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CURVATURE OF CURVILINEAR 4–WEBS AND PENCILS OF ONE FORMS

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ABSTRACT. A curvilinear n-web $W = (F_1, \ldots, F_n)$ is a configuration of n curvilinear foliations F_i on a surface. When $n = 3$, Bott connections of F_i extend naturally to a unique affine connection, which is called Chern connection. For $3 < n$, this is the case if and only if the modulus of tangents to the leaves of F_i at a point is constant. An n-web is *associative* if the modulus is constant and *weakly associative* if Chern connections of all 3-subwebs have equal curvature form. We give a geometric interpritation of the curvature form in terms of fake billiard in §2, and prove that a weakly associative n-web is associative if Chern connections of triples of the members are not flat, and then the foliations are members of a pencil (linear family of dim 2) of 1-forms. This result completes the classification of weakly associative 4-webs initiated by Poincaré, Mayrhofer and Reidemeister for the flat case.

A*curvilinear n-web* on a surface S is an n-tpule of foliations of codimension 1, $W = (F_1, \ldots, F_n)$. In this paper we assume that S is real analytic and connected and F_i is defined by a real meromorphic 1-form ω_i: of which coefficients are locally fractions of real analytic functions. W is *non singular* at a $p \in S$ if ω_i and $\omega_i \wedge \omega_j$ are analytic and non zero at p for $i \neq j$. $\Sigma(W)$ denotes the set of those p where W is singular. W is *diffeomorphic* to an n-web $W' = (F'_1, \ldots, F'_n)$ on S' if there exists an analytic diffeomorphism of S to S' sending F_i to F'_i for $i = 1, \ldots, n$. An m-subweb of W is an m-tuple of members of W.

First let $n = 3$ and assume W is non singular at p. Since the defining 1-forms $\omega_i, i = 1, 2, 3$ on a surface are linearly dependent, we may assume $\omega_1 - 2\omega_2 + \omega_3$ vanishes identitically on a neighbourhood of p. Then there exists a unique 1-form θ on the complement of $\Sigma(W)$ such that $d\omega_i = \theta \wedge \omega_i$ for $i = 1, 2, 3$ [1]. The exterior derivative $d\theta$ is independent of the forms ω_i defining F_i as well as the permutation of the suffix i. $d\theta$ is called the *web curvature form* of W and denoted $K(W)$. Bott connections of F_1, F_2, F_3 defined by the transverse dynamics extend to unique affine connection without torsion the so-called Chern connection on the complement of $\Sigma(W)$ (see §1 for the definition). And the leaves of F_i are geodesics of the connection. Chern connection has the connection form $\Theta = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}$ with respect to the coframe ω_1, ω_2 and the curvature form $d\Theta = \begin{pmatrix} K(W) & 0 \\ 0 & K(W) \end{pmatrix}$ [1,5,10]. A 3-web is *hexagonal* (or *flat*) if the web curvature vanishes identitically. It is classically known that a hexagonal 3-web is locally diffeomerphic to the 3-web by parallel lines on the plane (see Fig 0. and §1).

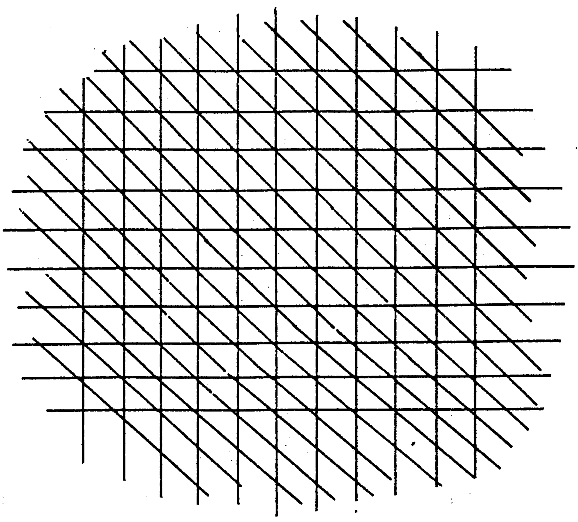


Fig. 0

A non singular 4-web $W = (F_1, \ldots, F_4)$ possesses the relative and absolute invariants: *web curvature forms of 3-subwebs*and the cross ratio of tangents to the leaves of W passing to a point which is a special case of the basic affinor in higher dimensional webs (see [5] for the definition). The higher covariant derivatives of the cross ratio generate all other absolute invariants [3,4].

We call n curvilinear foliations as well as an n-web are *associative* if their Bott connections extend to equal affine connection, in other words, all 3-subwebs have equal Chern connection on the complement of the singular locus. It is easy to see that n foliations $(3 < n)$ are associative if and only if the modulus of tangents to the leaves passing through a point is constant. Clearly if an n-web is associative, it is weakly associative, i.e. Chern connections of 3-subwebs have equal curvature form. But the converse is not true in general. Poincaré [3], Mayrhofer [8,9] and Reidemeister [11] proved

Theorem 0. *Let W be a germ of non singular 4-web on a surface. Assume that all 3-subwebs are hexagonal. Then W is diffeomorphic to a germ of the 4-web formed by 4 pencils of lines on the plane (Fig.1).*

