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# 点集合と超平面による切り口の次元について

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## 1. Introduction

Let $S\subset \mathbb{R}^{3}$ be the space described in K.Sitnikov[6] satisfting the relation $1=\mu\dim S<\dim S=2$, where $\mu\dim$ (resp. dim) denotes the metric (resp. covering) dimension. As easily seen, the space $S$ has a remarkable property that $\mu\dim(S\cap H)=\mu\dim S$ for every plane $H$ in $\mathbb{R}^{3}$. Motivated by this, we will be concerned with the problems whether there exists a point set $X$ in Euclidean $n$-space $\mathbb{R}^{n}$ satisfying (A) or both of the following two conditions:

(A) $\mu\dim(X\cap H)=\mu\dim X$ *for every hyperplane* $H$ *in* $\mathbb{R}^{n}$.
(B) $\dim(X\cap H)=\dim X$ *for every hyperplane* $H$ *in* $\mathbb{R}^{n}$.

Here by a *hyperplane* in $\mathbb{R}^{n}$, we mean an $(n-1)$-dimensional affine subspace of $\mathbb{R}^{n}$.

The first result is the following which improves [2,Lemma 4]:

**Theorem 1.** *For arbitrary integers* $m$ *and* $n$ *with* $0\leq m\leq n-1\geq 1$, *there exists a point set* $X_{m}^{n}$ *in* $\mathbb{R}^{n}$ *such that*
i) $\mu\dim X_{m}^{n}=m$ *and* $\dim X_{m}^{n}=\min\{2m,n-1\}$, *and*
ii) $\mu\dim(X_{m}^{n}\cap H)=m$ *for every hyperplane* $H$ *in* $\mathbb{R}^{n}$.

Let us note that if a non-empty space $X$ in $\mathbb{R}^{n}$ satisfies the condition (A), then necessarily $n\geq 2$ and $\dim X\leq n-1$. Moreover since $\dim X\leq 2\mu\dim X$ by a Katětov's inequlity[4], the space $X_{m}^{n}$ in Theorem 1 is one which admits the maximal differece between dim and $\mu$dim among those spaces $X$ in $\mathbb{R}^{n}$ satisfying $\mu\dim X=m$ and the condition (A).

In contrast with Theorem 1, it will be shown that there exists $Y_{k}^{n}$ in $\mathbb{R}^{n}$ with $\mu\dim Y_{k}^{n}=\dim Y_{k}^{n}=k$ satisfying the condition (A) (and also (B)) for arbitrary integers $n$ and $k$ with $0\leq k\leq n-1\geq 1$(Theorem 2).

Now suppose that a space $X$ in $\mathbb{R}^{n}$ satisfies both (A) and (B) with $\dim X=k$ and $\mu\dim X=m$. Then as above, it must be satisfied that $n\geq 2$ and $m\leq k\leq\min\{2m,n-1\}$, and also that either $k<n-1$ or $k=n-1=m$; indeed, if $\dim X=n-1$, then $X\cap H$ must have non-empty interior in a hyperplane $H$ by (B), which implies $\mu\dim X=n-1$.

The following is the main result which extends [3,Theorem]:

**Main theorem.** *Let $n, m$ and $k$ be arbitrary integers such that $0 \leq m \leq n-1 \geq 1$ and $m \leq k \leq \min\{2m, n-1\}$. Then there exists a point set $X^n_{m,k}$ in $\mathbb{R}^n$ such that*
i) $\mu\dim X^n_{m,k} = m$ *and* $\dim X^n_{m,k} = k$,
ii) $\mu\dim (X^n_{m,k} \cap H) = m$ *for every hyperplane $H$ in $\mathbb{R}^n$, and*
iii) *if either $k < n-1$ or $k = n-1 = m$, then* $\dim (X^n_{m,k} \cap H) = k$ *for every hyperplane $H$ in $\mathbb{R}^n$.*

