MANIFOLDS WITH FINITE FUNDAMENTAL GROUP AS CODIMENSION-2 FIBRATORS

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1.Introduction

D. Coram and P.Duvall in [CD1] introduced approximate fibrations as a generalization of both Hurewicz fibrations and cell-like maps. A proper map $p: M \to B$ between locally compact ANRs is called an *approximate fibration* if its map has the following approximate homotopy lifting property : Given an open cover ε of B, an arbitrary space X, and two maps $\mu: X \to M$ and $\Psi: X \times I \to B$ such that $\Psi_0 = p\,\mu$, there exists a map $\Phi: X \times I \to M$ such that $\Phi_0 = \mu$ and $p\,\Phi$ is ε -close to Ψ.
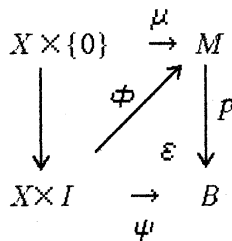
The following results of D. Coram and P.Duvall motived our work.

Fact. 1.1 [CD1]. *Let $p: M \to B$ be an approximate fibration. Then*
  (i) *a path $\omega: I \to B$ induces a shape equivalence from $p^{-1}(\omega(0))$ to $p^{-1}(\omega(1))$:*
  (ii) *each fiber is a pointed fundamental ANR (a pointed FANR).*

An FANR is the shape analog of an ANR. It is known that a space S is a pointed FANR if and only if S has the shape of a CW complex. The term is defined along with the term " shape " in [MS]. Our purpose in this paper is to examine a converse to the part (ii), that is, given a compact ANR N and an upper semicontinuous decomposition G of a locally compact ANR M satisfying that each element of G is shape equivalent to N, is the decomposition map $p: M \to M/G$ an approximate fibration ? Because of the following fact, we would like to focus attention on that both M and N are manifolds.

Fact. 1.2 [DH, Theorem 3.1.]. *Let $p: M \to B$ be an approximate fibration of a connected m-manifold (without boundary) onto an ANR B. Then B is a k-dimensional generalized manifold; moreover, if M is orientable, then the fiber of p has the shape of a Poincare duality space of formal dimension m - k.*

A closed connected manifold N^n is called a *codimension-2 fibrator* (respectively, *codimension-2 orientable fibrator*) if whenever G is an upper semicontinuous decomposition of an arbitrary (respectively, orientable) $(n+2)$-manifold M satisfying that each element of G is shape equivalent to N, then the decomposition map $p: M \to B$ is an approximate fibration.

Main Question 1.3. *What is a manifold which is a codimension-2 fibrator or codimension-2*

orientable fibrator .

In [D1], Daverman showed that all simply connected manifolds, closed surfaces with nonzero Euler characteristic, and real projective n-spaces (n $>$ 1) are codimension-2 fibrators. Nonfibrators in codimension-2 include all closed manifolds admitting a fixed point free cyclic action having a orbit space homotopy equivalent to itself. Therefore S^1, the torus and the Klein bottle are nonfibrators. For example, there exists a map p from S^3 to S^2 that every fiber of p has the shape of S^1 and p is not an approximate fibration. See [CD3].

Since all simply connected manifolds and real projective n-spaces (n $>$ 1) are codimension-2 fibrators, it is quite natural to ask the following :

Question 1.4[D1, Question 6.3]. *Is every closed n-manifold with finite fundamental group a codimension-2 fibrator ?*

Question 1.5[D3, Question 8.2]. *Are Lens spaces codimension-2 fibrators ?*

2.Closed hopfian manifolds and continuity sets

Throughout this paper, all spaces are locally compact, metrizable ANRs , and all manifolds are finite dimensional, connected, and boundaryless. Whenever we allow boundary, the object will be called a manifold with boundary. Homology is computed with integer coefficients unless another coefficient module is mentioned.

By the *degree* of a map between closed n-manifolds, we mean more precisely the absolute degree ; namely the nonnegative integer determining the induced homomorphism's effect on nth (integral or Z_2) homology, up to sign.

Recall that a group H is *hopfian* if every epimorphism $H \to H$ is an isomorphism and that a manifold M is *aspherical* if $\pi_i(M)$ is the trivial group for all i $>$ 1. Call a closed manifold N *hopfian* if it is orientable and every degree one map $N \to N$ is a homotopy equivalence.