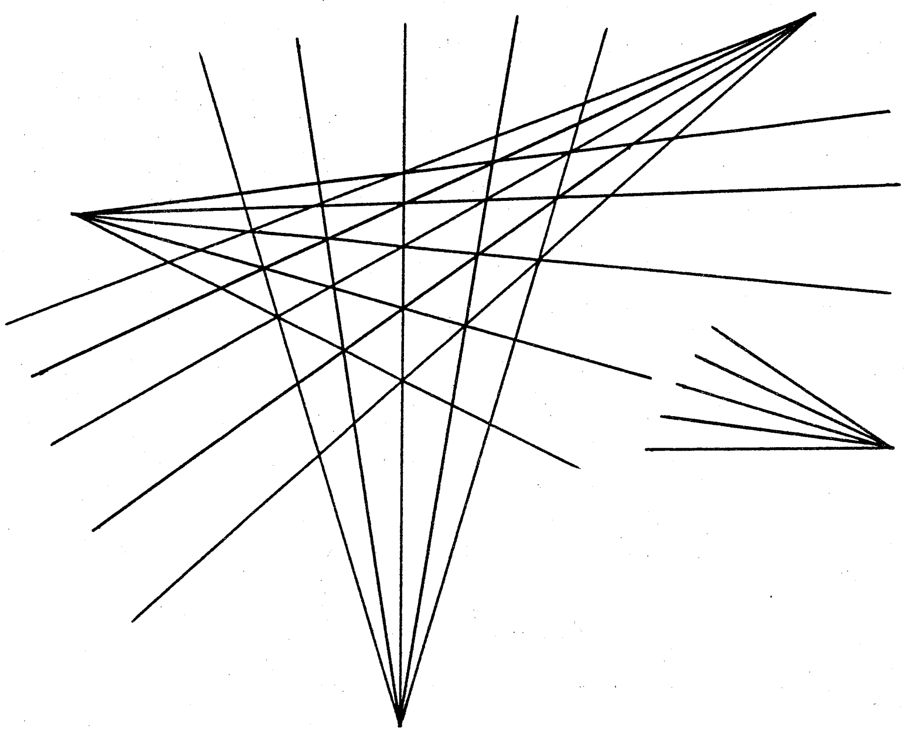


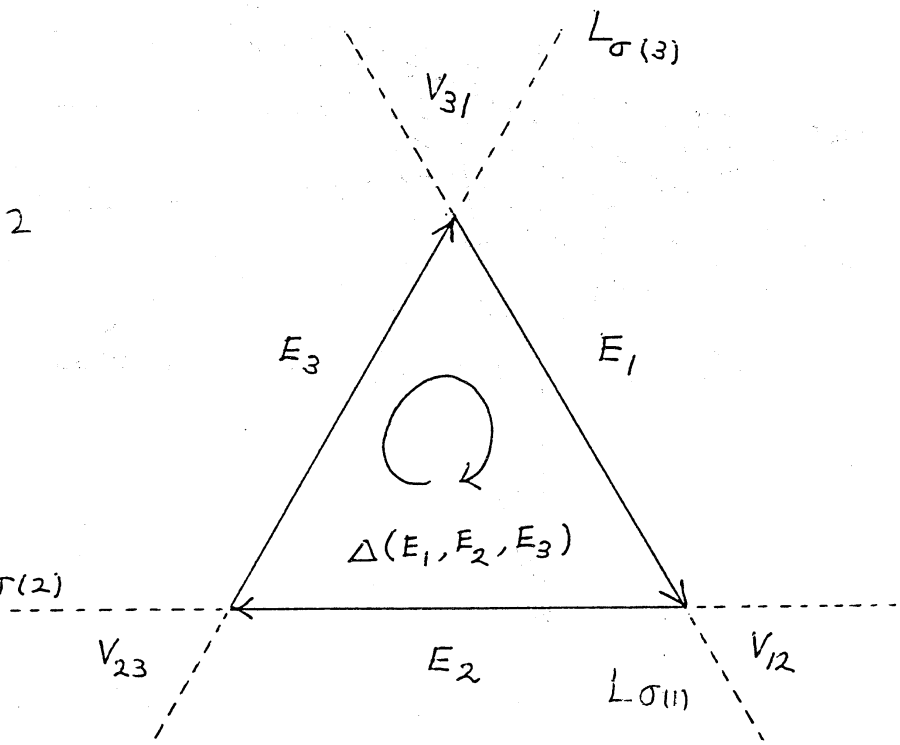
Fig. 1

3-subwebs of the 4-web in the theorem are hexagonal (curvature vanishes), but their Chern connections are not equal. Henaut[7] gives a simple proof of Theorem 0. Goldberg [4] proved a similar result by a different approach. This paper is devoted to finding all weakly associative n-webs.

Before stating our result we prepare some notions. A *pencil of meromorphic one forms* $P = \{\omega_t\}, \omega_t = (1-t)\omega_0 + t\omega_1, t \in \mathbb{R}$ defined on S is *non singular* at p if ω_s and $\omega_s \wedge \omega_t$ are analytic and non zero at p for distinct s, t. We denote the set of those p where P is singular by $\Sigma(P)$. The *web* *curvature form* $K(P)$ *for* P is the 2-form $d\theta$ defined on the complement of $\Sigma(P)$, where θ is the unique 1-form called the *connection form of* Psuch that $d\omega_t = \theta \wedge \omega_t$. Clearly all members of P are associative and all triples of the members form 3-webs which have equal web curvature form $K(P)$. Cerveau [2] and Ghys [4] and the author [10] applied the web geometry of 3-webs of codimension 1 to classify codimension 2 foliations of 3 manifolds.

Let $W = (F_1, F_2, F_3)$ be a non singular 3-web on S. In this paper *geodesics mean the leaves of the foliations. A geodesic triangle* is a smooth triangle $\Delta = \Delta(E_1, E_2, E_3)$ with the edges E_i in a leaf $L_{\sigma(i)} \in F_{\sigma(i)}$ for $i = 1, 2, 3$ transversal at the vertices $V_{jk} = E_j \cap E_k, j \neq k$. Here σ is a permutation of $\{1, 2, 3\}$ and the convention $E_{i+3} = E_i$ is used. The *orientation* of the Δ and the edge E_i are given by $\partial\Delta = E_1 + E_2 + E_3$ and $\partial E_i = V_{i,(i+1)} - V_{(i-1),i}$ (see Fig. 2). Define $\sigma(\Delta) = 1$ or -1 alternatively if the orientation is clockwise or anti-clockwise.

Fig. 2



From later on we assume the permutation σ is trivial unless otherwise stated.

Let $W = (F_1, F_2, F_3, F_4)$ be a 4-web. A *Schläfli configuration* is a quadraple of geodesic triangles $\Delta_1 = \Delta(E_2, E_3, E_4), \Delta_2 = \Delta(E_1', E_3', E_4')$, $\Delta_3 = \Delta(E_1'', E_2'', E_4'') \subset \Delta_2$ and $\Delta_4 = \Delta(E_1''', E_2''', E_3''') \subset \Delta_1$ with the following properties (see Fig. 3).

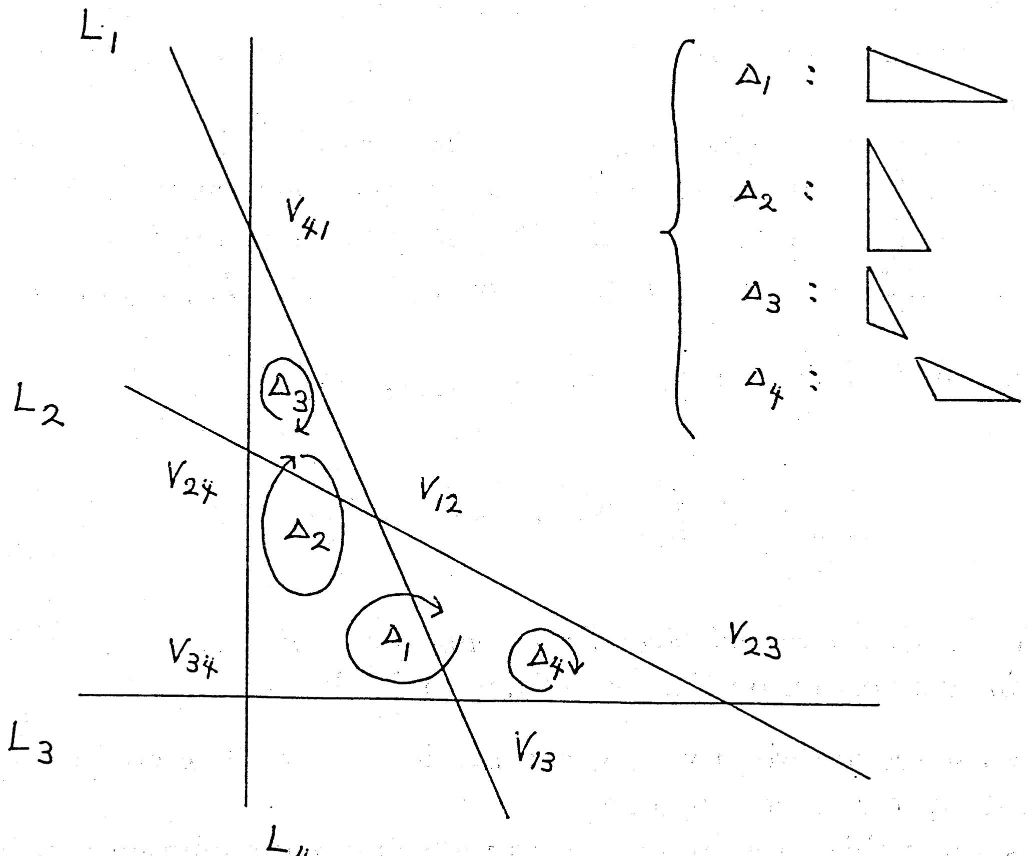


Fig. 3

(1) The edges with suffix i are contained in a common leaf $L_i \in F_i$ for $i = 1, ..., 4$,

(2) Δ_j, Δ_k has the common vertex $V_{m,n} = E_m \cap E_n, where\ \{j,k,m,n\} = \{1,2,3,4\}$,

(3) $\Delta_2 + \Delta_4 = \Delta_1 + \Delta_3$, where Δ_i denotes also the underlying set of Δ_i.

(4) The 3-subweb W_i is non singular on a neighbourhood of Δ_i for $i = 1, \ldots, 4$.

In other words a Schläfli configuration is formed by leaves of $F_1, \ldots, F_4$ in general position. The goal of this paper is to prove the following generalization of Theorem 0.

Theorem 1. *Assume 3-subwebs of a 4-web* $W = (F_1, F_2, F_3, F_4)$ *are non hexagonal. Then the following conditions are equivalent.*

(1) F_i *is defined by a 1-form* ω_{t_i} *in a pencil of meromorphic 1-forms* $P = \{\omega_t\}$.

(2) The cross ratio $C(F_4, F_3, F_2, F_1)$ *of tangents to the leaves of* $F_i, i = 1, ..., 4$ *passing through a point is constant on the complement of* $\Sigma(W)$.

(3) $F_1, \dots, F_4$ *are weakly associative: The web curvature form* $K(F_1, ..., \hat{F}_i, ..., F_4)$ *of the 3-subweb* $W_i = (F_1, ..., \hat{F}_i, ..., F_4)$ *is independent of* i.

(4) For any Schläfli configuration $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$,

$$(\bigstar) \qquad \sum_{i=1}^{4}(-1)^i \int_{\Delta_i} K(F_1, ..., \hat{F}_i, ..., F_4) = 0.$$

(5) $F_1, \dots, F_4$ *are associative: Bott connections of* $F_1, \dots, F_4$ *extend to equal affine connection on the complement of* $\Sigma(W)$.