## 2. Preliminaries

By $\mathbb{I}$ we denote the closed interval $[-1,1]$. Also, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denote the sets of natural numbers, integers and rationals, respectively. Thus $\mathcal{F} = \{z + \mathbb{I}^n : z \in \mathbb{Z}^n\}$ is a collection of congruent $n$-cubes whose interiors cover $\mathbb{R}^n$. Similarly, $\mathcal{F}_i = \{(1/i)z + [0,1/i]^n : z \in \mathbb{Z}^n\}$, $i \in \mathbb{N}$, is a cover of $\mathbb{R}^n$ by $n$-cubes whose interiors are pairwise disjoint and sides are of length $1/i$. We set

$$F_i^{(j)} = \cup\{\tau^{(j)} : \tau \in \mathcal{F}_i\},\ 0 \leq j \leq n, \text{ where } \tau^{(j)} \text{ denotes the union of } j\text{-faces of } \tau.$$

Let $\alpha = \{a_i\}$ be a sequence of points in $\mathbb{R}^n$ and $m, n$ integers with $0 \leq m \leq n-1$. Then we define

$$S^n_m(\alpha) = \mathbb{R}^n - \cup\{a_i + F_i^{(n-m-1)} : i \in \mathbb{N}\}.$$

**Fact 1**(cf. [2, Lemma 4].) $\mu\dim S^n_m(\alpha) = m$ *for every sequence $\alpha$ of points in $\mathbb{R}^n$.*

Indeed, for every $i$, $S^n_m(\alpha)$ admits a continuous map $f$ onto the $m$-skeleton of the decomposition of $\mathbb{R}^n$ by $n$-cubes which is dual to $\mathcal{F}_i$, satisfying $\|x - f(x)\| < \sqrt{n}/2i$ for every $x$. This implies $\mu\dim S^n_m(\alpha) \leq m$(cf. [7, Corollary 2]), and the opposite inequality is obvious because $S^n_m(\alpha)$ contains a (rectilinear) $m$-simplex. The following is a special case of [8, Theorem 3].

**Fact 2.** *If a sequence $\alpha = \{a_i\}$ of points in $\mathbb{R}^n$ satisfies the condition*

$$\dim((a_i + F_i^{(n-m-1)}) \cap (a_j + F_j^{(n-m-1)})) \leq k \quad \textit{whenever } i \neq j,$$

*then* $\dim S^n_m(\alpha) \geq n - k - 2$.

For every finite set $A$ of $\mathbb{R}^n$ and every integer $k$ with $0 \leq k \leq n-1$, we set

$$A^{[k]} = \{[v_0, ..., v_j] : v_0, ..., v_j \in A,\ j \leq k\}$$

where $[v_0, ..., v_j]$ denotes the plane(i.e., the affine subspace) determined by points $v_0, ..., v_j$. Then we say that $p \in \mathbb{R}^n$ is *in a general position* (or *g.p.*, for short) *relative to* $A$, if $p \notin \cup A^{[n-1]}$.

We denote by $\pi_k : \mathbb{R}^n \to \mathbb{R}$, $1 \le k \le n$, the projection of $\mathbb{R}^n$ into the $k$-th factor.

**Fact 3.**(cf.[2, Lemma 4]) *If* $\alpha = \{a_i\}$ *in* $\mathbb{R}^n$ *satisfies the condition that*

$$(\pi_k(a_i) + (1/i)\mathbb{Z}) \cap (\pi_k(a_j) + (1/j)\mathbb{Z}) = \emptyset \quad \textit{whenever } i \ne j,$$

*for every* $k = 1, ..., n$, *then* $\dim S_m^n(\alpha) = \min\{2m, n-1\}$.

This follows from Fact 2 and the fact that if $\alpha$ satisfies the condition in Fact 3, then

$$\dim((a_i + F_i^{(n-m-1)}) \cap (a_j + F_j^{(n-m-1)})) = \max\{n - 2m - 2, -1\}.$$

Sitnikov's space $S$ cited above is of type $S_1^3(\alpha)$ with $\alpha$ satisfying the condition in Fact 3 for $(m, n) = (1, 3)$. Also the space $X_m^n$ with $m > 0$, which will be given in the proof of Theorem 1, is of form $S_m^n(\alpha)$.