Theorem 2.1[D4. Theorem 2.2]. *A closed, orientable n-manifold N is a hopfian manifold if any of the following conditions holds* :

(1) $n \leqq 4$ *and* $\pi_1(N)$ *is hopfian* ;

(2) $\pi_1(N)$ *is hopfian and its integral group ring,* $Z\pi_1(N)$*, is Noetherian* ;

(3) $\pi_1(N)$ *contains a nilpotent subgroup of finite index* ; *or*

(4) $\pi_i(N)$ *is trivial,* $1 < i < n - 1$*, and* $\pi_1(N)$ *is hopfian.*

All closed, orientable n-maniflods N with $\pi_1(N)$ finite and asperical n-manifold N 'with $\pi_1(N')$ hopfian are hopfian manifolds.

Given a orientable closed n-manifold N^n, an upper semicontinuous decomposition G of a orientable manifold M^m is N-*like* if each element of G is shape equivalent to N For simplicity or familiarity, we shall assume that each element of an N^n-like upper semicontinuous decomposition G is an ANR having the homotopy type of N^n ; experts can easily modify the proofs to treat the more general situation. See [D1].

Let N^n be a closed n-manifold, M^{n+2} be an $(n + 2)$-manifold and G be an N^n-like upper semicontinuous decomposition of M . For each $g \in G$ there exist a neighborhood U_g of g in M and a retraction $R_g : U_g \to g$. We will define the *continuity set* $C \subset M/G$ of the decomposition map $p : M \to$

M/G:

$C = \{ p(g) \in M/G$: there exist a neighborhood U_g of g in M and a retraction $R_g : U_g \to g$ such that $R_g \mid g' : g' \to g$ is a degree one map for all $g' \in G$ with $g' \subset U_g \}$.

D. Coram and P. Duvall in [CD3] have shown that C is a dense, open subset of M/G. The following result comes from [CD2]. See [D4, Theorem 2.1].

Proposition 2.2. *Suppose N^n is a closed orientable hopfian n-manifold and M^{n+2} is an (n+2)-manifold. Then the N-like decomposition map $p : M \to B$ is an approximate fibration over its continuity set C.*

The following results of the decomposition spaces is found in [D1].

Theorem 2.3. *Let N^n be a closed n-manifold, M^{n+2} be an (n+2)-manifold and G be an N^n-like upper semicontinuous decomposition of M. If both N and M are orientable, then the decomposition space $B = M/G$ is a 2- manifold and $D = B \setminus C$ is locally finite in M/G, where C represents the continuity set of the decomposition map $p : M \to B$. If either N or M is nonorientable, B is a 2- manifold with boundary (possibly empty) and $D' = (Int\ B) \setminus C'$ is locally finite in B, where C' represents the mod 2 continuity set.*

3. Closed hopfian manifolds with finite first homology group

The setting throughout this section is that N^n is a closed hopfian n-manifold, G is an N-like upper semicontinuous decomposition of an orientable $(n+2)$-manifold M^{n+2}, the decomposition space M/G is denoted by B and the induced quotient map is denoted by $p : M \to B$.

Proposition 3.1 [Im2, Lemma 3.2]. *Let g be an element of G satisfying $g \neq g_0$ and $g \subset U_{g_0}$. If the restriction $R_{g_0} \mid g : g \to g_0$ of $R_{g_0} : U_{g_0} \to g_0$ induces an H_1-isomorphism $(R_{g_0} \mid g)_* : H_1(g) \to H_1(g_0)$, then $R_{g_0} \mid g' : g' \to g_0$ is a degree one map for all $g' \in G$ with $g' \subset U_{g_0}$.*

It is obtained from the above propositions and Theorem 2.3 that every closed hopfian manifold N satisfying $H_1(N) = 0$ is a codimension-2 orientable fibrator. One may notice that every simply connected closed manifold is a codimension-2 fibrator. For further details, see Theorem 2.3 and Corollary 2.4 of [D1]. The proof of Theorem 2.5 in [C] leads to the following conclusion.

Theorem 3.2. *Every closed hopfian manifold N^n satisfying $H_1(N)$ is finite is a codimension-2 orientable fibrator.*

Corollary 3.3 [cf. D2]. *Let N^n be a closed orientable n-manifold. We consider the following two conditions:*

(a) *$\pi_1(N)$ is finite;*

(b) *$\pi_1(N)$ is hopfian, $H_1(N)$ is finite and N is aspherical.*

Then N is a codimension-2 orientable fibrator.

Let S^n, P^n and $L(p,q)$ denote the n-sphere, real projective n-space and Lens space of type (p,q), respectively.