In the last section we prove a generalization of the theorem for m-webs of $\mathbb{R}^n$, $n < m$, of codimension one.

All results in this paper remain valid replacing real analyticity with C^3-smoothness. The argument is local, so from now on we assume S is a connected domain of $\mathbb{R}^2$.

1. Bott connection and Chern connection. Bott connection of a non singular foliation is defined by the differential of the transverse dynamics. To state more precisely in our setting, recall the integrability condition

$$d\omega_i = \theta \wedge \omega_i,$$

where ω_i is the defining one form of F_i and $\omega_1 - 2\omega_2 + \omega_3 = 0$. The 1-form θ defines the (partial) connection of the normal bundle of the foliation F_i along the leaves as follows. Let L be a leaf of F_i, $p, q \in L$, and $C \subset L$ a smooth curve joining p to q. The parallel transport $T(X)$ of a vector X normal to L at p along C is defined by the relation

$$\omega_i(T(X)) = \exp(\int_C \theta) \cdot \omega_i(X).$$

To extend Bott connection to an affine connection on S, consider an (infinitesimally) small geodesic triangle Δ with vertex p. By the transverse dynamics along C. Δ is transported to a unique (infinitesimally small) geodesic triangle Δ' with vertex q (Fig. 4).

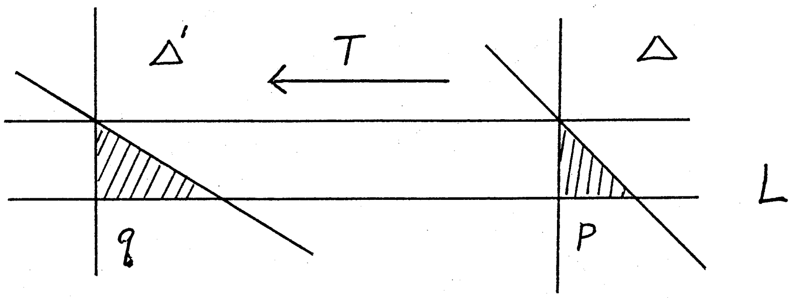


Fig. 4

This transportation determines a linear map of the tangent spaces T_pS to T_qS. The linear map is defined also for all piecewise geodesics by composition of those linear maps along the geodesic pieces. It is easy to see this transportation determines an affine connection, and the connection form with respect to the coframes ω_1, ω_2 is $\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}$. This connection is called Chern connection of the 3-web W. Chern connection is in other words the unique common extension of Bott connections of F_1, F_2, F_3. The structure group of the connection is $\mathbb{R}^*$: the group of similar transformations, and the holonomy map along a closed cycle C is

$$\exp(\int_C \theta) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Assume that θ is closed, i.e. the web curvature form vanishes identically. Then $\tilde{\omega}_i = \exp(-\int \theta) \cdot \omega_i$ is closed and $\tilde{\omega}_1 - 2\tilde{\omega}_2 + \tilde{\omega}_3 = 0$. By integrating the equation, we obtain the deveroping map $(\int \tilde{\omega}_1, \int \tilde{\omega}_2, \int \tilde{\omega}_3)$ of S to the hyperplane $H = \{(u_1, u_2, u_3) \in \mathbb{R}^3 \mid u_1 - 2u_2 + u_3 = 0\}$, which sends the leaves of the web to the lines defined by $u_i =$ const. in H. Thomsen (c.f. [3]) proved that the Hexagonality of 3-webs is equivalent to the closure condition of the piecewise geodesic hexagon as in Fig.5.

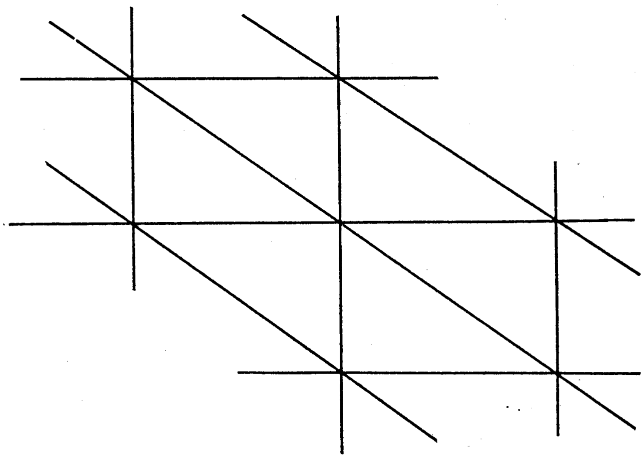


Fig. 5

In general this hexagon is not closed (Fig.6) .

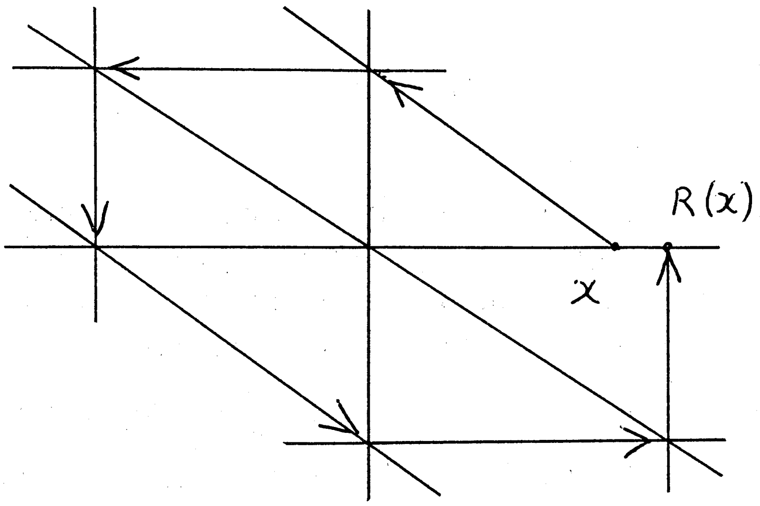


Fig. 6

Now assume that the foliations F_1, F_2, F_3 are defined by the level functions x, y and an $f(x,y)$ such that $f(t,0) = f(0,t) = t$ and $f(t,t) = 2t$: $f(x,y) = x + y + k(x-y)xy + \cdots$. Then the web curvature form for the 3-web of this form is presented as

$$K(W) = \frac{\partial^2}{\partial x \partial y}\left(\log \frac{f_x}{f_y}\right) dx \wedge dy$$

and the return map $R(x)$ as in Fig.6 is written in a coordinate x on the leaf L centered at p as follows.

$$R(x) = x + kx^3 + \cdots .$$

To see the for of R it suffices to notice $R'(x) = 1 + 3kx^2 + \cdots$ is the linear term of the holonomy at x anti-clockwisely along the "non-closed hexagon" and the area of the "hexagon" is proportional to $3x^2$. In the next section we interprete the curvature in terms of the fake billiard.

2. Transverse dynamics and fake billiard. Let $W = (F_1, F_2, F_3)$ be a non singular 3-web on a surface and assume all leaves are simply connected. Let L_i be a leaf of F_i, $p, q \in L_i$ and $j, k \neq i$. The *transverse dynamics*

$$T_{pq}^{jk} : L_j(p), p \to L_k(q), q$$

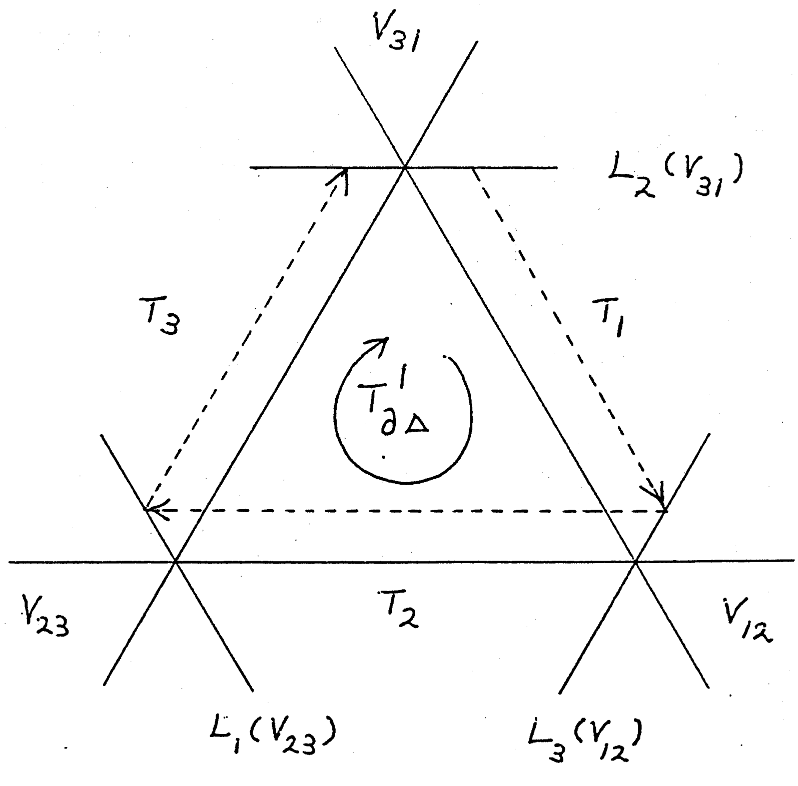
is a germ of diffeomorphism of the leaf $L_j(p)$ of F_j passing through p to the leaf $L_k(q)$ of F_k passing through q, which assigns to $x \in L_j(p)$ sufficintly close to p the unique intersection point $y \in L_i(x) \cap L_k(q)$. *Fake billiard* along a boundary of an oriented geodesic triangle $\Delta = \Delta(E_1, E_2, E_3)$ is the return map

