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BLOW ANALYTIC MODULI OF ANALYTIC FUNCTIONS OF TWO VARIABLES

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Dedicated to the Memory of Professor Etsuo Yoshinaga

1. Introduction

Let $f_1, f_2 : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}, 0)$ be germs of real analytic functions. We say that f_1, f_2 are blow analytic equivalent if there exist maps Φ, ϕ, β_1, β_2 such that the diagram is commute:

$$\begin{array}{ccccc}
(\mathcal{M}_1, \beta_1^{-1}(0)) & \xrightarrow{\beta_1} & (\mathbb{R}^n, \mathbf{0}) & \searrow^{f_1} & \\
\Phi \downarrow & & \downarrow \phi & & (\mathbb{R}, 0) \\
(\mathcal{M}_2, \beta_2^{-1}(0)) & \xrightarrow[\beta_2]{} & (\mathbb{R}^n, \mathbf{0}) & \nearrow_{f_2} &
\end{array}$$

where ϕ is a homeomorphis, Φ is an analytic isomorphism and $\beta(i = 1, 2)$ are compositions of blowing ups with smooth centers. T.C.Kuo proved the following theorem in [1].

Theorem 1.1. *Let $F(x,p) : (\mathbb{R}^n \times P, \mathbf{0} \times P) \to (\mathbb{R}, 0)$ be real analytic and let P be a subanalytic set. Suppose that for $p \in P$ fixed, $F_p : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0)$, $F_p(x) = F(x,p)$ has an isolated singular point. Then there exists a finite filtraton of P by subanalytic sets $P^{(i)}(i = 0, \dots l)$*

$$P = P^{(0)} \supset P^{(1)} \supset \cdots \supset P^{(l)} = \phi$$

such that

(1) *$dim P^{(i)} > dim P^{(i+1)}$, $P^{(i)} - P^{(i+1)}$ is smooth,*
(2) *for $p, p' \in P^{(i)} - P^{(i+1)}$, F_p and $F_{p'}$ are blow analytic equivalent.*

Let $\mathcal{A}_n := \{f | f : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0); \text{analytic}\}$. If $f \in \mathcal{A}_n$ has an isolated singularity, then f is finitely determined, that is, there exists an integer $r \in \mathbb{N}$ such that the analytic type of f is determined by the Taylor polynomial of f with degree r. Let $J^r(n,1)$ be the set of r-jets of the element of $\mathcal{A}_n$ and $L^r(n,n)$ be the set of r-jets of isomorphisms of $(\mathbb{R}^n, \mathbf{0})$. Then the Lie group $L^r(n,n)$ acts on $J^r(n,1)$. Since $\text{codim} Orb(j^r f) < +\infty$, There exists an analytic map $F : (\mathbb{R}^n \times \mathbb{R}^{\mu-1}, \mathbf{0} \times \mathbf{0}) \to (\mathbb{R}, 0)$ such that F is transeversal to $Orb(j^r(f))$. We call F the transeversal family of f. It is well known that we have

$$F(x,p) = f(x) + \sum_{i=1}^{\mu-1} p_i \alpha_i(x),$$

where $\alpha_1, \dots \alpha_{\mu-1}$ are a basis of $\mathfrak{M}^2/\mathfrak{M}(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$.

Let $\mathcal{S}_f := \{p \in \mathbb{R}^{\mu-1} | f$ is blow analytic equivalent to $F_p\}$ as a germ at the origin. By Kuo's theorem above mentioned, it follows that $\mathcal{S}_f$ consists of finitely union of smooth manifolds. We propose the following problems:

Problems.

(1) Estimate the dimension of $\mathcal{S}_f$.
(2) $\mathcal{S}_f$ is smooth?
(3) Classify $\mathcal{A}_n$ by the dimension of $\mathcal{S}_f$.
(4) Prove that the upper semi-comtinuity of the dim$\mathcal{S}_f$.

