig.16). By a infinitely flat perturbation of $\tilde{F}$, we get a non-stretchable pseudo-line arrangements as a perturbation of F. Originally this example was found by Levi [13].

Based on Grünbaum's conjecture, I conjecture that the deformation by all line arrangements of straight s-lines is topologically versal for $s \leq 8$.

(2) It seems that few results are known for the concrete topological classification of real rational curves of degree 6 in $\mathbf{RP}^2$ (besides trivial observations). Here we just mention that, using Petrovsii-Marin's inequality [15], we see the left picture of Fig.17 is not realised as an isotopy type of real rational curve of degree 6. But this method can not applied to the right picture of Fig.17. I do not know whether it exist or not.

Let $C \subset \mathbf{P}^2$ be a real rational M-curve of degree 6. Then after blowing up at the 10 double points and taking ramified double coverings along the strict transform, we get a real $K3$ surface with the cohomology classes of doubled exceptional divisors. I wish to ask the relation to the Nikulin's theory [16].

(3) Ozawa pointed out that the space C_n of convex curves $S^1 \to \mathbf{R}^2$ with Gauss index n is connected for any n. I wish to know the topology of C_n and the subspace of curves isotopic to Fourier M curves of degree n.

(4) Pecker [17] constructed (not necessarily rational) M-curves by using Chebyshev polynomials. See also [21]. I wish to know the relation of Pecker's construction and the perturbations of a blossom b_{2n-1}.

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2 Proofs of Results

Proof of Proposition 1.4: Let $C \subset \mathbf{P}^2$ be a generic algebraic front (= algebraic curve with ordinary double points and ordinary cusps as singularities) of degree m. Recall that the Plücker-Klein's formula [14][25]:

$$\begin{aligned} k &= m(m-1) - 2d - 3r, \\ m &= k(k-1) - 2t - 3w, \\ w &= 3m(m-2) - 6d - 8r, \\ r &= 3k(k-2) - 6t - 8w. \end{aligned}$$

(Here we need the first equality.) And if C is defined over $\mathbf{R}$,

$$m + w' + 2t'' = k + r' + 2d''.$$

Here k is the degree of the dual $C^\vee$ in the dual projective plane $\mathbf{P}^{2*}$, d is the number of double points, r that of cusps, t double tangents and w inflection points of C. The genericity condition demands that all appearing numbers are finite. We denote by w' (resp. r') the number of real inflection points (resp. real cusp points). For real curves, remark that there are two kinds of real double points (resp. real double tangents): t'' designates the number of isolated double tangents (= real tangent lines to imaginary points of C) and d'' the number of isolated double points of C.

Remark that, if there is a Fourier M-curve with an inflection point, then there is a Fourier M-curve with the same degree satisfying the genericity conditions of Plücker-Klien formula for the image in $\mathbf{P}^2$.

Now, for a generic Fourier curve of degree n, consider the corresponding real rational curve and its image in $\mathbf{P}^2$. Then $m = 2n, d = \ell = (2n-1)(n-1), r = 0$. So $k = 2(2n-1)$. Since $r' = 0, d'' = 0$, we have $2n + w' + 2t'' = 2(2n-1)$, therefore $w' + 2t'' = 2n - 2$. But, for Fourier curve of degree n, the line of infinite is perturbed to $(n-1)$ real isolated double tangents. Thus $t'' \geq n-1$. Therefore we see $w' = 0$. □

To prove Theorem 1.7 (2), (3), (4), we use the following non-trivial result:

Lemma 2.1 *Two simple admissible pseudo-line arrangements of* $(2n-1)$ *strings are connected be a one-parameter family of admissible pseudo-line arrangements with only bifurcations of death-barth triangles (without tangent points).*

The proof of Lemma 2.1 will be given in a forthcoming paper. (In the case of line arrangements, Ringel [19] showed the result already.)

Proof of Theorem 1.7: Let F be an admissible representative of $\mathcal{A}$. First we remark that, if we add the line at infinity to $\mathcal{A}$, then we get a pseudo-line arrangement $\tilde{\mathcal{A}}$ in the projective plane $\mathbf{RP}^2$ [13]. (The orientation of the line at infinity is determined from an admissible F). We are going to show that the Gauss diagram $D(c(F))$ of the closure of F determines the Gauss diagram $D(\tilde{\mathcal{A}})$.