$$T_{\partial\Delta}^i = T_{i+2} \circ T_{i+1} \circ T_i : L_{i+1}(v_{i-1,i}), v_{i-1,i} \to L_{i+1}(v_{i-1,i}), v_{i-1,i}$$

where T_j denotes the transverse dynamics along the edge E_j

$$T_{v_{j-1,j}\, v_{j,j+1}}^{j+1,j+2} : L_{j+1}(v_{j-1,j}), v_{j-1,j} \to L_{j+2}(v_{j,j+1}), v_{j,j+1},$$

Fig. 7



Clearly $T^i_{\partial\Delta}, i = 1,2,3$ are conjugate with each other : $T_{i+1} \circ T^{i+1}_{\partial\Delta} \circ T_i = T^i_{\partial\Delta}$. We denote the derivative of $T^i_{\partial\Delta}$ at $v_{i-1,i}$ by $dT_{\partial\Delta}$.

Lemma 1. *Fake billiard along an oriented geodesic triangle $\Delta = \Delta(F_1, F_2, F_3)$ has the following derivative at the origin.*

$$\partial T_\Delta = -exp\left(-\sigma(\Delta)\int_\Delta K(F_1,F_2,F_3)\right),$$

where $\sigma(\Delta) = 1$ or -1 respectively Δ is the clockwise orientation or anti-clockwise.

Proof. Assume Δ and F_1, F_2, F_3 are defined by the level functions f, x and y as in Fig.7 bis.

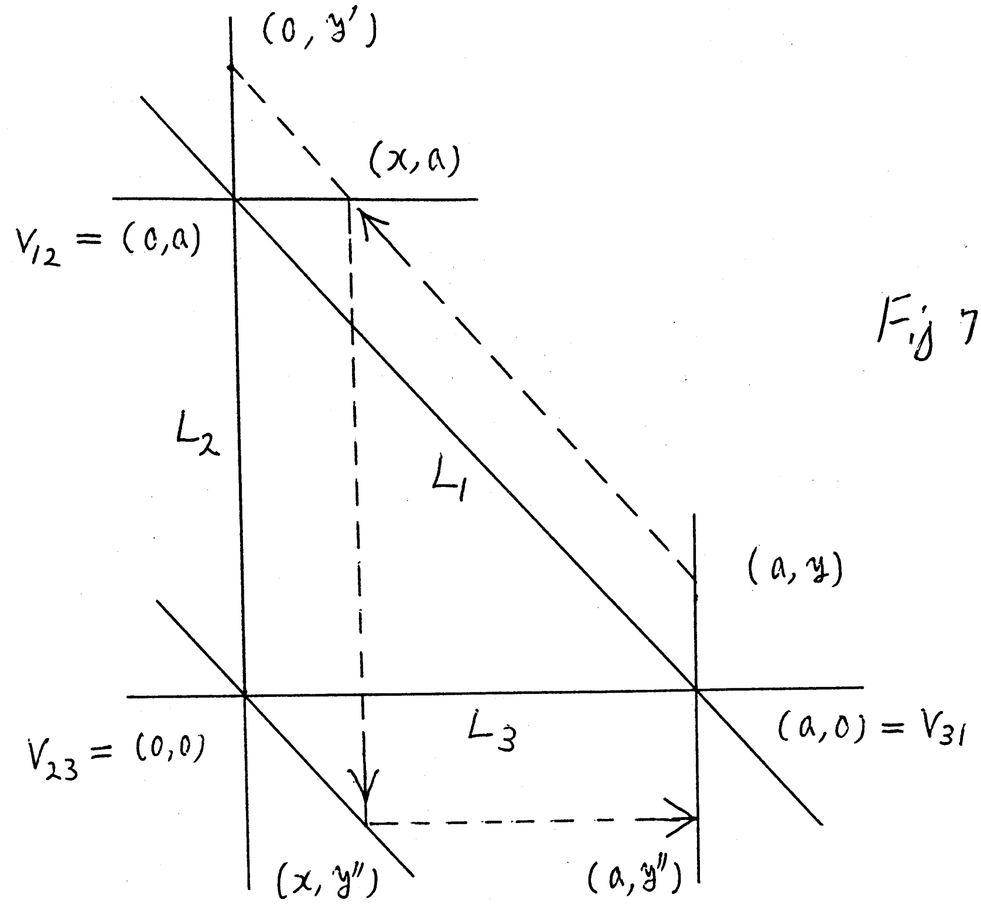


Fig 7.bis

Then $\sigma(\Delta) = -1$. Let $v_{31} = (a,0), v_{12} = (0,a)$ and

$$T^{22}_{v_{31}v_{12}}(a,y) = (0,y').$$

Then we obtain

$$\log\left(\frac{dy'}{dy}(0)\right) = \int_{v_{31}v_{12}} \left(-\frac{f_x}{f_y}\right)_y dx$$
$$= \int_{v_{31}v_{12}} \left(\frac{f_x}{f_y}\right)_y \frac{dx}{dy}\, dy$$
$$= \int_{v_{31}v_{12}} \frac{\left(\frac{f_x}{f_y}\right)_y}{\frac{f_x}{f_y}}\, dy$$
$$= \int_{v_{31}v_{12}} \left(\log\frac{f_x}{f_y}\right)_y dy.$$

Let $T^{23}_{v_{12}v_{12}}(0, y') = (x, a)$. Then

(2) $$\log\frac{dx}{dy'}(y' = a) = \log\frac{f_x}{f_y}(0, a).$$

Let $T^{32}_{v_{12}v_{23}}(x, a) = (x, y'')$. Then

(3) $$\log\frac{d''}{dx}(x = 0) = \log\frac{f_x}{f_y}(0, 0).$$

From (2) and (3), we obtain

(4) $$\log\frac{dy''}{dy'} = \log\frac{dy''}{dx} - \log\frac{dy'}{dx}$$
$$= \log\frac{f_x}{f_y}(0,0) - \log\frac{f_x}{f_y}(0,a)$$
$$= \int_{v_{12}v_{23}} \left(\log\frac{f_x}{f_y}\right)_y dy.$$

From (1) and (4)

$$\log\left(-\frac{dy''}{dy}\right) = \int_{v_{31}v_{12}v_{23}} \left(\log\frac{f_x}{f_y}\right)_y dy$$
$$= \int_\Delta \left(\log\frac{f_x}{f_y}\right)_{xy} dx \wedge dy$$
$$= \int_\Delta K(W)$$

This completes the proof.

Let $W = (F_1, F_2, F_3, F_4)$ be a non singular 4-web on a surface. Define the cross ratio by

$$C(k_1, k_2, l_1, l_2) = \frac{(a_1 - b_1)(a_2 - b_2)}{(a_1 - b_2)(a_2 - b_1)}$$

for the lines $k_i = \{y = a_i x\}, l_i = \{y = b_i x\}, i = 1, 2$, and define the cross ratio of tangents to the leaves $L_i(p) \in F_i$ passing through p by

$$C(p) = C(T_p L_4(p), T_p L_3(p), T_p L_2(p), T_p L_1(p)).$$

From now on we assume $1 < C(k_1, k_2, l_1, l_2) < \infty$ (this holds uniformly on the connected component of the surface).