2. Complex case

Let $f : (\mathbb{C}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a germ of an analytic function with isolated singularity and let $F : (\mathbb{C}^n \times \mathbb{C}^{\mu-1}, \mathbf{0} \times 0) \to (\mathbb{C}, 0)$ be the traseversal family of f. Let

$$\mathcal{S}_f := \{p \in \mathbb{C}^{\mu-1} | (\mathbb{C}^n, f^{-1}(0)) \text{ is relatively topological equivalent to } (\mathbb{C}^n, F_p^{-1}(0))\}$$

Let $f : (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$ be a germ of an analytic function and let $f(x,y) = \sum a_{i,j} x^i y^j$ be the Taylor expansion of f with respect to a coordinate system (x,y) of $\mathbb{C}^2$. *The Newton polygon* $\Gamma_+(f;(x,y))$ with respect to the coordinate system (x,y) is the set $\bigcup_{i,j \neq 0} \{(i,j) + \mathbb{R}^2_+\}$. *Newton boundary* $\Gamma(f;(x,y))$ of f is the union of caompact faces of the boundary of the Newton polygon of f. For a compact face γ of $\Gamma(f;(x,y))$, we define f_γ by $f_\gamma(x,y) = \sum_{(i,j)\in\gamma} a_{i,j} x^i y^j$.

The author prove the following result in [2].

Theorem 2.1. *Let $f(x,y)$ be a germ of a quasihomogeneous function of two complex variables with isolated singularity. Then we have $\mathcal{S}_f =$ a linear space in $\mathbb{C}^{\mu-1}$ genarated by a basis of $\mathbb{C}\{x_1, \dots, x_n\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ in $\Gamma_+(f;(x,y))$.*

M.Oka proved the following result in [3].

Theorem 2.2. *Let $F(x,t) : (\mathbb{C}^2 \times \mathbb{C}, \mathbf{0} \times 0) \to (\mathbb{C}, 0)$ be analytic and suppose that $f_t = F|_{\mathbb{C}^2 \times \{0\}}$ has an isolated singularty. If the Milnor number of f_t is constant independent* *of* $\forall t$ *and* f_0 *is convenient, then there exists a coordianate system* $\phi_t(x,y) = (x(t), y(t))$ *which is analytic in* t *and satisfies the following conditons:*

(1) $\phi_t(0) = 0$, $\phi_0(x,y) = (x,y)$

(2) $\Gamma(f_t;\phi_t) = \Gamma(f_0;\phi_0)$

M.Oka and Kushnirenko proved the following result in [4],[5].

Theorem 2.3. *Suppose that* $f : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$ *has an isolated singularity and has a non-degenerate Newton boundary. Then the Milnor number of* f *is the number of a basis of* $\mathbb{C}\{x_1, \dots, x_n\}/(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ *in* $\Gamma_+(f)$.

Lê Dũng Tráng and C.P.Ramanujan proved the following result in [6].

Theorem 2.4. *Let* $F : (\mathbb{C}^n \times \mathbb{R}, \mathbf{0} \times 0) \to (\mathbb{C}, 0)$ *be analytic and* $F_t = F|_{\mathbb{C}^n \times \{t\}}$ *has an isolated singularity for* $\forall t \in \mathbb{R}$. *If the Milnor numbers of* F_t *are independent of* t, *then the relative topological types of* $(\mathbb{C}^n, F_t^{-1}(0))$ *are independent of* t.

From the above three results, it follows that

Theorem 2.5. *Let* f *be a germ of complex analytic function with isolated singularity at the origin,* $f(0) = 0$ *and suppose that* f *has the non-degenarate Newton boundary. Then we have* S_f *=a linear space in the moduli space of the transversal family of* f *genarated by a basis of* $\mathbb{C}\{x_1, \dots, x_n\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ *in* $\Gamma_+(f;(x,y))$.

We will draw an analogy in case of real analytic functions in what follows.

3. Real case

In what follows, suppose that $f \in \mathcal{A}_n$ has an isolated singularity. Let $F : (\mathbb{R}^n \times I, \mathbf{0} \times I) \to (\mathbb{R}, 0)$ be analytic and $F_0 = f$, where I is the open interval $(-1, 1)$.