The circle of the Gauss diagram of a closure of a psedo-line arrangement is divided into $(2n-1)$-arcs. On each arc, there are $(2n-2)$ intersection points(= end points of chords). If we fix a division point, which is not an intersection point, then we get a decomposition of the circle into $(2n-1)$-arcs, each of which posseses $(2n-2)$ intersection points on it.

Assume that there exists a division point P (for another possible pseudo-line arrangement) lies on the string L_1 of F and $\ell, \ell > 0$, intersection points next to P on L_1. Consider the end point Q of L_1, the start point R of L_2 and the next division point S on L_2. Let T be the intersection point of L_1 and L_2. Assume T lies before P. Then we have the cycle $TPQRT$. Consider the ℓ strings intersecting to L_1 along PQ. They must intersect to L_2 along RT. Since PS is a string for another arrangement, T must come after S. Besides of L_1, the ℓ strings intersect to L_2 on TS. Then we have $(2n-2-\ell)+(\ell+1)=2n-1$ intersection points on L_2. (Fig.18.) This leads a contradiction. So T must lie after P. Then S must lie before T. However, then, considering $(2n-2-\ell)$ strings through RS, we have again a contradiction.

(1): If $c(F)$ and $c(F')$ are isotopic (resp. homeomorphic), then there exists an orientation preserving (resp. not necessarily orientation preserving) isomorphisms of diagrams $D(c(F))$ and $D(c(F'))$. Then the isomorphism induces that of $D(\tilde{\mathcal{A}})$ and $D(\tilde{\mathcal{A}}')$. By a theorem of Carter [6], we see $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ are isotopic (resp. homeomorphic). Since the line at infinity is distinguished, we can take homeomorphisms preserving the line at infinity. This implies the required isotopy (resp. homotopy) in $\mathbf{R}^2$.

(2), (3) & (4): Under the passage of a death-barth triangle, the values $\mathrm{St}(c(F))+\mathrm{St}(c'(F))$ (resp. $J^{\pm}(c(F))$) remains constant. By concrete computations based on the results of Viro [26] and Shumakovitch [20], the constant is equal to $n-1$ (resp. $(n-1)(n-2)$, $-(n-1)(n+1)$). □

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Goo ISHIKAWA
Department of Mathematics, Hokkaido University,
Sapporo 060, JAPAN
e-mail: ishikawa@math.hokudai.ac.jp

Fig. 1.

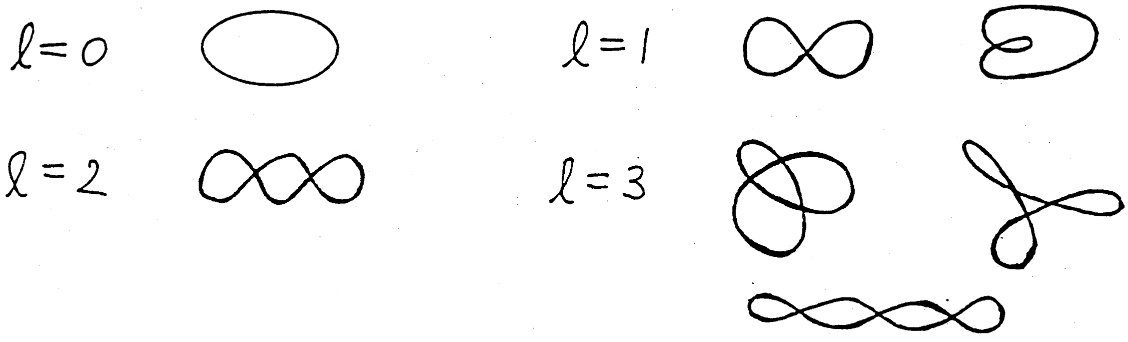


Fig. 2.