Lemma 2. *Let $\Delta_3 = \Delta(E_1, E_2, E_4)$ be a geodesic triangle of the 3-subweb (F_1, F_2, F_4) (Fig.8). Then the following fake billiard along $\partial\Delta_3$*

$$\tilde{T}_{\Delta_3} = T^{34}_{v_{41} v_{12}} \circ T^{33}_{v_{24} v_{41}} \circ T^{43}_{v_{12} v_{24}} : L_4(v_{12}), v_{12} \to L_4(v_{12}), v_{12}$$

has the following derivative at v_{12}

$$d\tilde{T}_{\Delta_3} = \frac{C(v_{41})}{C(v_{41}) - 1}(1 - C(v_{24}))\; dT_{\Delta_3}.$$

Fig.8

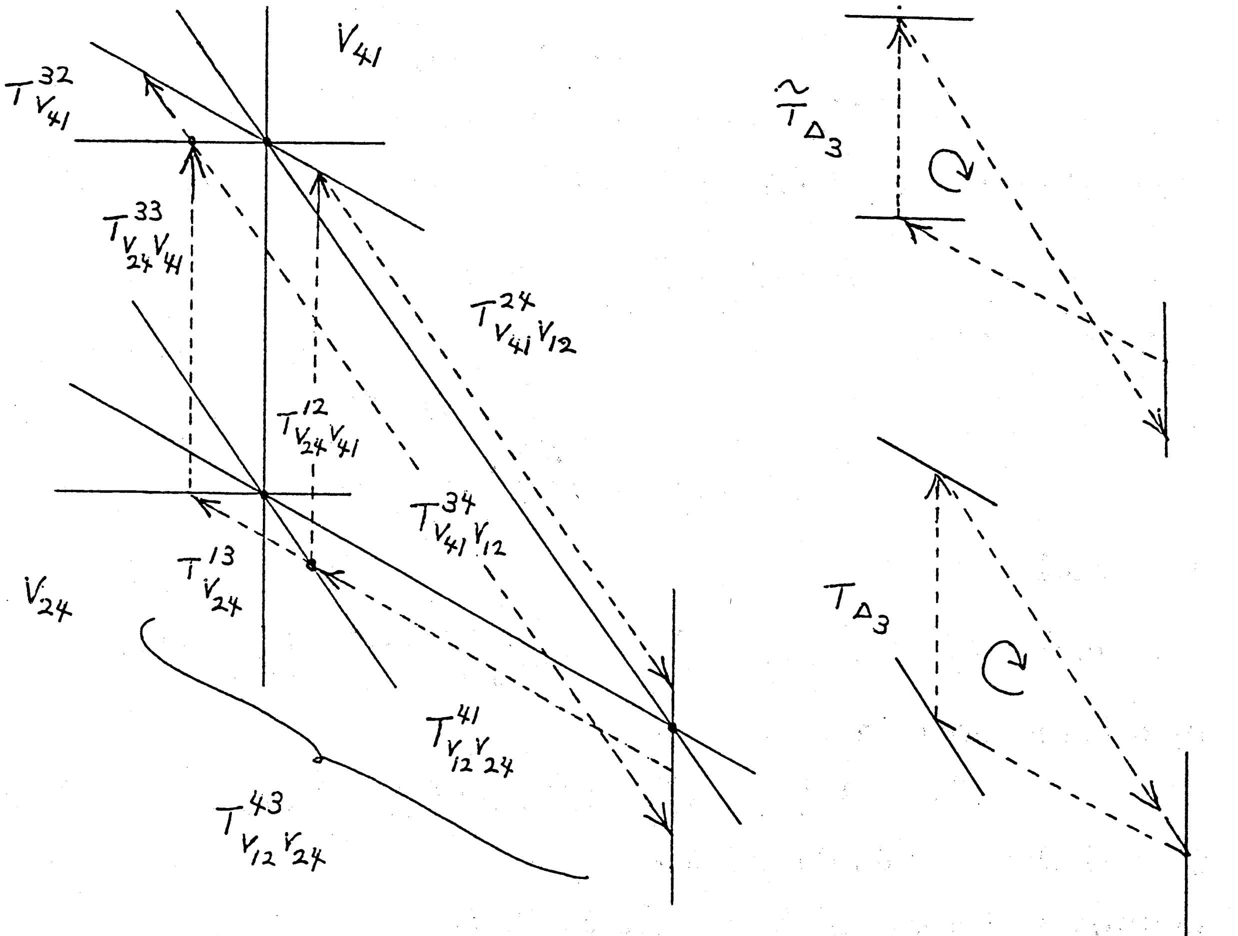

v_{41}
$T^{32}_{v_{41}}$
$T^{33}_{v_{24}v_{41}}$
$T^{24}_{v_{41}v_{12}}$
$T^{12}_{v_{24}v_{41}}$
$T^{34}_{v_{41}v_{12}}$
$T^{13}_{v_{24}}$
v_{24}
$T^{41}_{v_{12}v_{24}}$
$T^{43}_{v_{12}v_{24}}$
$\tilde{T}_{\Delta_3}$
T_{Δ_3}

Fig. 8

Proof. Let f_i be a local level function of F_i defined on a neighbourhood of Δ_3. Let

$$T^{32}_{v_{41}} : L_3(v_{41}), v_{41} \to L_2(v_{41}), v_{41}$$
$$T^{13}_{v_{24}} : L_1(v_{24}), v_{24} \to L_3(v_{24}), v_{24}$$

denote the transverse dynamics respectively along the leaves of F_1, F_2 such that

$$f_1 \circ T^{32}_{v41} = f_1 \ , \quad f_2 \circ T^{13}_{v24} = f_2.$$

It is easy to see

$$d\left(f_4 \circ T^{13}_{v_{24}}\right) = (1 - C(v_{v_{24}}))\ df_4,$$
$$d\left(f_4 \circ T^{32}_{v_{41}}\right) = \frac{(C(v_{v_{41}})}{(C(v_{v_{41}}) - 1)}\ df_4,$$

from which

$$d(f_4 \circ\ T^{32}_{v_{41}} \circ T^{33}_{v_{24},v_{41}} \circ T^{13}_{v_{24}}) = \frac{(C(v_{v_{41}})}{(C(v_{v_{41}}) - 1)}(1 - C(v_{v_{24}}))\ df_4.$$

By definition we obtain

$$\tilde{T}_{\Delta_3} = T^{24}_{v_{41},v_{12}} \circ T^{32}_{v_{41}} \circ T^{33}_{v_{24},v_{41}} \circ T^{13}_{v_{24}} \circ T^{41}_{v_{12},v_{24}},$$

from which we obtain the statement.

Lemma 3. *Let $\Delta_1, \Delta_2, \Delta_3$ be as in Fig.9. Then*

$$d\tilde{T}_{\Delta_3} = dT_{\Delta_2} \cdot dT_{-\Delta_1} \cdot C(v_{34}),$$

where $-\Delta_1$ denotes the triangle Δ_1 with reverted orientation.

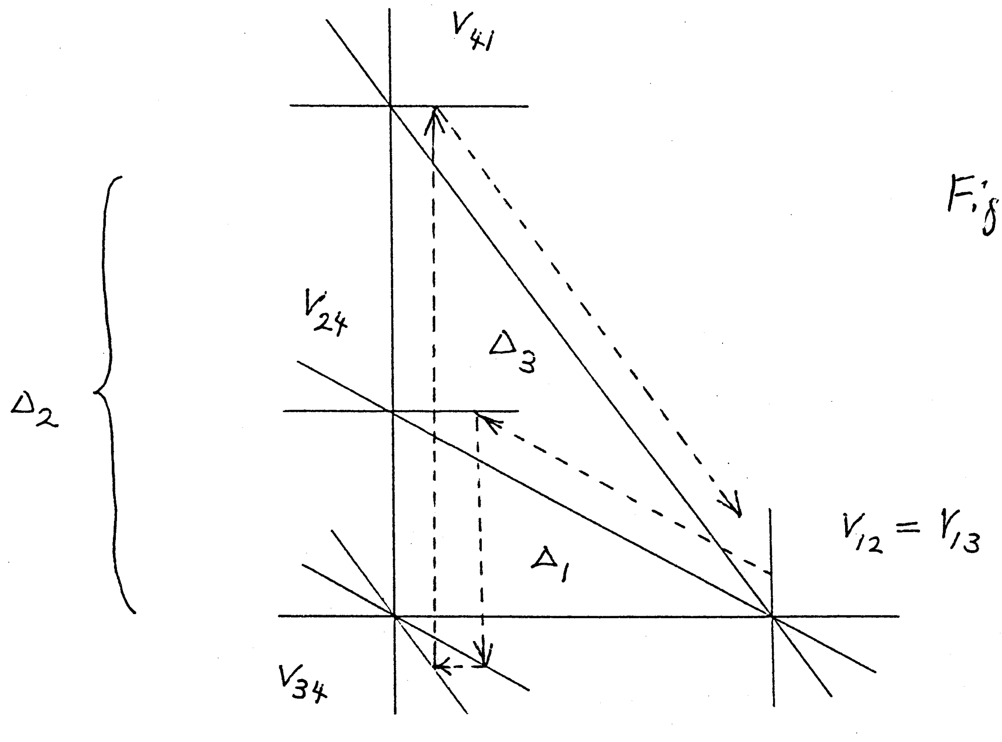


Fig. 9

Proof. Let $T_{\Delta_2}v_{13}, T_{-\Delta_1}v_{13}, T_{\Delta_3}v_{24}$ denote fake billiard along $\partial\Delta_2, -\partial\Delta_1$, $\partial\delta_3$ starting at v_{13}, v_{24}. It is easy to see fake billiard $T_{\Delta_2 v_{13}} \circ T_{-\Delta_1 v_{13}}$ is conjugate with

