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INDICES AND RESIDUES OF HOLOMORPHIC VECTOR FIELDS ON SINGULAR VARIETIES

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My talk at the RIMS conference summarized the recent joint work with D. Lehmann and M. Soares [LSS] (see also [LS2]).

We give a differential geometric definition of the residues, which include the index defined in [GSV] (see also [Se], [BG], [G], [SS]) as a special case, of a holomorphic vector field tangent a singular variety and also integral formulas to compute them. The method is a generalization of the one initiated in [L].

Let V be a pure p dimensional reduced subvariety of a complex manifold W of dimension n. Assume that V is a local complete intersection. Thus the normal bundle $N_{V'}$ of its regular part V' extends (canonically) to a vector bundle N_V on V and we have a commutative diagram of vector bundles on V with an exact row

$$\begin{array}{ccccccccc} & & & & TW|_V & \xrightarrow{\ \pi\ } & N_V & & \\ & & & & \uparrow{\scriptstyle \text{incl.}} & & \uparrow{\scriptstyle \text{incl.}} & & \\ 0 & \longrightarrow & TV' & \longrightarrow & TW|_{V'} & \longrightarrow & N_{V'} & \longrightarrow & 0. \end{array}$$

Suppose, furthermore, that V is a "strong" local complete intersection in the sense of [LS1], i.e., N_V still extends to a (C^∞) vector bundle on a neighborhood of V in W. This class of varieties include, beside the non-singular ones, every hypersurface with a natural holomorphic extension of N_V (the line bundle on W determined by the divisor V), every complete intersection with a trivial extension of N_V and every complete intersection in the projective space with a holomorphic extension of N_V depending only on the degrees of polynomials defining V. See [LS1] for more details.

Suppose we have a holomorphic vector field X on W leaving V invariant and define the singular set Σ to be the set of singular points of X on V and singular points of V; $\Sigma = (\mathrm{Sing}(X) \cap V) \cup \mathrm{Sing}(V)$. For each compact component of Σ, we may define the residues, which are localized characteristic classes of the virtual tangent bundle $TW|_V - N_V$ of V.

First we consider the case of isolated singularities. Let P be an isolated point of Σ and $f_1, \dots, f_q$, $q = n - p$, local defining functions for V near P. The invariance condition for V by X is given by

$$Xf_i = \sum_{j=1}^{q} c_{ij} f_j, \quad i = 1, \dots, q,$$

with c_{ij} holomorphic functions near P ([Sa], [BR]). We set $C = (c_{ij})$, a $q \times q$ matrix. Then we have the following lemma ([LS1] Theorem 2).

Lemma 1. *There exists a local coordinate system $(z_1, ..., z_n)$ near P in W such that, if we express X as*

$$X = \sum_{i=1}^{n} a_i(z_1, \dots, z_n) \frac{\partial}{\partial z_i},$$

the sequence $(a_1, ..., a_p, f_1, ..., f_q)$ is regular, i.e., the set of common zeros of the holomorphic functions $a_1, ..., a_p, f_1, ..., f_q$ consists only of P.

Letting $J = \frac{\partial(a_1, \dots, a_n)}{\partial(z_1, \dots, z_n)}$ be the Jacobian matrix, we denote by $[c(-J) \cdot c(-C)^{-1}]_k$ the holomorphic function given as the coefficient of t^k in the formal power series expansion of $\det\left(I - t\frac{\sqrt{-1}}{2\pi}J\right) \cdot \det\left(I - t\frac{\sqrt{-1}}{2\pi}C\right)^{-1}$ in t.

Theorem 1. *We take a coordinate system as in Lemma 1 and set*

$$\mathrm{Ind}_{V,P}(X) = \int_\Gamma \frac{[c(-J) \cdot c(-C)^{-1}]_p dz_1 \wedge dz_2 \wedge \cdots \wedge dz_p}{a_1 a_2 \cdots a_p}.$$

Here Γ denotes the p-cycle in V given by

$$\Gamma = \{\, z \mid |a_1(z)| = \cdots = |a_p(z)| = \varepsilon,\ f_1(z) = \cdots = f_q(z) = 0 \,\},$$

for a small positive number ε, which is oriented so that $d\theta_1 \wedge \cdots \wedge d\theta_p$ is positive, $\theta_i = \arg a_i$. Then
(i) $\mathrm{Ind}_{V,P}(X)$ *coincides with the index defined in [GSV].*
(ii) *If V is compact and if Σ consists of isolated points, we have*