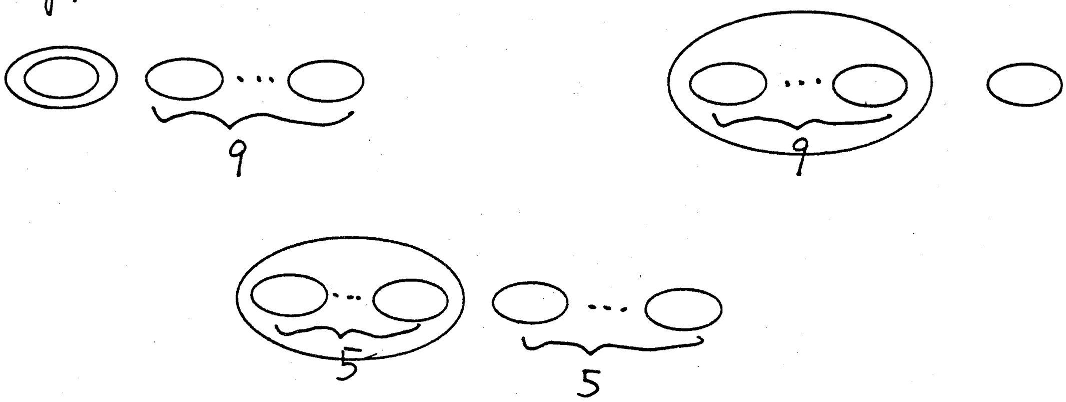


Fig. 3.

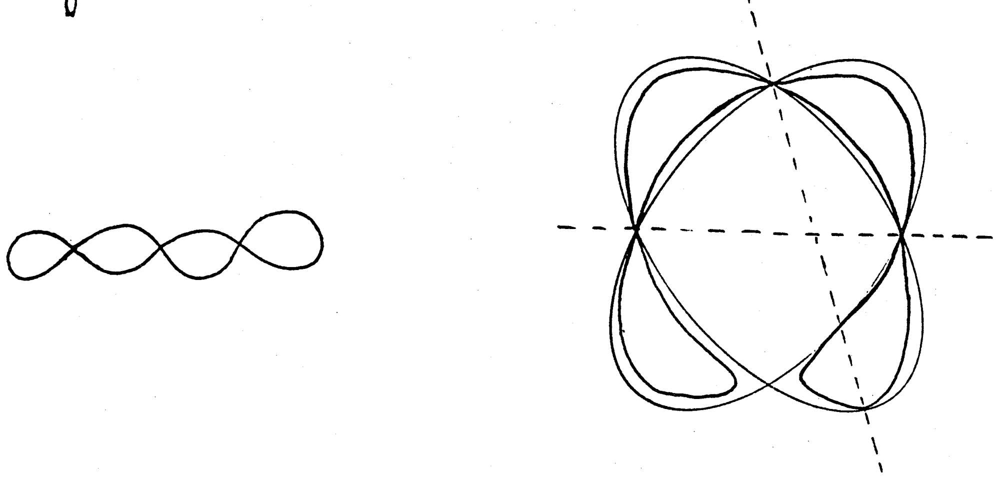
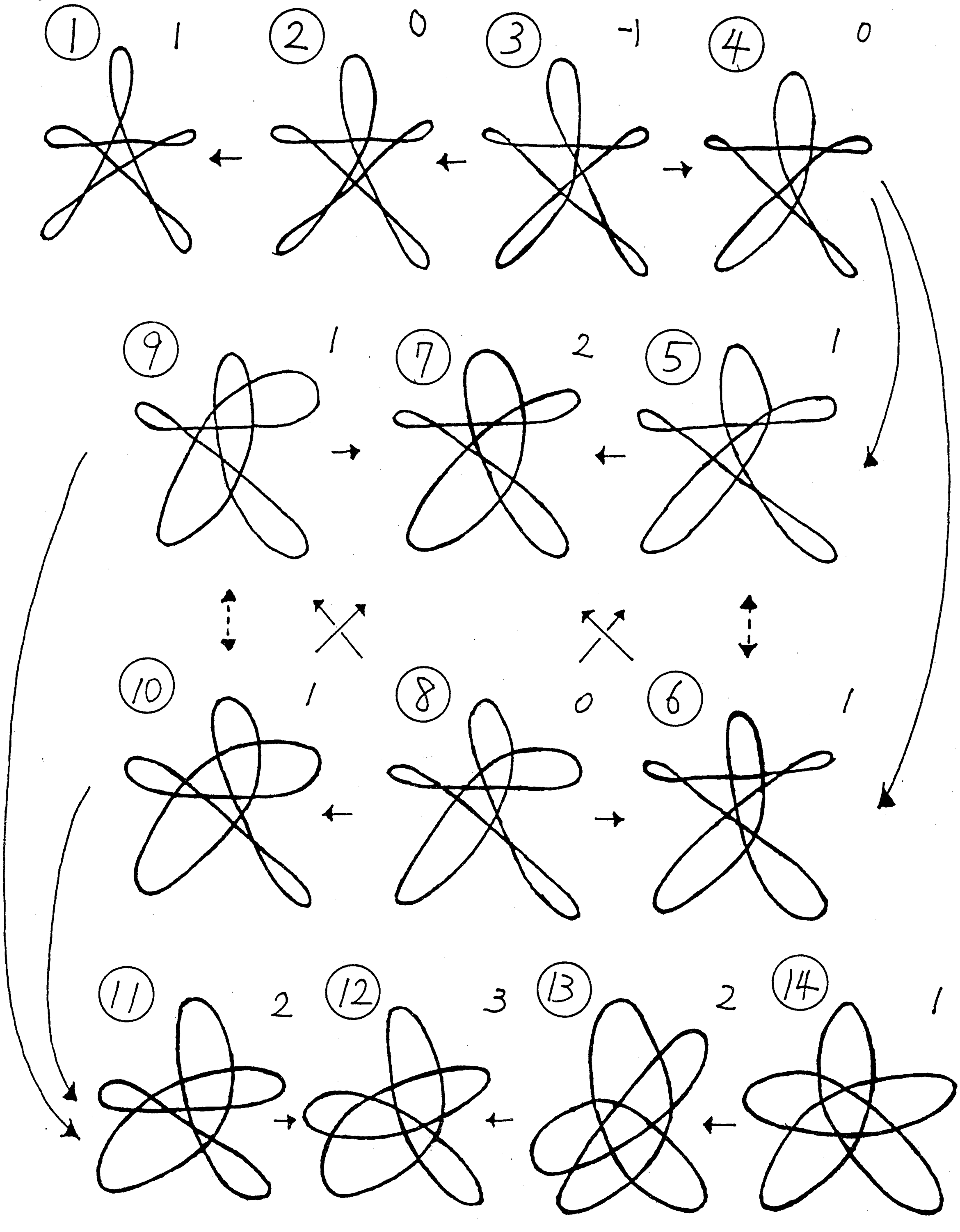