$$\tilde{T}_{\Delta_3 v_{24}} \circ T^{13}_{v_{34}v_{24}} \circ T^{41}_{v_{13}v_{34}} \circ T^{24}_{v_{34}v_{13}} \circ T^{32}_{v_{24}v_{34}} = \tilde{T}_{\Delta_3 v_{24}} \circ T^{13}_{v_{34}v_{24}} \circ T^{21}_{v_{34}} \circ T^{32}_{v_{24}v_{34}},$$

where $T^{21}_{v_{34}}$ is defined by $f_3 \circ T^{21}_{v_{34}} = f_3$, f_3 being the defining function of F_3. Differentiating the equality we obtain the statement with the equality.

$$C(v_{34}) \cdot d\left(f_4 \circ T^{21}_{v_{34}}\right) = df_4.$$

Lemma 4. *Let $\Delta_1, \Delta_2, \Delta_3$ be as in Fig.9. Then*

$$dT_{\Delta_1} \cdot dT_{-\Delta_2} \cdot dT_{\Delta_3} = -\frac{C(v_{41}) - 1}{C(v_{41})} C(v_{34}) \frac{1}{C(v_{24}) - 1}$$

Proof. The statement follows from Lemmas 2 and 3.

Proposition 5. *Let $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ be a Schläfli configuration. Then*

$$\sum (-1)^i \int_{\Delta_i} K(F_1, ..., \hat{F}_i, ..., F_4)$$
$$= \log \frac{C(v_{41}) - 1}{C(v_{41})} \cdot \frac{C(v_{23}) - 1}{C(v_{23})} \cdot \frac{1}{C(v_{24}) - 1} \cdot \frac{1}{C(v_{13}) - 1} \cdot C(v_{34}) \cdot C(v_{12})$$

Proof. We may assume $v_{34} = (0, 0)$, F_3, F_4 are defined by the coordinate functions y, x and F_1, F_2 by functions f, g respectively. Let $v_{12} = (a, b)$ and $P_1 = (0, b), P_2 = (a, 0)$. Let $\square$ denote the geodesic rectangle with the vertices v_{12}, P_1, v_{34}, P_2, and let Δ'_1 (resp. Δ'_2) denote the geodesic triangle with the vertices v_{12}, v_{24}, P_1 (resp. v_{12}, v_{13}, P_2) and let $\Delta''_1 = \Delta_4 + \Delta'_2, \Delta''_2 = \Delta_3 + \Delta'_1$ (see Fig.10).

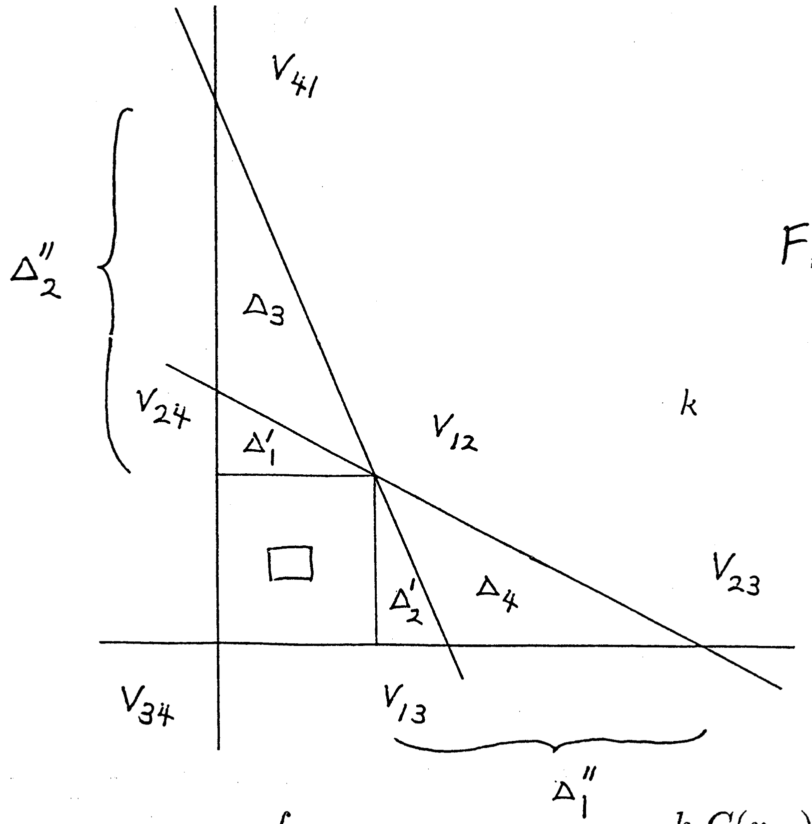


Fig. 10

Then

$$\int_{\square} K(F_1, F_3, F_4) - \int_{\square} K(F_2, F_3, F_4) = \log \frac{k\ C(v_{12})}{C(P_1)C(P_2)}$$

The alternative sum in the equality of the proposition is

$$-\{\int_{\Delta_1'} + \int_{\square} + \int_{\Delta_1''} K(F_2, F_3, F_4)\} + \{\int_{\Delta_2'} + \int_{\square} + \int_{\Delta_2''} K(F_1, F_3, F_4)\}$$
$$- \int_{\Delta_3} K(F_1, F_2, F_4) + \int_{\Delta_4} K(F_1, F_2, F_3)$$
$$= \{- \int_{\Delta_1'} K(F_2, F_3, F_4) + \int_{\Delta_2''} K(F_1, F_3, F_4) - \int_{\Delta_3} K(F_1, F_2, F_4)\}$$
$$+ \{- \int_{\square} K(F_2, F_3, F_4) + \int_{\square} K(F_1, F_3, F_4)\}$$
$$- \{- \int_{\Delta_1'} K(F_1, F_3, F_4) + \int_{\Delta_1''} K(F_2, F_3, F_4) - \int_{\Delta_4} K(F_1, F_2, F_3)\}.$$

By Lemma 1 and Lemma 4

$$= \log \{- \frac{C(v_{41})(C(v_{41}) - 1)}{(C(v_{24}) - 1)C(v_{41})}\}\{+ \frac{C(v_{12})C(v_{34})}{C(P_1)C(P_2)}\}\{- \frac{C(v_{41})(C(v_{23}) - 1)}{(C(v_{13}) - 1)C(v_{23})}\}$$
$$= \log \{\frac{C(v_{41}) - 1}{C(v_{41})} \cdot \frac{C(v_{23}) - 1}{C(v_{23})} \cdot \frac{1}{C(v_{24}) - 1} \cdot \frac{1}{C(v_{13}) - 1} \cdot C(v_{34}) \cdot C(v_{12})\} \blacksquare$$

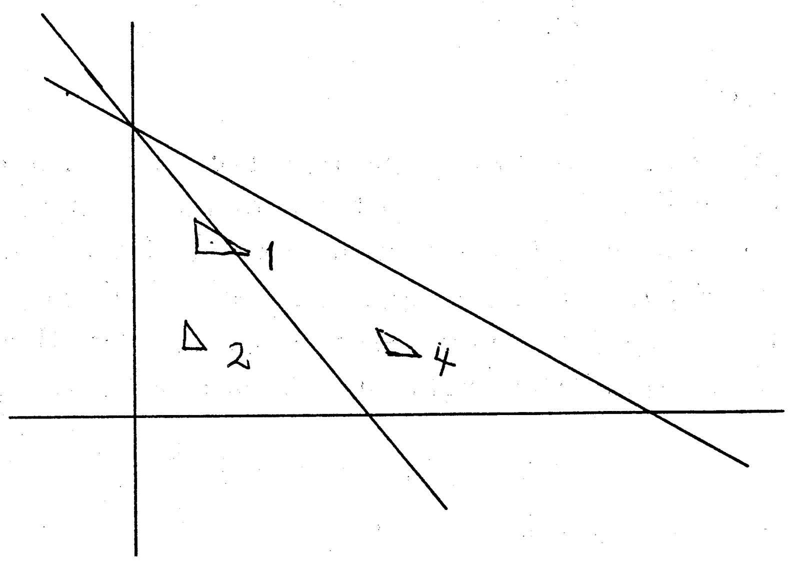
3. Proof of Theorem 1. The implications $(1) \to (2), (3) \to (4), (1) \to (5)$ are clear. The implication $(1) \to (3)$ follows from the uniqueness of the 1-form $\theta(P)$.