$$\sum_{P \in \Sigma} \mathrm{Ind}_{V,P}(X) = \int_V c_p(TW|_V - N_V).$$

To state more general results, we briefly recall the Chern-Weil theory of characteristic classes. Let $E \to M$ be a complex vector bundle of rank r on a (C^∞) manifold M. For a connection ∇ for E and a Chern polynomial $\varphi \in \mathbb{C}[c_1, \dots, c_r]$, homogeneous of degree d ($\deg c_i = i$), we have a closed $2d$-form $\varphi(\nabla)$ on M representing the characteristic class $\varphi(E)$ in the de Rham cohomology. Moreover, if we have a finite number of connections $\nabla_0, \dots, \nabla_k$ for E, there is a $2d-k$-form $\varphi(\nabla_0, \dots, \nabla_k)$ such that

$$\sum_{i=1}^{k} \varphi(\nabla_0, \dots, \hat{\nabla}_i, \dots, \nabla_k) + (-1)^k d\varphi(\nabla_0, \dots, \nabla_k) = 0$$

(see [B2]).

Now let V, W, X and Σ be as before. The key fact in localizing the characteristic classes of the virtual tangent bundle $TW|_V - N_V$ is that the bundles $TW|_V$ and N_V admit an "X-action" on $V - \Sigma$ in the sense of [B1]: for $TW|_V$, $Y \mapsto [X, Y]$ and for N_V, $\pi(Y) \mapsto \pi([X, Y])$. Thus there exist "special connections" for $TW|_V$ and N_V.

Lemma 2 (Vanishing theorem). *Let $\nabla_1, \dots, \nabla_s$ be special connecctions for $TW|_{V-\Sigma}$ and $\nabla_1, \dots, \nabla_{s'}$ special connecctions for $N_{V-\Sigma}$. Also, let $\varphi \in \mathbb{C}[c_1, \dots, c_n]$ and $\varphi' \in \mathbb{C}[c_1, \dots, c_q]$ be homogeneous Chern polynomials. If $\deg \varphi + \deg \varphi' = p$, then we have*

$$\varphi(\nabla_1, \dots, \nabla_s) \wedge \varphi'(\nabla'_1, \dots, \nabla'_{s'}) = 0.$$

This lemma in particular implies that the cup product $\varphi(TW|_V) \smile \varphi'(N_V)$ of characteristic classes vanishes over $V - \Sigma$. Thus this product "localizes" near Σ, in the sense that it has a natural lift to $H^{2p}(V, V - \Sigma)$ giving rise to residues in $H_0(\Sigma)$ by duality when Σ is compact. In fact this is done as follows.

Let Σ_0 be a compact connected component of Σ and U_0 an open neighborhood of Σ_0 in W such that $V_0 - \Sigma_0$ is in the regular part of V, $V_0 = U_0 \cap V$. Also, let $\tilde{\mathcal{T}}$ be a compact (real) manifold of dimension $2n$ with boundary in U_0 such that Σ_0 is in the interior of $\tilde{\mathcal{T}}$ and that the boundary $\partial\tilde{\mathcal{T}}$ is transverse to V. We write $\mathcal{T} = \tilde{\mathcal{T}} \cap V$ and $\partial\mathcal{T} = \partial\tilde{\mathcal{T}} \cap V$. We take an arbitrary connection ∇_0 for TW on U_0 and a special connection ∇ for $TW|_{V_0 - \Sigma_0}$. Take also ∇'_0 and ∇' similarly for an extension of N_V and $N_V|_{V_0 - \Sigma_0}$.

Let

$$\rho : \mathbb{C}[c_1, \dots, c_p] \longrightarrow \mathbb{C}[c_1, \dots, c_n] \otimes \mathbb{C}[c'_1, \dots, c'_q]$$

be the homomorphism which assigns, to c_i, the i-th component of the element $(1 + c_1 + \cdots + c_n)(1 + c'_1 + \cdots + c'_q)^{-1}$ (with the terms of sufficiently large degree truncated). For a polynomial $\varphi \in \mathbb{C}[c_1, \dots, c_p]$, we may write $\varphi = \sum_i \varphi_i \varphi'_i$ with $\varphi_i \in \mathbb{C}[c_1, \dots, c_n]$ and $\varphi'_i \in \mathbb{C}[c'_1, \dots, c'_q]$

Lemma 2. *Let φ be a polynomial in $\mathbb{C}[c_1, \dots, c_p]$ homogeneous of degree p. If we define the residue $\operatorname{Res}_\varphi(TW|_V, N_V; \Sigma_0)$ by*

$$\operatorname{Res}_\varphi(TW|_V, N_V; \Sigma_0) = \sum_i \left(\int_{\mathcal{T}} \varphi_i(\nabla_0)\varphi'_i(\nabla'_0) - \int_{\partial\mathcal{T}} (\varphi_i(\nabla)\varphi'_i(\nabla', \nabla'_0) + \varphi_i(\nabla, \nabla_0)\varphi'_i(\nabla'_0)) \right),$$

then
(i) *This number does not depend on the choices of* $\tilde{\mathcal{T}}$, ∇, ∇_0, ∇', *and* ∇'_0.
(ii) *Assume* V *to be compact and let* $(\Sigma_\alpha)_\alpha$ *be the partition of* Σ *into connected components. Then, we have*

$$\sum_\alpha \mathrm{Res}_\varphi(TW|_V, N_V; \Sigma_\alpha) = \int_V \varphi(TW|_V - N_V).$$