Corollary 3.4. *For m, n > 1, any finite product of* S^m, P^{2n-1} *and L (p,q) is a codimension-2 orientable fibrator.*

4. Products of codimension-2 orientable fibrators

In [Im1] , Im proved that any finite product of closed, orientable surfaces with negative Euler characteristic is a codimension-2 orientable fibrator. It is quite natural to ask the following :

Question 4.1. Is a product of codimension-2 orientable fibrators a codimension-2 orientable fibrator ?

In this section, we will state that a product of most known codimension-2 orientable fibrators is a codimension-2 orientable fibrator, that is, a product of any closed manifold with finite fundamental group and any closed orientable surface F with nonzero Euler characteristic is a codimension-2 orientable fibrator.

Definition 4.2. A group G is said to be *residually finite* if for each $g \in G \setminus \{1\}$, there exist a finite group H and a homomorphism $F : G \rightarrow H$ with $F(g) \neq 1$.

Hempel [H2] showed that for each surface F, $\pi_1(F)$ is residually finite.

Lemma 4.3 [H2, Lemma 15. 17]. *If G is a finitely generated, residually finite group, then G is hopfian.*

It is clear that every finite group is residually finite and hopfian.

Corollary 4.4. *Any finite product of finitely generated, residually finite groups is hopfian.*

Lemma 4.5. *If* N^n *is a closed orientable m-manifold with finite fundamental group and if F be a closed orientable surface with negative Euler characteristic* $\chi(F)$, $N \times F$ *is a hopfian* $(n+2)$-*manifold.*

As in the proof of [Im 2, Theorem 3.4], the following theorem can be proved.

Theorem 4.6. *Suppose a closed orientable n-manifold* N^n *satisfies that* $\pi_1(N)$ *is finite and F be a closed orientable surface with negative Euler characteristic. Then an (n+2)-manifold $N \times F$ is a codimension-2 orientable fibrator.*

5. Applications

When $p : M \rightarrow B$ is an approximate fibration, for then, just as with Hurewicz fibrations, there is a homotopy exact sequence

$$\cdots \rightarrow \pi_{i+1}(B) \rightarrow \pi_i(p^{-1}(b)) \rightarrow \pi_i(M) \rightarrow \pi_i(B) \rightarrow \cdots$$

Theorem 5.1. *Let N^n be a closed hopfian n-manifold with finite first homology group, G be an N-like upper semicontinuous decomposition of an orientable $(n + 2)$-manifold M^{n+2} and $p : M \to B = M/G$ be the decomposition map. If $\pi_1(M) = 1 = \pi_2(M)$, the inclusion map $p^{-1}(b) \to M$ is a homotopy equivalence for each $b \in B$.*

Proof. From Theorem 3.2, $p : M \to B$ is an approximate fibration. Thus there is a homotopy exact sequence

$$\cdots \to \pi_2(M) \to \pi_2(B) \to \pi_1(p^{-1}(b)) \to \pi_1(M) \to \pi_1(B) \to 1.$$

It follows from $\pi_1(M) = 1 = \pi_2(M)$ that $\pi_1(B) = 1$ and $\pi_2(B) \approx \pi_1(p^{-1}(b))$. We see $H_2(B) \approx \pi_2(B) \approx \pi_1(p^{-1}(b)) \approx H_1(p^{-1}(b))$, therefore, $H_2(B) \approx \pi_2(B)$ is a finite abelian group. Since B is a 2-manifold, $H_2(B)$ is either 0 or Z, thus $H_2(B) \approx \pi_2(B) \approx 0$. Thus B is homeomorphic to E^2. We recognize from the above homotopy exact sequence that the inclusion map $p^{-1}(b) \to M$ is a homotopy equivalence. $\square$

Corollary 5.2. *Let N^n be a closed hopfian n-manifold with finite first homology group and an $(n + 2)$-manifold M^{n+2} satisfying $\pi_1(M) = 1 = \pi_2(M)$. If N is not homotopy equivalent to M, then there is no N-like upper semicontinuous decomposition of M. Particularly, there is no N-like upper semicontinuous decomposition of E^{n+2} or S^{n+2}.*

We call a closed n-manifold N a *codimension k PL fibrator* if, for all orientable PL $(n + k)$-manifold M, B is a polyhedron, and PL maps $p : M \to B$ satisfying that each $p^{-1}b$ collapses to an n-complex homotopy equivalent to N, p is an approximate fibration. The following result stems from [D5, Theorem 2.10].

Theorem 5.3. *If the closed orientable n-manifold N has a closed $(k - 1)$-connected universal cover, then N is a codimension-k PL fibrator.*

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