Definition 3.1. *We say that* F *admits a blow analytic trivialization along* I *if there exist a local homeomorphism* ϕ, *an analytic isomorphism* Φ *and succesive blowing ups* $\beta_i (i = 1, \dots \gamma)$ *such that the following diagram commutes:*

$$\begin{array}{ccccccc}
(\mathcal{M}_\gamma, \beta^{-1}(o \times I)) & \xrightarrow{\beta_\gamma} \cdots \xrightarrow{\beta_0} & (\mathbb{R}^n \times I, \mathbf{0} \times I) & & \xrightarrow{f} & & (\mathbb{R}, 0) \\
\uparrow \Phi & & \uparrow \phi & \searrow^{proj} & & & \uparrow f \\
 & & & & I & & \\
 & & & \nearrow_{proj\, f} & & & \\
(\mathcal{M}_\gamma, \beta^{-1}(o \times I)) & \xrightarrow{\beta_\gamma} \cdots \xrightarrow{\beta_0} & (\mathbb{R}^n \times I, \mathbf{0} \times I) & & \longrightarrow & & (\mathbb{R}^n, 0),
\end{array}$$

where $\beta_i : \mathcal{M}_i \to \mathcal{M}_{i-1}$ *is the blowing up with a smooth center* $S_{i-1} \subset \mathcal{M}_i (\mathcal{M}_0 = \mathbb{R}^n \times I)$ *and the composition map* $S_i \xrightarrow{inclusion} \mathcal{M}_i \xrightarrow{\beta_i} \cdots \xrightarrow{\beta_1} \mathcal{M}_0 = \mathbb{R}^n \times I \to I$ *is a submersion.*

We have the following result.

Theorem 3.1. *Suppose that* $F : (\mathbb{R}^2 \times I, \mathbf{0} \times I) \to (\mathbb{R}, 0)$ *admits a blow analytic trivialization along* I *and* $F_0(x,y) = f(x,y)$ *is convenient. Then there exists a coordinate system* (x', y') *Of* $\mathbb{R}^2$ *(which is a small perturbation of the original coordinate* (x,y)*) and a real analytic family of local coordinates* $\varphi_t(x', y') = (x(t), y(t))$ *(*$|t| << 1$*) such that*

(1) $\varphi_t(0,0) = (0,0)$ *and* $\varphi_0(x', y') = (x', y')$

(2) $\Gamma(F_t; \varphi_t) = \Gamma(f; (x', y'))$.

In Kuo's theorem, we can replace the condition (ii) to

(1) For $P^{(i)} - P^{(i+1)} \ni \forall p, p'$(close enough), F_p and $F_{p'}$ are jointed by a blow analytically trivial homotopy.

In fact, he have proved Theorem 1.1 under the above condition in [1]. From our result and Kuo's result, we have

Assertion 3.1. *dim(the topological component of* $P^i - P^{(i+1)}$ *which contains* f*)* $\leq$ *the number of a basis of* $\mathcal{A}_2/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ *in* $\Gamma_+(f)$

The following result is deduced as the special case from the result in T.Fukui and E.Yoshinaga [6].

Theorem 3.2. *Let* $F : (\mathbb{R}^2 \times I, \mathbf{0} \times I) \to (\mathbb{R}, 0)$ *be analytic. Suppose that the Newton boundary of* F_t *is independent of* t *and non-degenerate. Then* F *admits a blow analytic trivialization along* I.

From this result and the above Assertion, we have

Assertion 3.2. *Let* $f; (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}, 0)$ *be a germ of analytic function with isolated singularity. If* f *has a non-degenerate Newton boundary, then dim(the topological component of* $P^{(i)} - P^{(i+1)}$ *which contains* f = *the number of a basis of* $\mathcal{A}_2/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ *in* $\Gamma_+(f)$.