Note that if Σ_0 is in the regular part of V, the residue $\mathrm{Res}_\varphi(TW|_V, N_V; \Sigma_0)$ coincides with that of P. Baum and R. Bott ([BB1], [BB2]) of X for φ at Σ_0.

Now we suppose Σ_0 consists of an isolated point P. In general, for an $r \times r$ matrix A, we define $c_i(A)$, $i = 1, \ldots, r$, by

$$\det\left(I + t\frac{\sqrt{-1}}{2\pi}A\right) = 1 + tc_1(A) + \cdots + t^r c_r(A).$$

Thus, for a polynomial φ in $\mathbb{C}[c_1, \ldots, c_r]$, we may also define $\varphi(A)$, which is a holomorphic function, if A is a matrix with holomorphic entries.

Theorem 2. *If we take a coordinate system* $(z_1, \ldots, z_n)$ *as in Lemma 1, for a homogeneous polynomial* φ *of degree p, we have*

$$\mathrm{Res}_\varphi(TW|_V, N_V; P) = \sum_i \int_\Gamma \frac{\varphi_i(-J)\varphi'_i(-C)dz_1 \wedge \cdots \wedge dz_p}{a_1 \cdots a_p},$$

where Γ *denotes the p-cycle as in Theorem 1.*

Note that $\mathrm{Res}_{c_p}(TW|_V, N_V; P) = \mathrm{Ind}_{V,P}(X)$.

As we have seen in the above theorems, we encounter integrals of the form

$$\int_\Gamma \frac{h(z)\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_p}{a_1 a_2 \cdots a_p},$$

where Γ denotes a p-cycle as in Theorem 1. We give a formula for this integral in the case V is a hypersurface and the system $(a_1, \ldots, a_p)$ is "non-degenerate" in the following sense. We denote by $\mathcal{O}_n$ the ring of germs of holomorphic functions at the origin 0 in $\mathbb{C}^n$ and let $(z_1, \ldots, z_n)$ be a coordinate system near 0 in $\mathbb{C}^n$. Also, let $a_1, \ldots, a_{n-1}$ be germs in $\mathcal{O}_n$ vanishing at 0 and V a germ of hypersurface with isolated singularity at 0 in $\mathbb{C}^n$ with defining function f. We further assume:
(i) $\det\left(\frac{\partial(a_1,\ldots,a_{n-1})}{\partial(z_1,\ldots,z_{n-1})}\right)(0) \neq 0$, thus $(a_1, \ldots, a_{n-1}, z_n)$ form a coordinate system.
(ii) Each a_i, $i = 1, \ldots, n-1$, depends only on $z_1, \ldots, z_{n-1}$.
(iii) In the coordinate system $(a_1, \ldots, a_{n-1}, z_n)$, f is regular in z_n. We denote by ℓ the order of f in z_n.

Note that the condition (iii) implies that $(a_1, \dots, a_{n-1}, f)$ is a regular sequence. Denoting by Γ the $(n-1)$-cycle in V given by

$$\Gamma = \{ z \in V \mid |a_1(z)| = \cdots = |a_{n-1}(z)| = \varepsilon \},$$

for a small positive number ε, which is oriented so that $d\theta_1 \wedge \cdots \wedge d\theta_{n-1}$ is positive, $\theta_i = \arg a_i$, we have the following formula.

Proposition. *In the above situation, we have, for a holomorphic function h near 0,*

$$\left(\frac{1}{2\pi\sqrt{-1}} \right)^{n-1} \int_\Gamma \frac{h(z)\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n-1}}{a_1 a_2 \cdots a_{n-1}} = \frac{\ell \cdot h(0)}{\det \left(\frac{\partial(a_1, \dots, a_{n-1})}{\partial(z_1, \dots, z_{n-1})} \right)(0)}.$$

The above formula is proved in [LS2] under a weaker condition.

Let W, V, X and Σ be as before. Here we assume that V is a hypersurface. For an isolated point P in Σ, under the additional conditions above, we may compute the residues in Theorem 2, by the formula in the above Proposition.