## 3. Point sets $X_m^n$ and $Y_k^n$

For a sequence $\alpha = \{a_i\}$ of points in $\mathbb{R}^n$, we consider the condition:

$(C_i)$ *Every point* $p \in (a_i + F_i^{(0)}) \cap \mathbb{I}^n$ *is g.p.relative to* $\cup\{(a_j + F_j^{(0)}) \cap \mathbb{I}^n : j < i\}$.

**Lemma 1.** *There exists a sequence* $\alpha = \{a_i\}$ *of points in* $\mathbb{Q}^n$ *satisfying* $(C_i)$ *for all* $i \ge 2$.

Let us note that $(C_i)$ implies

(1) $(\pi_k(a_i) + (1/i)\mathbb{Z}) \cap (\pi_k(a_j) + (1/j)\mathbb{Z}) = \emptyset$ for every $j < i$ and $k = 1, \ldots, n$.

Since each $F_i^{(n-m-1)}$ can be expressed as the countable union of $(n-m-1)$-planes, there exist $(n-m-1)$-planes $B_{i,s}^{n-m-1}$ such that

(2) $a_i + F_i^{(n-m-1)} = \cup\{B_{i,s}^{n-m-1} : s \in \mathbb{N}\}$, $i \in \mathbb{N}$.

**Lemma 2.** *Suppose a sequence* $\alpha = \{a_i\}$ *of points in* $\mathbb{R}^n (n \ge 2)$ *satisfies* $(C_i)$ *for every* $i \ge 2$. *Then for every hyperplane* $H$ *in* $\mathbb{R}^n$ *with* $H \cap \operatorname{Int} \mathbb{I}^n \ne \emptyset$,

$$\Lambda = \{i \in \mathbb{N} : B_{i,s}^{n-m-1} \cap \mathbb{I}^n \ne \emptyset,\ B_{i,s}^{n-m-1} \subset H \textit{ for some } s \in \mathbb{N}\}$$

*consists of at most* $n$ *elements.*

**Lemma 3.** *Let* $m$ *and* $n$ *be integers with* $0 \le m \le n - 1 \ge 1$ *and* $\alpha$ *a sequence of points in* $\mathbb{R}^n$ *satisfying* $(C_i)$ *for every* $i \ge 2$. *Then we have*

i) $\mu\dim S^n_m(\alpha) = m$ *and* $\dim S^n_m(\alpha) = \min\{2m, n-1\}$.
ii) $\mu\dim(S^n_m(\alpha) \cap H) = m$ *for every hyperplane* $H$ *in* $\mathbb{R}^n$ *in case* $m > 0$.

**Proof of Theorem 1.** Let $\alpha$ be an arbitrary sequence of points in $\mathbb{R}^n$ which satisfies $(C_i)$ for all $i \geq 2$. We choose a point $q_{i,s}$ from each hyperplane $B^{n-1}_{i,s}$ so that $Q = \{q_{i,s} : i, s \in \mathbb{N}\}$ is discrete in $\mathbb{R}^n$. Then we define

$$X^n_m = \begin{cases} S^n_m(\alpha) & (0 < m \leq n-1) \\ S^n_0(\alpha) \cup Q & (m = 0) \end{cases}$$

Obviously we have $\dim X^n_0 = 0$ because $\dim S^n_0(\alpha) = \dim Q = 0$ and $Q$ is closed. Then it is evident that $X^n_m$ satisfies all of the required conditions in view of Lemma 3. □

Let $N^n_k$ be the space of those points in $\mathbb{R}^n$ at most $k$ of whose coordinates are rationals. It is known that $\dim N^n_k = k$(cf.[1]) and $\mu\dim N^n_k = k$ because $N^n_k$ contains a $k$-simplex. Moreover for every hyperplane $H$ in $\mathbb{R}^n$, we have

(3) $k - 1 \leq \mu\dim(N^n_k \cap H) \leq \dim(N^n_k \cap H) \leq k,\ 0 \leq k \leq n.$

Also it is obvious that $N^n_m \subset S^n_m(\alpha)$ for every sequence $\alpha$ of points in $\mathbb{Q}^n$. Let $A^{n-k-1}_i$ be the $(n-k-1)$-planes such that