Fig. 4.



Adjacency $J^{+} = 2,\ J^{-} = -8$ ↔ Mirror

They are amphicheiral except for 5 6 9 10

Fig.5

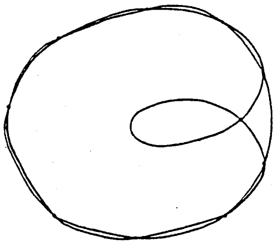
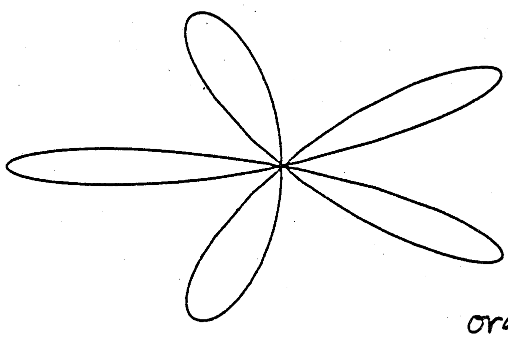


Fig.6

```
ParametricPlot[{Cos[3t] - Cos[2t], Sin[3t] + Sin[2t]},
{t, -Pi, Pi}, Axes -> None]
```



ordinary
quintuple point

Out[3]=
-Graphics-

bouquet b_5
(flower, "sakura")