In addition to the problems mentioned in the section 1, we propose the following problem:

Problem

(5)Blow analytic constancy implies blow analytic triviality?

If this is true, then we have

Conjecture. *Let* $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}, 0)$ *be a germ of analytic functions with isolated singularity. If* f *has a non-degenerate Newton boundary, then we have* $\dim S_f$ *=the number of a basis of* $\mathcal{A}_2/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ *in* $\Gamma_+(f)$.

we expect the conjecture will be true in genaral.

4. Outline of a proof of the result

Let $F : (\mathbb{R}^2 \times I, \mathbf{0} \times I)$ be analytic and $F_0(x,y) = f(x,y)$. Suppose that F admits a blow analytic trivialization along I and satisfies the commutative diagram mentioned in the previous section:

$$\begin{array}{ccccccc}
(\mathcal{M}_\gamma, \beta^{-1}(o \times I)) & \xrightarrow{\beta_\gamma} \cdots \xrightarrow{\beta_0} & (\mathbb{R}^2 \times I, \mathbf{0} \times I) & \xrightarrow{f} & & & (\mathbb{R}, 0) \\
\uparrow \Phi & & \uparrow \phi & \searrow^{proj} & I & & \uparrow f \\
(\mathcal{M}_\gamma, \beta^{-1}(o \times I)) & \xrightarrow{\beta_\gamma} \cdots \xrightarrow{\beta_0} & (\mathbb{R}^2 \times I, \mathbf{0} \times I) & \nearrow_{proj_f} \xrightarrow{\quad} & & & (\mathbb{R}^2, 0),
\end{array}$$

Note that β_0 is $\sigma \times id_I : \mathcal{N} \times I \to \mathbb{R}^2 \times I$, where σ is the blowing up of $\mathbb{R}^2$ with center the origin and $\mathcal{N} = \{([\xi,\eta],(x,y)) | \xi y - \eta x = 0\} \subset \mathbb{R}P^1 \times \mathbb{R}^2$. Let (x,y) be the coordinate system of $\mathbb{R}^2$ which is obtained by the coordinate transformation

$$\begin{cases} x = x \\ y = y - \pi(p)x \end{cases}$$

where $p \in \sigma^{-1}(0) \cong S^1$ is a point which is not contained in the centers of the blowing ups $\beta_1, \dots, \beta_\gamma$ and $\pi = proj : \mathbb{R}P^1 \times \mathbb{R}^2 \to \mathbb{R}P^1$.

We set $(u_{0\pm}, v_{0\pm}) = (u'_{0\pm}, v'_{0\pm}) := (x,y)$, $o_{0\pm} = o'_{0\pm}$ =the origin of $\mathbb{R}^2$ and set $\mathcal{N}_{0\pm} = \mathbb{R}^2(x,y)$. Next we define inductively real analytic manifold $\mathcal{N}_{k\pm}$, real analytic maps $\sigma_{k\pm} : \mathcal{N}_{k\pm} \to \mathcal{N}_{(k-1)\pm}$ according to the sign $+$, $-$ respectively as follows. Let $\mathbb{R}^2(u_{k\pm}, v_{k\pm})$,$\mathbb{R}^2(u'_{k\pm}, v'_{k\pm})$ be copies of $\mathbb{R}^2$ and set $\mathcal{N}_{k\pm} = \mathbb{R}^2(u_{k\pm}, v_{k\pm}) \cup \mathbb{R}^2(u'_{k\pm}, v'_{k\pm})$. Let $o_{k\pm}$(resp. $o'_{k\pm}$) be the origin of the patch $\mathbb{R}^2(u_{k\pm}, v_{k\pm})$ (resp. $\mathbb{R}^2(u'_{k\pm}, v'_{k\pm})$) and let $\sigma_{k+} : \mathcal{N}_{k+} \to \mathcal{N}_{(k-1)+}$ (resp. $\sigma_{k-} : \mathcal{N}_{k-} \to \mathcal{N}_{(k-1)-}$) be the blowing up of $\mathcal{N}_{(k-1)+}$(resp. $\mathcal{N}_{(k-1)-}$) with center o_{k+} (resp. $o"_{k-}$) defined by