(4) $N^n_k = \mathbb{R}^n - \cup\{A^{n-k-1}_i : i \in \mathbb{N}\},\ 0 \leq k \leq n-1.$

We denote by $\mathcal{H}_0 = \{H_i : i \in \mathbb{N}\}$ the family of all hyperplanes in $\mathbb{R}^n$ which are determined by points in $\mathbb{Q}^n$. Moreover we set $\mathcal{A}^{n-k-1}_H = \{A^{n-k-1}_i : A^{n-k-1}_i \subset H\}$ for arbitrary hyperplanes $H$ in $\mathbb{R}^n$. Since every non-empty open set in $A^{n-k-1}_i$ contains points in $\mathbb{Q}^n$ densely, we have

**Lemma 4.** *If* $U$ *is a non-empty open set in a hyperplane* $H$ *in* $\mathbb{R}^n$ *such that* $U \cap (\cup\mathcal{A}^{n-k-1}_H)$ *is dense in* $U$ *for some* $k$ *with* $0 \leq k \leq n-1$, *then* $H \in \mathcal{H}_0$.

**Theorem 2.** *Let* $n$ *and* $k$ *be integers such that* $0 \leq k \leq n-1 \geq 1$. *Then there exists a space* $Y^n_k$ *in* $\mathbb{R}^n$ *such that*
i) $N^n_k \subset Y^n_k \subset N^n_{k+1}$,
ii) $\mu\dim Y^n_k = \dim Y^n_k = k$, *and*
iii) $\mu\dim(Y^n_k \cap H) = \dim(Y^n_k \cap H) = k$ *for every hyperplane* $H$ *in* $\mathbb{R}^n$.

**Proof.** First we choose a sequence $\{z_i\}$ of points in $\mathbb{Z}^n$ such that $H_i \cap \mathrm{Int}(z_i + \mathbb{I}^n) \neq \emptyset$ and $\{z_i + \mathbb{I}^n : i \in \mathbb{N}\}$ is discrete in $\mathbb{R}^n$. Then we can take a $k$-simplex $\sigma^k_i \subset N^n_{k+1} \cap (z_i + \mathbb{I}^n) \cap H_i$ for every $i$. We set $Y^n_k = N^n_k \cup \cup\{\sigma^k_i : i \in \mathbb{N}\}$. Then obviousely i) and ii) are satisfied. To prove iii), let $H$ be an arbitrary hyperplane with $H \notin \mathcal{H}_0$. Then by Lemma 4, $\cup\mathcal{A}^{n-k-1}_H$ is not dense in $H$. Hence there exists a non-empty open set $U$ in $H$ such that $U \cap (\cup\mathcal{A}^{n-k-1}_H) = \emptyset$. Then it is clear that $N_k^n \cap U$ contains a $k$-simplex and hence $\mu\text{dim}\,(Y_k^n \cap H) = \dim\,(Y_k^n \cap H) = k$.□

In the above proof, it has been proved that

(5) $\mu\text{dim}\,(N_k^n \cap H) = \dim\,(N_k^n \cap H)$

holds if $H \notin \mathcal{H}_0$; however, as shown in the following, the condition $H \notin \mathcal{H}_0$ can be dropped.

**Theorem 3.** *For every hyperplane $H$ in $\mathbb{R}^n$, $\mu\text{dim}\,(N_k^n \cap H) = \dim(N_k^n \cap H)$, $0 \le k \le n-1$.*

**Proof.** Let $H$ be the hyperplane defined by $\sum_{i=1}^{n} a_i x_i = b$. We set $\lambda = \{i : a_i \neq 0\}$; here we may assume $n \in \lambda$. If in particular, $\lambda = \{n\}$, then $N_k^n \cap H$ is a copy of $N_{k-1}^{n-1}$ or $N_k^{n-1}$ according as $b/a_n \in \mathbf{Q}$ or not, and (5) follows. Hence we can assume that $s = |\lambda| \ge 2$. Then we claim that

(6) $\dim\,(N_k^n \cap H) = \mu\text{dim}\,(N_k^n \cap H) = k$.