Fig.7

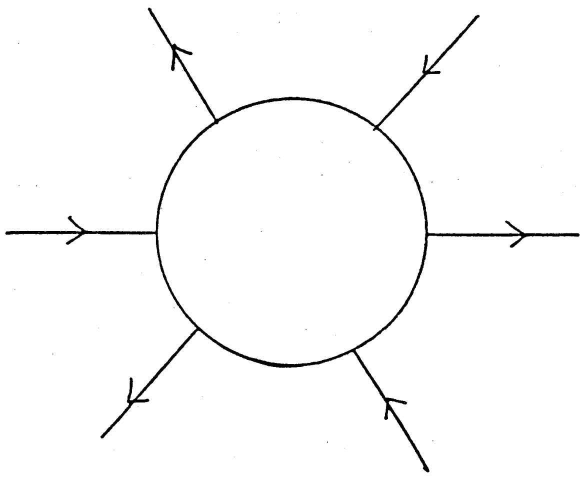
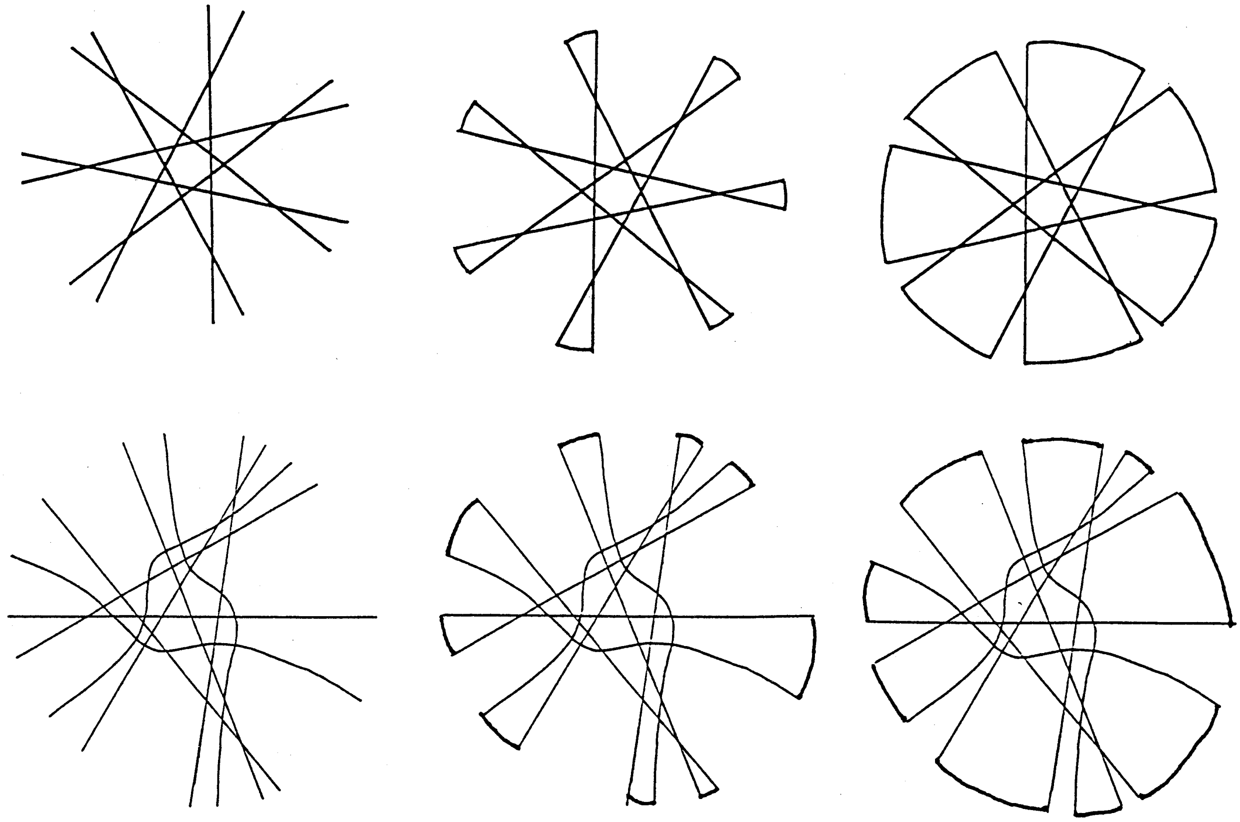