$$\begin{aligned}
\sigma_{k+}(u_{k+}, v_{k+}) &= (u_{k+}, u_{k+}v_{k+}) = (u_{(k-1)+}, u_{(k-1)+}v_{(k-1)+}) \\
&= (u'_{k+}v'_{k+}, v'_{k+}) = \sigma_{k+}(u'_{k+}, v'_{k+})
\end{aligned}$$

$$\begin{aligned}
\sigma_{k-}(u_{k-}, v_{k-}) &= (u_{k-}, u_{k-}v_{k-}) = (u'_{(k-1)-}, u'_{(k-1)-}v'_{(k-1)-}) \\
&= (u'_{k-}v'_{k-}, v'_{k-}) = \sigma_{k-}(u'_{k-}, v'_{k-})
\end{aligned},$$

and we set $\sigma_{0\pm} = id : \mathbb{R}^2 \to \mathbb{R}^2$.

At first we consider the analytic map $\sigma_{k+} : \mathcal{N}_{k+} \to \mathcal{N}_{(k-1)+}$. We set

$$\begin{aligned}
m_{k+} &= \text{the order of } f \circ \sigma_{0+} \circ \cdots \circ \sigma_{k+}(u_{k+}, v_{k+}) \\
\Delta'_{k+} &= \{(u_{k+}, v_{k+}) \in \Gamma(f \circ \sigma_{0+} \circ \cdots \circ \sigma_{k+}; (u_{k+}, v_{k+})) | u_{k+} + v_{k+} = m_{k+}\}
\end{aligned}$$

Let Δ_{k+} be the face of $\Gamma(f;(x,y))$ corresponding to Δ'_{k+} by the map

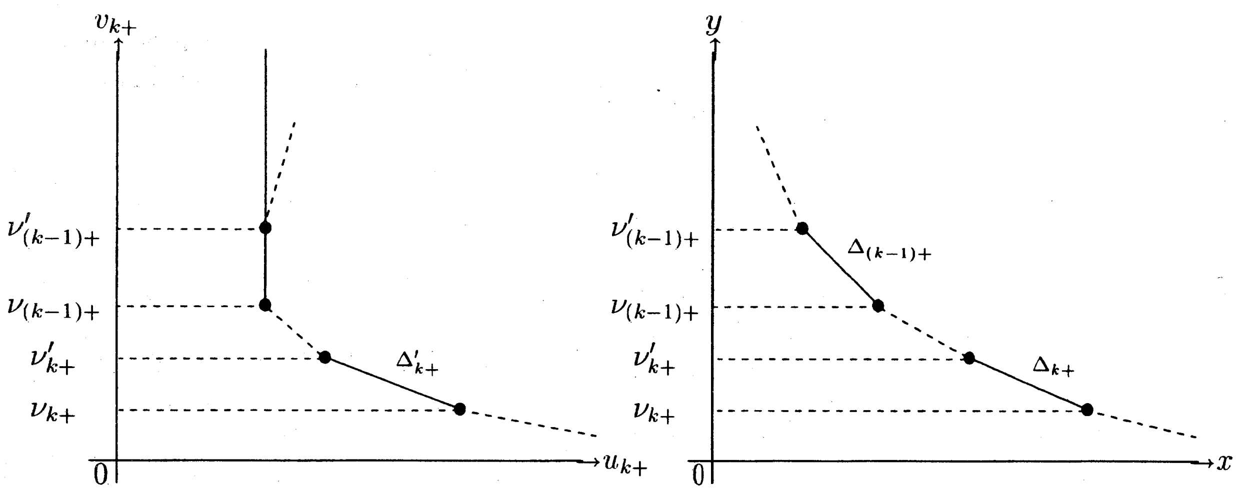
$$\begin{pmatrix} u_{k+} \\ v_{k+} \end{pmatrix} \mapsto \begin{pmatrix} u_{k+} - kv_{k+} \\ v_{k+} \end{pmatrix}.$$