Let $\pi_0 : \mathbb{R}^n \to \mathbb{R}^s = \prod\{\mathbb{R}_i : i \in \lambda\}$, $\mathbb{R}_i = \mathbb{R}$, be the projection. Also by $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ we denote the projection defined by $\pi(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$. Then it is obvious

(7) $\pi|_H : H \to \mathbb{R}^{n-1}$ is a uniform isomorphism, and

(8) if $A_i^{n-k-1} \subset H$, then $A_i^{n-k-1} \subset \pi_0^{-1}(p) \subset H$ for some $p \in \mathbf{Q}^s \cap \pi_0(H)$.

Let us set

$$\mathcal{A} = \{\pi(\pi_0^{-1}(p)) \cap \mathbb{I}^{n-1} : p \in \pi_0(H) \cap \mathbf{Q}^s\} \cup \{\pi(A_i^{n-k-1} \cap H) \cap \mathbb{I}^{n-1} : A_i^{n-k-1} \not\subset H\},$$

where $\mathbb{I}^{n-1}$ is the $(n-1)$-cube in $\mathbb{R}^{n-1}$. Then $\mathcal{A}$ is a countable family of closed sets in $\mathbb{I}^{n-1}$ and clearly we have

$$\dim\,(\pi(\pi_0^{-1}(p)) \cap \mathbb{I}^{n-1}) \le n - s \le n-2 \quad \text{for } p \in \pi_0(H) \cap \mathbf{Q}^s, \text{ and}$$

$$\dim\,(\pi(A_i^{n-k-1} \cap H) \cap \mathbb{I}^{n-1}) \le n-k-2 \quad \text{if } A_i^{n-k-1} \not\subset H.$$

Morever it is impossible that $\pi(\pi_0^{-1}(p)) \cap \mathbb{I}^{n-1}$ intersects with all of the $(n-2)$-faces of $\mathbb{I}^{n-1}$; indeed, $\pi(\pi_0^{-1}(p))$ is pararell to at least one of the $(n-2)$-faces. The same is true as for $\pi(A_i^{n-k-1} \cap H) \cap \mathbb{I}^{n-1}$. Also the following are valid:

$$\pi(\pi_0^{-1}(p)) \cap \pi(\pi_0^{-1}(q)) = \emptyset \quad \text{if } p \neq q \quad (p,\ q \in \mathbf{Q}^s \cap \pi_0(H)).$$

$$\dim(\pi(A_i^{n-k-1}\cap H)\cap\pi(A_j^{n-k-1}\cap H))\leq n-k-3 \text{ if } A_i^{n-k-1}\cap H\neq A_j^{n-k-1}\cap H$$
$$(A_i^{n-k-1}\not\subset H \text{ and } A_j^{n-k-1}\not\subset H).$$

$$\dim(\pi(\pi_0^{-1}(p))\cap\pi(A_i^{n-k-1}\cap H))\leq n-k-3 \text{ if } A_i^{n-k-1}\cap H\not\subset\pi_0^{-1}(p)$$
$$(p\in\mathbf{Q}^s\cap\pi_0(H) \text{ and } A_i^{n-k-1}\not\subset H).$$

Hence by [5, Theorem 2] we obtain

$$\mu\dim(\mathbb{I}^{n-1}-\cup\mathcal{A})\geq n-1-(n-k-3)-2=k.$$

Since $\mathbb{I}^{n-1}-\cup\mathcal{A}\subset\pi(N_k^n\cap H)$ by (8), we have $\mu\dim(N_k^n\cap H)=\mu\dim\pi(N_k^n\cap H)\geq k$ by virtue of (7), which proves (6).□

## 4. Proof of Main theorem

Hereafter we fix integers $n,m$ and $k$ satisfying the following:

$$0\leq m\leq n-1\geq 1 \text{ and } m\leq k\leq\min\{2m,n-1\}.$$

**Lemma 5**(cf.[3]). *Let $\alpha$ be a sequence of points in $\mathbb{R}^n$. Then*
*i) $\mu\dim(S_m^n(\alpha)\cap N_k^n)=m$, and*
*ii) if in particular, $\alpha$ satisfies the condition $(C_i)$ for all $i\geq 2$, then $\dim(S_m^n(\alpha)\cap N_k^n)=k$.*