Proof of (5) $\to$ (3). For each 3-subweb $W' = (F_i, F_j, F_k)$ if Bott connections of F_i, F_j, F_k extend to equal affine connection, it is Chern connection of W'. Therefore common extension of all Bott connections is Chern connection of 3-subwebs.

Proof of (2) $\to$ (1). Let ω_1, ω_2 be meromorphic 1-forms defining F_1, F_2 respectively. Then ω_3 is presented as $\omega_3 = \lambda\omega_1 + \mu\omega_2$ with meromorphic functions λ, μ on the surface S. Now we may assume $\lambda = \mu = 1/2$ replacing ω_1, ω_2. Similarly assume F_4 is defined by $\omega_4 = \lambda'\omega_1 + \mu'\omega_2$. Then the cross ratio $C(F_4, F_3, F_2, F_1) = -\lambda'/\mu'$ of the leaves of F_4, F_3, F_2, F_1 is constant $c \neq 0$ by assumption. Therefore we may assume $\omega_4 = c\omega_1 + \omega_2$.

Proof of (4) $\to$ (3). Consider the Schläfli configuration such that $v_{12} = v_{24} = v_{41}$ as in Fig.11.

Fig.11

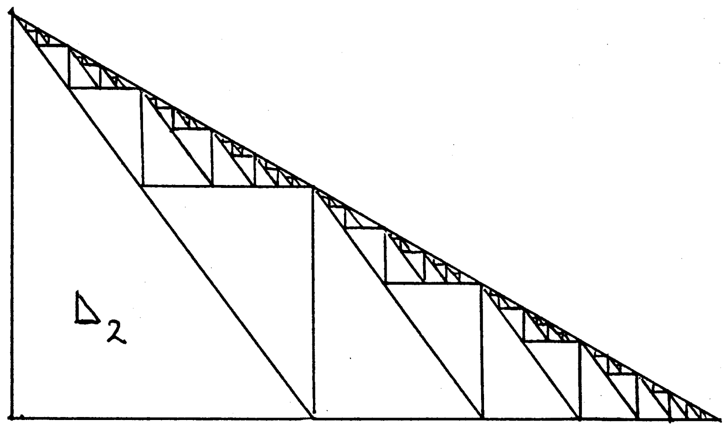


By the hypothesis (4)

$$\int_{\Delta_1} K(W_1) = \int_{\Delta_2} K(W_2) + \int_{\Delta_2} K(W_2),$$

where $K(W_i)$ denotes the web curvature form of the 3-subweb of W forgetting the i-th foliation. This tells that the integral of the curvature form over a geodesic triangle can be calculated by decomposing into small geodesic triangles. Now decompose the Δ_1 into infinitely many geodesic triangles of the same type as Δ_2 as in Fig.12.

Fig.12



Then it follows that

$$\int_{\Delta_1} K(W_1) = \int_{\Delta_1} K(W_2),$$

from which $K(W_1) = K(W_2)$.

Proof of (3), (4) $\to$ (2). We may ssume W is non singular and F_1, F_2, F_3, F_4 are defined by level fuctions f, g and the coordinate functions y, x respectively. By a suitable coordinate transformation of the y we may assume $f(t,0) = f(0,t) = g(t,0) = t$, and applying Poincaré linearlization theorem to the dynamics $t \to g(0,t)$ we may assume $g(0,t) = kt$. Here k is the cross ratio $C(F_4, F_3, F_2, F_1)$ at the origin, and f, g are of the form

$$f(x,y) = x + y + mxy + O,$$
$$g(x,y) = x + ky + nxy + O',$$

O, O' being the remainder terms of x, y of order 3 which vanishe identitically on the x and y axes. It is easy to see that only similar transformations $(x,y) \to (cx, cy)$ respect this normal form. Therefore the ratio $(m : n)$ as well as k gives rise to an absolute invariant of 4-webs. By definition, the cross ratio at (x,y) is

$$C(x,y) = C(F_4, F_3, F_2, F_1)(x,y) = \frac{(k+nx)(1+my)}{(1+ny)(1+mx)} + O''.$$

Denote $C(x,0) = C(x)$ and $C(0,y) = D(y)$ for simplicity. On the x-axis this restricts to

$$C(x) = \frac{k+nx}{1+mx} + \cdots = k + (n-km)x + \cdots .$$

By Proposition 5 applied to the Schläfli configuration as in Fig.11 we obtain

$$\frac{C(kx)}{C(x)} = \frac{C(kx)-1}{C(x)-1}.$$

With the initial condition $C(0) = k, C'(0) = n - km$ this equation admits a unique solution

$$C(x,0) = C(x) = k \ + \ \frac{(n-km)x}{1 - \frac{(n-km)x}{k-1}}.$$

Similarly we obtain

$$C(0,y) = D(y) = k \ + \ \frac{k(m-n)y}{1 - \frac{k(m-n)y}{k-1}}.$$

By the hypothesis (3) $K(W_1) = K(W_2)$. Hence

$$\frac{\partial^2}{\partial x \partial y}\log\left(\frac{f_x}{f_y}\right) = \frac{\partial^2}{\partial x \partial y}\log\left(\frac{g_x}{g_y}\right),$$

from which

$$\frac{\partial^2}{\partial x \partial y}\log C(x,y) = \frac{\partial^2}{\partial x \partial y}(\log(-\frac{f_x}{f_y}) - \log(-\frac{g_x}{g_y})) = 0.$$

Therefore

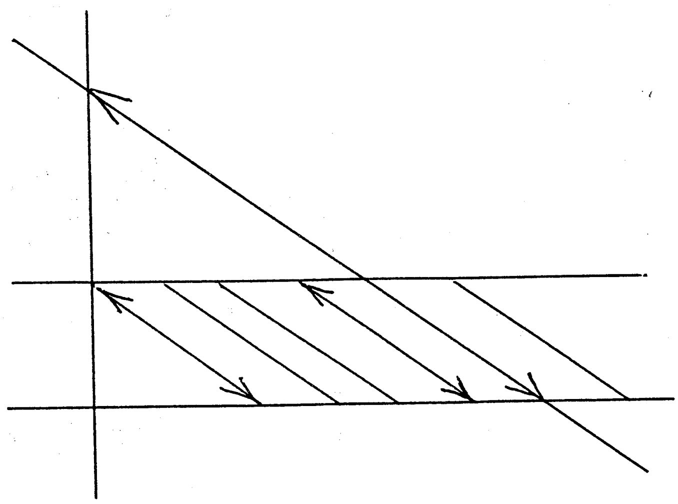
$$C(x,y) = C(x)D(y)/k.$$

Assume a 4-web $W' = (F_1', \ldots, F_4')$ (not necessarily of the above normal form) is defined by the level functions f, g, y, x, and assume the cross ratio function $C'(x,y)$ is a product of two linear fractions $C'(x), D'(y)$ of x and y. Let ϕ and ψ be the diffeomorphisms of the x-axis and the y-axis, which normalizes W' to the above normal form. Since the cross ratio is an absolute invariant, we obtain

$$C'(x,y) = C(\phi(x), \psi(y)).$$

First assume $n \neq m, km$. Then $C(x), D(y)$ are not constant and it follows from the above equality that ϕ, ψ are also linear fractions. Since for the normal form the transverse dynamics of F_1, F_2 sending the x-axis to the y-axis respecting the origin are linear maps, those dynamics for W' are linear fractions. This argument applies to germs of the normal form W at all points on a neighbourhood of the origin, and implies that the transverse dynamics sending horizontal lines to vertical lines are all linear fractions. It is easy to see that this implies also the transverse dynamics of F_1, F_2 among the horizontal lines as well as the vertical lines are also linear fractions in the coordinates x and y (Fig.13).