Next we set

$$\nu_{k+} = \min\{y | \text{there exists } x \text{ such that } (x,y) \in \Delta_{k+}\}$$
$$\nu'_{k+} = \max\{y | \text{there exists } x \text{ such that } (x,y) \in \Delta_{k+}\}$$

and there exists $\gamma \in \mathbb{N}$ such that

$$\nu_{0+} \geq \cdots \geq \nu_{(\gamma-1)+} = 0, \qquad \nu'_{0+} \geq \cdots \geq \nu'_{(\gamma-1)+}$$



We have the following lemmata.

Lemma 4.1. *Suppose that $F : (\mathbb{R}^2 \times I, \mathbf{0} \times I) \to (\mathbb{R}, 0)$ admits a blow analytic trivialization along I and suppose that*

$$f_{\Delta_0+}(x,y) = x^\alpha y^\beta \Big(\sum_{i=0}^{\gamma} a_i x^{\gamma-i} y^i\Big), \qquad a_0 a_\gamma \neq 0$$

as a germ at the origin. Then there exist germs $\varepsilon(t)$, $\delta(t)$ and $a_i(t)$ $(i = 1, \dots, \gamma)$, at $t = 0$, of real analytic functions wich saatisfy the following conditions,

(1) $F_{t,\Delta_t}(x,y) = (x - \delta(t)t)^\alpha (y - \varepsilon(t)x)^\beta (\sum_{i=0}^{\gamma} a_i(t) x^{\gamma-i} y^i)$
(2) $\varepsilon(0) = \delta(0) = 0$ *and* $a_i(0) = a_i$ $(i = 1, \dots, \gamma)$
(3) $\sum a_i(t) x^{\gamma-i} y^i$ *does not divide by* $(x - \delta(t)y)$ *or* $(y - \varepsilon(t)x)$ *in* $\mathbb{R}\{x,y\}$,

where Δ_t is the homogeneous face of $\Gamma(F_t;(x,y))$.

By Lemma 4.1 we can find germs of real analytic functions $\varepsilon_1(t)$, $\delta_1(t)$ and $a_i(t)$ ($i = 1, \dots, \gamma_1$) as in Lemma 4.1 so that

$$F_{t,\Delta_t}(x,y) = (x - \delta_1(t)y)^\alpha (y - \varepsilon_1(t)x)^\beta \Big(\sum_{i=0}^{\gamma_1} a_i(t) x^{\gamma_1 - i} y^i \Big).$$

We set

$$\varphi_{1+}(x,y,t) = (x_1(t), y_1(t), t) = (x - \delta_1(t)y, y - \varepsilon_1(t)x, t).$$

Then $(x_1(t), y_1(t), t)$ is a real analytic family of coordinates of $\mathbb{R}^2$ and

$$\{(x_1,y_1) \in \Gamma(F_t;(x_1,y_1)) | x_1 + y_1 = \text{the order of } F_t \circ \varphi_{1+,t}^{-1}(x_1,y_1)\} = \Delta_{0+}$$

for $\forall t$ ($|t| << 1$), where $\varphi_{1+,t} = \varphi_{1+}|_{\mathbb{R}^2 \times \{t\}}$.

We have the following lemma by means of induction.

Lemma 4.2. *Suppose that* $F \circ (\sigma_{(k-1)+} \times id_J) \circ \cdots \circ (\sigma_{0+} \times id_J) : (\mathcal{N}_{(k-1)+}, o_{(k-1)+}) \to (\mathbb{R}, 0)$ *admits a blow analytic trivialization along* J, *where* $J = (-\varepsilon, \varepsilon)$ *and suppose that* $\Gamma(F_t; (x_{(k-1)+}, y_{(k-1)+})) \cap \{\nu_{(k-1)+} \le y_{(k-1)+} \le \nu'_{(k-1)+}\} = \Delta_{(k-1)+}$ *for* $\forall t \in J$ *and* $f_{\Delta_{(k-1)+}}$ *has no power of* x *only. Then* $F \circ (\sigma_{k+} \times id_J) \circ \cdots \circ (\sigma_{0+} \times id_J) : (\mathcal{N}_{k+}, o_{k+}) \to (\mathbb{R}, 0)$ *admits a blow analytic trivialization along* J *and there exist a positive number* δ_{k+} *and a bianalytic map*