**Lemma 6.** *Let $k\geq m+1$ and $H$ a hyperplane in $\mathbb{R}^n$. Then $\mu\dim(S_m^n(\alpha)\cap N_k^n\cap H)=m$ for every $\alpha$ satisfying $(C_i)$ for all $i\geq 2$.*

Henceforth we fix an $\alpha=\{a_i\}$ in $\mathbf{Q}^n$ satisfying $(C_i)$ for all $i\geq 2$. Let us define

$$F_{i,j}=(a_i+F_i^{(n-m-1)})\cap(a_j+F_j^{(n-m-1)}),\ i\neq j.$$

In case $2m\leq n-2$, there exists a collection $\mathcal{F}_{i,j}$ of $(n-2m-2)$-planes such that $\cup\mathcal{F}_{i,j}=F_{i,j}$, and we set $\mathcal{F}_{i,j}=\emptyset$ if $2m\geq n-1$.

**Lemma 7.** *For every hyperplane $H$ in $\mathbb{R}^n$ with $H\notin\mathcal{H}_0$, we have*

$$\dim(S_m^n(\alpha)\cap N_k^n\cap H)=k$$

*if either $k\leq n-2$ or $k=m$.*

**Lemma 8.** $k-1\leq\dim(S_m^n(\alpha)\cap N_k^n\cap H)\leq k$ *for every hyperplane $H$ in $\mathbb{R}^n$.*

**Proof of Main theorem.** As in the proof of Theorem 2 we take a sequence $\{z_i\}$ of points in $\mathbb{Z}^n$ such that $H_i\cap\mathrm{Int}(z_i+\mathbb{I}^n)\neq\emptyset$ and $\{z_i+\mathbb{I}^n : i\in\mathbb{N}\}$ is discrete in $\mathbb{R}^n$. Then by

Lemma 8 we can define $J_i$ with $\dim J_i = k$ where $J_i = S^n_m(\alpha) \cap N^n_k \cap (z_i + \mathbb{I}^n) \cap H_i$ or $J_i = S^n_m(\alpha) \cap N^n_{k+1} \cap (z_i + \mathbb{I}^n) \cap H_i$. We set

$$X^n_{m,k} = \begin{cases} (S^n_m(\alpha) \cap N^n_k) \cup \cup\{J_i : i \in \mathbb{N}\} & (m+1 \leq k \leq n-2) \\ S^n_m(\alpha) \cap N^n_{n-1} & (m+1 \leq k = n-1) \\ Y^n_k & (m = k \leq n-1) \end{cases}$$

where $Y^n_k = N^n_k \cup \cup\{\sigma^k_i : i \in \mathbb{N}\}$ and each $\sigma^k_i$ is a $k$-simplex contained in $N^n_{k+1} \cap H_i \cap (z_i + \mathbb{I}^n)$ (cf. Theorem 2); here it is possible to choose $\sigma^k_i$ so that $\sigma^k_i \subset S^n_k(\alpha) \cap N^n_{k+1} \cap H_i \cap (z_i + \mathbb{I}^n)$ (cf. Lemma 8). Then we have

$$S^n_m(\alpha) \cap N^n_k \subset X^n_{m,k} \subset S^n_m(\alpha) \cap N^n_{k+1},$$

which implies $\mu\dim X^n_{m,k} = k$ by Lemma 5. Also we have $\dim X^n_{m,k} = k$ by Lemma 5 and Theorem 2; we note that each $J_i$ is closed and $\{J_i : i \in \mathbb{N}\}$ is discrete in $X^n_{m,k}$ in case $m+1 \leq k \leq n-2$. Thus the condition i) of Main theorem is satisfied. Moreover the remaining conditions ii) and iii) follow from Lamma 6, Lemma 7 and Theorem 2 directly. This completes the proof of Main theorem.□

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