Fig.13



Therefore we may assume that the level functions f, g are linear fractions in x, y when it is restricted to the horizontal and the vertical lines defined by the coordinate functions y, x respectively. We may write as

$$f = \frac{a(x)y + b(x)}{1 - c(x)y} = b + (a + bc)y + (ac + bc^2)y^2 + \cdots,$$

b being a linear fraction of x. The Schwarzian derivative of f in x is

$$[f : x] = \frac{f_{xxx}}{f_x} - \frac{3}{2}\left(\frac{f_{xx}}{f_x}\right)^2$$

and write it as a series in y with function coefficients

$$S_0 + S_1 y + S_2 y^2 + \cdots .$$

The initial term is Schwarzian derivative of b, and S_1, S_2 are Schwarzian derivatives of the first two and three terms of the expantion of f. Since f is a linear fraction in x for all fixed y, all these coefficients vanish. It then follows that a, b and c are rational functions of x, hence f is a rational function of x, y. Again since f is linear fraction in x and y, f is of the form

$$f(x,y) = \frac{c_1 xy + c_2 x + c_3 y + c_4}{c_1' xy + c_2' x + c_3' y + c_4'}.$$

Now we will show that the curvature form $K(F_1, F_3, F_4)$ of the 3-subweb (F_1, F_3, F_4) vanishes at the origin. (This can be also seen by straight forword calculation.) Recall the rotation map R defined as in §1, Fig 6. Since R is defined by composing the various transverse dynamics respecting the origin, all of which are linear fractions by the above form of f, p is also a linear fraction of x. On the other hand the rotation map has the expantion $R(x) = x + k(0,0)x^3 + \cdots$ and the second order term is missing. Therefore R is the identity and in particular the web curvature vanishes at the origin. This argument applies at all point in the domain of definition S to imply that the curvature form vanishes identically. This contradicts the hypothesis of the theorem.

Next assume $n = m$ and $n \neq km$. Then by the same argument as the above case $D(y)$ is constant and

$$C(x,y) = C(x,0) = C(x) = k + \frac{(n-km)x}{1 - \frac{(n-km)x}{k-1}}$$

is not constant. In this case exchange the roles of F_4 and F_1 (or F_2) in the above argument. Then it reduces to the first case $n \neq m, km$, since C is not constant on the leaves of F_1, F_2 and also F_3.

Similar argument applies to the case $n = km$ and $n \neq m$. The rest is the case $m = n = 0$. Clearly this implies that the cross ratio function is locally constant. By the analyticity, the cross ratio is constant on the domain of definition. This completes the proof of Theorem 1.

4. Proof of Theorem 0 and associative webs and weakly associative webs of codimension 1 in higher dimension.

First we give an elementary proof of Theorem 0. By the form of f, the foliation F_3 is a pencil of conics with two base points p_1, p_2. By a Möbius transformation $(\phi(x), \psi(y))$ sending p_1 to $(0,0)$ and p_2 to (∞, ∞), we may assume F_3 is a pencil of lines with base point $(0,0)$. In particular a leaf of F_3 meet a leaf of F_1, F_2 at a single point. Now consider the 4-th foliation F_4. The same argument as in the previous section applies to imply that F_4 is a pencil of conics and a leaf meets each member of the pencils of lines F_1, F_2, F_3 at a single point. It then follows that the leaves of F_4 are lines hence F_4 is a pencil of lines. This completes the proof of Theorem 0.

Let $W = (F_1, \ldots, F_{n+1})$ be a non singular (in general position) $(n+1)$-web of codimension 1 on an open subset of $\mathbb{R}^n$. Chern connection γ_i of W is an extention of Bott connection of the i-th foliation F_i (see [5] for the definition). In the case $n = 3$ twice the average of the curvature forms of $\gamma_1, \ldots, \gamma_{n+1}$ had been already found by Blaschke[1], so we call it Blaschke curvature form. Let F_i be given by the i-th coordinate function of $\mathbb{R}^n$ for $i = 1, \ldots, n$ and let F_{n+1} be defined by a function f. Define Blaschke curvature form by

$$d\Gamma = d(-\sum_{i=1,\ldots,n} (\log f_{x_i})_{x_i} dx_i) = \frac{1}{2} \sum_{i,j=1,\ldots,n} (\log \frac{f_{x_i}}{f_{x_j}})_{x_i x_j} dx_i \wedge dx_j.$$

It is easily seen that $d\Gamma$ restricts to the web curvature form of the 3-web on $x_i x_j$-plane cut out by x_i, x_j and f. Conversely this property characterizes Blaschke curvature form.

Proposition 6 [1]. *Blaschke curvature form restricts to the web curvature form of the 3-web on the intersections of the leaves of $n-2$ foliations cut out by the remaining 3 foliations.*

We call $n+2$ foliations of codimension 1 are *associative* if the modulus of tangent hyperplanes to the leaves of foliations passing through a point is constant. And we call $n+2$ foliations of codimension 1 are *weakly associative* if for all $n+1$-subwebs Blaschke curvature forms are equal. It is not difficult to see that if the modulus of the tangent hyperplane is constant, $n+2$ foliations are associative hence weakly associative. In the following we discuss the converse.

Assume that Blaschke curvature forms are equal. Then the curvilinear 4-web on the intersection of $n-2$ foliations cut out by the remaining 4 foliations is weakly associative by Proposition 2. Assume that those curvilinear 4-webs are not hexagonal i.e. all 3-subwebs are not hexagonal. By Theorem 1 the cross ratio of tangents to the leaves of curvilinear 4-web is constant on each leaves of the $(n-2)$-intersection. We claim this implies that the cross ratio of the 4-web is constant on the connected domain of definition. It then follows that the modulus of the tangent hyperplanes is constant on the domain of definition. To prove the claim it suffices to prove for the case $n = 3$, by induction on n. For simplicity assume that 5 foliations are defined by the coordinate functions x, y, z and f and g. By the above argument the cross ratio of the curvilinear 4-web on each level hyperplane of z is constant. Notice that the cross ratio of the curvilinear 4-web on z-level hyperplane is determined by those on the level hyperplanes of y and z. Those cross ratios are constant on the z-axis by the same argument. Therefore the cross ratio of the 4-web on the z-level planes is constant on z-axis hence the modulus of tangent hyperplanes as well as the cross ratio is locally constant. We can state the result in the following general form.

Theorem 7. *Assume m foliations $F_1, \ldots, F_m$, $n+2 \leq m$ of codimension 1 on an n-manifold are non singular, in general position and also the curvilinear 4-webs on the intersection of $n-2$ foliations cut out by the remaining 4 foliations are not hexagonal. Then the following conditions are euivalent.*
(1) $F_1, \ldots, F_m$ are associative: the modulus of tangent planes to the leaves of $F_1, \ldots, F_m$ passing through a point is constant.
(2) $F_1, \ldots, F_m$ are weakly associative: Blaschke curvature forms of $(n+1)$-subwebs are equal.

For the hexagonal case the statement is not true. In fact m pencils of hyperplanes on $\mathbb{R}^n$ satisfies (2) but (1). The author does not know if there exist other such examples. It seems imporant to classify all such webs, generalizing Theorem 0.

Clearly a non singular associative m-web of codimension 1 on an n-manifold, $n < m$, is defined by an m-tple of members of n-dimensional linear family of one forms $L = \{\omega_v\}_{v \in \mathbb{R}^n}$. It is easily seen that if the m points in the projectivization $PL = P^{n-1}$ are non degenerate and not contained in a quadric hypersurface, all members of L are integrable by Frobenius theorem. We say such m foliations are generic. In [10] the author proved

Proposition 8. *If all members of L are integrable and $3 \leq n$, there exists unique closed one form θ such that*

$$d\omega = \theta \wedge \omega.$$

By Proposition 8 we obtain

Proposition 9. *If linearly independent integrable one forms $F_1, \ldots, F_m$ (possiblly singular) on an n-manifold are weakly associative and generic, then the m-web $(F_1, \ldots, F_m)$ is parallelizable at non singular point.*

Here a non singular m-web $(F_1, \ldots, F_m)$ is *parallelizable* if it is locally diffeomorphic to the m-web by m foliations by parallel hyper planes in $\mathbb{R}^n$. In the paper [10] a more detailed suructure of the parallelizable webs is investigated.

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