Fig. 8



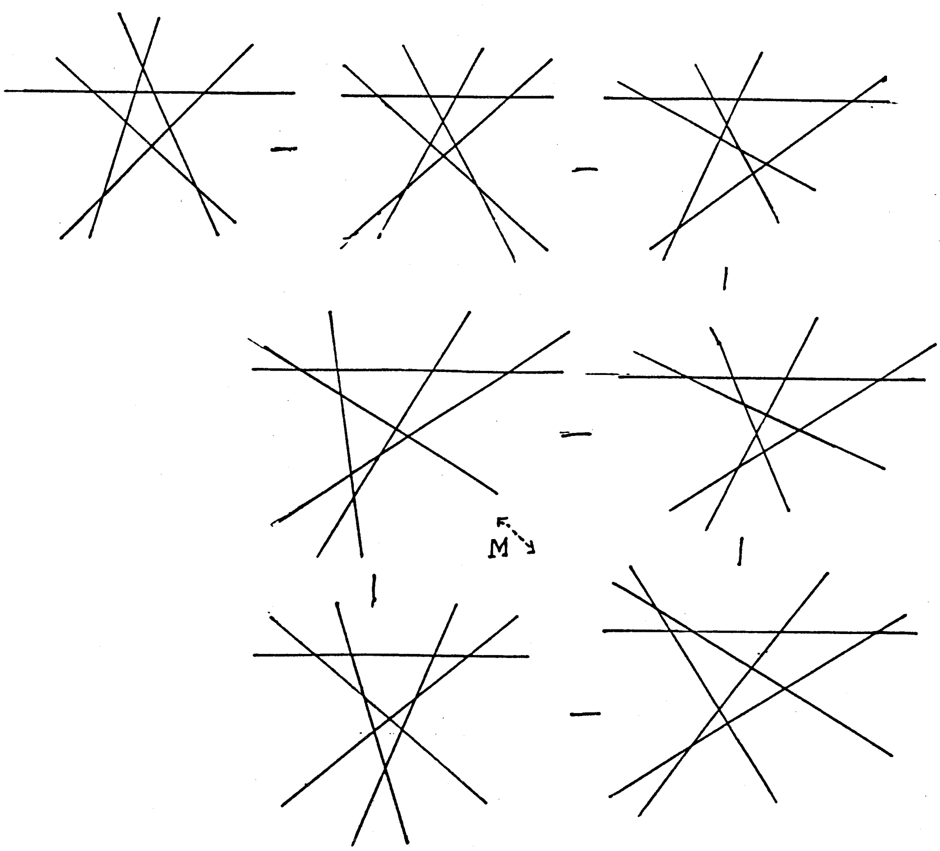


Fig 9

Fig 10

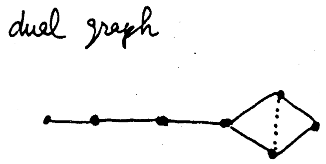


Fig 11

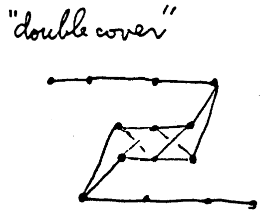
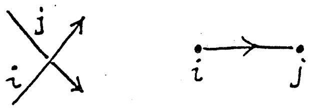
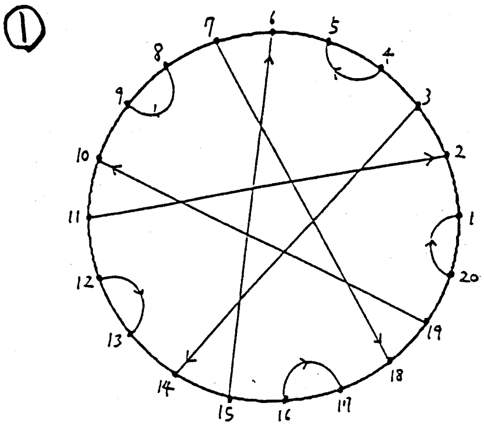


Fig. 12

<<Gauss Diagram>>

Connect the pairs of double points on the circle by chords. Give an orientation on each chord by the Gauss rule:





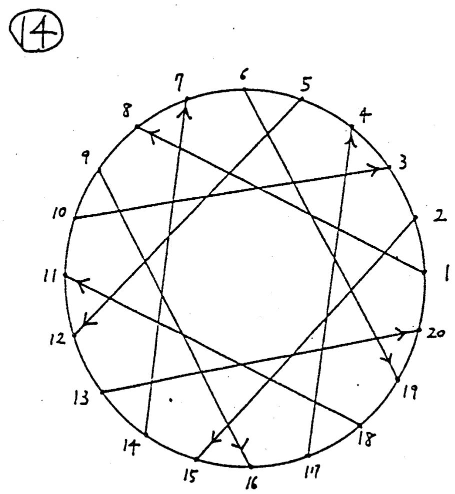


Fig. 13

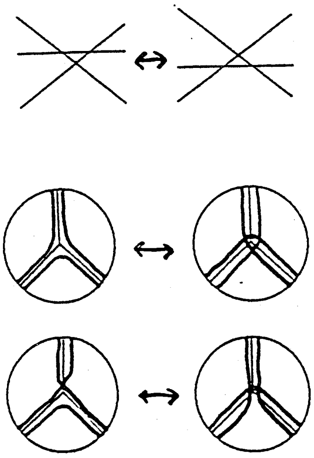


Fig. 14

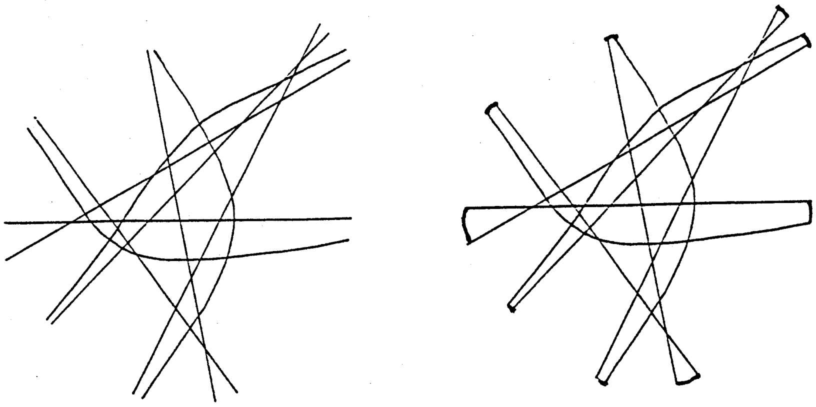


Fig. 15

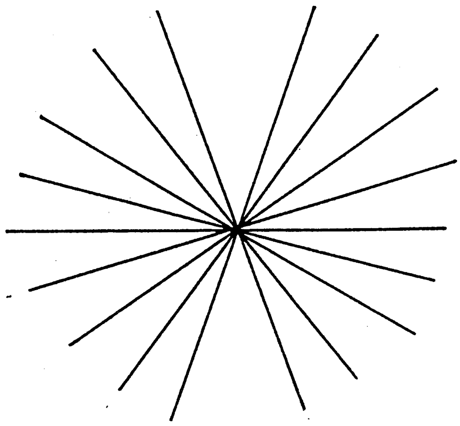


Fig. 16

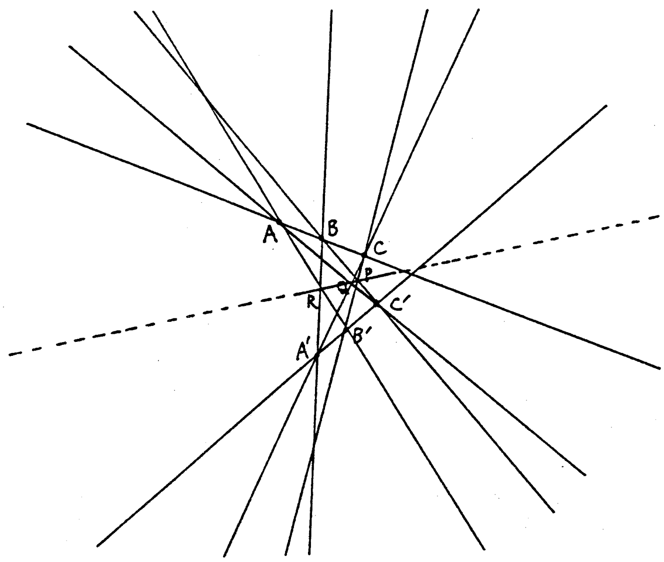


Fig. 17

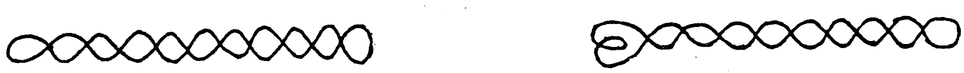


Fig. 18