$$\begin{aligned} \varphi_{k+} : \mathbb{R}^2(x,y) \times I_{\delta_{k+}} &\longrightarrow \mathbb{R}^2(x_{k+}, y_{k+}) \times I_{\delta_{k+}} \\ (x,y,t) &\longmapsto (x_{k+}(t), y_{k+}(t), t) \end{aligned}$$

such that

(1) $\varphi_{k+}(x,y,0) = (x,y,0)$, $\varphi_{k+}(0,0,t) = (0,0,t)$
(2) $\Gamma(F_t; (x_{k+}, y_{k+})) \cap \{\nu_{k+} \le y_{k+} \le \nu'_{k+}\} = \Delta_{k+}$
(3) $\Gamma(F_t; (x_{k+}, y_{k+})) \cap \{\nu_{1+} \le y_{k+}\} = \Gamma(F_t; (x_{1+}, y_{1+})) \cap \{\nu_{1+} \le y_1 +\}$,

where $I_{\delta_{k+}}$ *is the open interval* $(-\delta_{k+}, \delta_{k+})$.

Lemma 4.3. *If* F *admits a blow analytic trivialization along* I *and noncompact face of* $\Gamma_+(F_t;(x,y))$ *is independent of* t, *then we have*

$$\Gamma(F_t;(x,y)) = \Gamma(f;(x,y)) \qquad \textit{for } |t| << 1$$

We apply Lemma 4.2 for F and then for $k = 0, \dots, \gamma - 1$ and $|t| << 1$,

$$\Gamma(F_t;(x_{k+}, y_{k+}) \cap \{(x_{k+}, y_{k+}) | \nu_{k+} \le y_{k+} \le \nu'_{k+}\} = \Delta_{k+}$$

Next for the sign $-$, we proceed with the same argument as the sign $+$ and then for $l = 1, \dots, \xi - 1$ and $|t| << 1$,

$$\Gamma(F_t;(x_{l-}, y_{l-})) \cap \{(x_{l-}, y_{l-}) | \zeta_{l-} \le x_{l-} \le \zeta'_{l-}\} = \Delta_{l-}$$

$$\Gamma(F_t;(x_{l-}, y_{l-})) \cap \{\zeta_{1-} \le x_{l-}\} = \Gamma(F_t;(x', y')) \cap \{\zeta_{1-} \le x'\},$$

where $(x'(t), y'(t), t) = \varphi_{1-} \circ \varphi_{(\gamma-1)+} \circ \cdots \circ \varphi_{1+}(x, y, t)$ and

$$\zeta_{k-} = \min\{x | \text{there exists } y \text{ such that } (x, y) \in \Delta_{k-}\}$$
$$\zeta'_{k-} = \max\{x | \text{there exists } y \text{ such that } (x, y) \in \Delta_{k-}\}$$
$$\zeta_{0+} \geq \cdots \geq \zeta_{(\xi-1)+} = 0, \quad \zeta'_{0+} \geq \cdots \geq \zeta'_{(\xi-1)+}.$$

We set $\varphi(x, y, t) = (\varphi_t(x, y), t) = \varphi_{(\gamma-1)-} \circ \cdots \circ \varphi_{1-} \circ \varphi_{(\gamma-1)+} \circ \cdots \circ \varphi_{1+}(x, y, t)$ and then we have that the noncompact face of $\Gamma(F_t; \varphi_t)$ is independent of t for $|t| << 1$. Hence from Lemma4.3, $\Gamma(F_t; \varphi_t)$ is independent of t for $|t| << 1$. This completes the proof.

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