Let V be defined by f near P and $(z_1, \dots, z_n)$ a coordinate system about P. We write $X = \sum_{i=1}^n a_i \frac{\partial}{\partial z_i}$ and assume that the conditions (i), (ii) and (iii) are satisfied. Note that the eigenvalues of $\frac{\partial(a_1, \dots, a_{n-1})}{\partial(z_1, \dots, z_{n-1})}(0)$ are part of those of $J(0) = \frac{\partial(a_1, \dots, a_n)}{\partial(z_1, \dots, z_n)}(0)$. So let $\lambda_1, \dots, \lambda_{n-1}$ and $\lambda_1, \dots, \lambda_{n-1}, \lambda_n$ be the ones for these matrices. By (i), $\lambda_1, \dots, \lambda_{n-1}$ are all non-zero, while λ_n may be zero. Since $q = 1$ in this case, C is a 1×1 matrix. We set $\gamma = C(0)$.

In what follows, for complex numbers $\lambda_1, \dots, \lambda_r$, we define $c_i(\lambda_1, \dots, \lambda_r)$, $i = 1, \dots, r$, by

$$\prod_{i=1}^r (1 + t\lambda_i) = 1 + tc_1(\lambda_1, \dots, \lambda_r) + \cdots + t^r c_r(\lambda_1, \dots, \lambda_r).$$

Thus for a polynomial φ in $\mathbb{C}[c_1, \dots, c_r]$, we may define $\varphi(\lambda_1, \dots, \lambda_r)$.

By the above proposition, for a polynomial φ in $\mathbb{C}[c_1, \dots, c_{n-1}]$ homogeneous of degree $n-1$, the residue in Theorem 2 is given by

$$\mathrm{Res}_\varphi(TW|_V, N_V; P) = \ell \cdot \sum_{i=0}^{n-1} \frac{\varphi_i(\lambda_1, \dots, \lambda_n)\gamma^i}{\lambda_1 \cdots \lambda_{n-1}},$$

where, for each $i = 0, \dots, n-1$, φ_i is a polynomial in $\mathbb{C}[c_1, \dots, c_n]$, homogeneous of degree $n-i-1$, determined by $\rho(\varphi) = \sum_{i=0}^{n-1} \varphi_i \cdot (c_1')^i$. In particular, for $\varphi = c_{n-1}$, we have

$$\mathrm{Ind}_{V'P}(X) = \ell \cdot \frac{\lambda_1 \cdots \lambda_n - (\lambda_1 - \gamma) \cdots (\lambda_n - \gamma)}{\lambda_1 \cdots \lambda_{n-1}\, \gamma}.$$

If $\gamma = 0$, the right hand side in the above is understood to be the limit as γ approaches 0.

Example. Let V be a hypersurface in $\mathbb{C}^n = \{(z_1, \ldots, z_n)\}$ defined by a weighted homogeneous polynomial f of type $(d_1, ..., d_n)$ with isolated singularity at the origin 0. For the holomorphic vector field $X = \sum_{i=1}^n \frac{z_i}{d_i} \frac{\partial}{\partial z_i}$, we have $X(f) = f$ and thus V is invariant by X. We asssume that f is regular in z_n. This implies that d_n is a positive integer and f is regular in z_n of order d_n. If we let $a_i = \frac{z_i}{d_i}$, $i = 1, \ldots, n$, $(a_1, \ldots, a_{n-1}, f)$ is a regular sequence and the conditions (i), (ii) and (iii) are satisfied. We have $\ell = d_n$, $\lambda_i = \frac{1}{d_i}$ and $\gamma = 1$. Hence we have

$$\mathrm{Res}_\varphi(TW|_V, N_V; P) = \sum_{i=0}^{n-1} \varphi_i \left(\frac{1}{d_1}, \ldots, \frac{1}{d_n} \right) d_1 \cdots d_n,$$

where, for each $i = 0, \ldots, n-1$, φ_i is a polynomial in $\mathbb{C}[c_1, \ldots, c_n]$, homogeneous of degree $n - i - 1$, determined by $\rho(\varphi) = \sum_{i=0}^{n-1} \varphi_i \cdot (c_1')^i$. In particular, for $\varphi = c_{n-1}$, we have

$$\mathrm{Ind}_{V'P}(X) = 1 + (-1)^{n-1}(d_1 - 1)(d_2 - 1) \cdots (d_n - 1).$$

Note that, since X is transversal to the boundary of the Milnor fiber of f, $\mathrm{Ind}_{V,P}(X)$ is also equal to the Euler number $1 + (-1)^{n-1}\mu$ of the Milnor fiber, where μ denotes the Milnor number of f at 0. Thus we reprove the formula

$$\mu = (d_1 - 1)(d_2 - 1) \cdots (d_n - 1)$$

for the Milnor number ([MO] Theorem 1).

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