数理解析研究所講究録
955 巻 1996 年 133-144

Triangulations of integral polytopes, examples and problems

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We are interested in polytopes in real space of arbitrary dimension, having vertices with integral coordinates: integral polytopes. The recent increase of interest for the study of these polytopes and their triangulations has various motivations; let us mention the main ones:

- the beautiful theory of toric varieties has built a bridge between algebraic geometry and the combinatorics of these integral polytopes [12]. Triangulations of cones and polytopes occur naturally for example in problems of existence of crepant resolution of singularities [1,5].
- The work of the school of I.M. Gelfand on secondary polytopes gives a new insight on triangulations, with applications to algebraic geometry and group theory [13].
- In statistical physics, random tilings lead to some interesting problems dealing with triangulations of order polytopes [6,29].

With these motivations in mind, we introduce new tools: Generalizations of the Ehrhart polynomial (counting points "modulo congruence"), discrete length between integral points (and studying the geometry associated to it), arithmetic Euler-Poincaré formula which gives, in dimension 3, the Ehrhart polynomial in terms of the f-vector of a minimal triangulation of the polytope (Theorem 7).

Finally, let us mention the results in dimension 2 of the late Peter Greenberg, they led us to the study of "Arithmetical PL-topology" which, we believe with M. Gromov, D. Sullivan, and P. Vogel, has not yet revealed all its beauties. We thank these mathematicians for their interest, and Professor Ito for his kind invitation to the Seminar at R.I.M.S. in October 1995, where part of these results where given.

I Polytopes; counting integral points; triangulations.

I.1.

Definition 1. A polytope P in $\mathbb{R}^d$ is the convex hull of a finite number of points $\{A_1, \dots A_n\}$.

The set of vertices $\mathrm{Vert}(P)$ is a subset of $\{A_1, \dots A_n\}$.
The polytope P is called integral (resp. rational) if the A_i's can be chosen in $\mathbb{Z}^d$ (resp. in $\mathbb{Q}^d$).

Definition 2. The polytope P is said to be *elementary* if

$$\mathrm{Vert}(P) = P \cap \mathbb{Z}^d .$$

These polytopes have also been called "free-lattice polytopes".

Denote by

$$\mathbf{G}_d = \mathbb{Z}^d \ltimes GL(d, \mathbb{Z})$$

the group of affine unimodular maps (affine linear isomorphisms preserving the lattice $\mathbb{Z}^d$).

Lemma 1 and definition 3. Let σ be an integral simplex in $\mathbb{R}^d$. The following conditions are equivalent:

1) $\sigma = g(\sigma_{\mathrm{can}})$,
where g is in $\mathbf{G}_d$ and σ_{can} is the basic simplex with vertices the origin and

$$\left\{A_i = \Big(\underbrace{0, \dots, 0}_{i-1}, 1, 0, \dots, 0\Big); i = 1, \dots, d\right\}$$

2) The vertices of σ generate $\mathbb{Z}^d$.

If these conditions are satisfied, σ is said to be a *primitive* simplex, and the volume of σ is $\dfrac{1}{d!}$.

Elementary simplices.

Elementary simplices are well known in dimension up to three (see III.1). They coincide with primitive simplices in dimension 1 and 2. Some partial results are known in dimension 4 [26].

I.2. The Ehrhart polynomial. Let P be an integral polytope.

Theorem 1. [10,3] For any integer k, let

$$i_P(k) = \mathrm{card}\{k\,P \cap \mathbb{Z}^d\}\,.$$

1) i_P is a polynomial in k (k in $\mathbb{N}$),

2) The values of this polynomial at negative k are given by

$$i_P(-k) = (-1)^m \;\mathrm{card}(k\,P^0 \cap \mathbb{Z}^d) \tag{1}$$

where m is the dimension of P (dimension of the affine space generated by P), and P^0 denotes the relative interior of P.

The polynomial i_P is called the *Ehrhart polynomial* of P.

Properties of i_P.

The degree of i_P is the dimension of P. For example, for a polytope of dimension d

$$i_P(k) = 1 + a_1(P)\,k + \cdots + a_d(P)\,k^d$$

where

$$\begin{cases} a_d(P) = V(P), \text{ volume of } P \\ a_{d-1}(P) = \frac{1}{2}\sum V_{d-1}(F) = \frac{1}{2}V_{d-1}(P) \end{cases} \tag{2}$$

summation over all facets F of P, V_{d-1} denoting the volume of each facet with respect to the lattice induced by $\mathbb{Z}^d$ on the affine space generated by this facet.

Properties of other coefficients are still mysterious [3.15,17].

I.3. We introduce new counting functions.

Let m be an integer and:

$$\Pi_m : \mathbb{R}^d \to \mathbb{R}^d/m\mathbb{Z}^d$$

the quotient map.

Definition 4. For any couple of integers m and k, define:

$$i_P(k,m) = \mathrm{card}\,\Pi_m\left(k\,P \cap \mathbb{Z}^d\right) \tag{3}$$

The functions $i_P(k,m)$ count integral points in $k\,P$ "modulo m", and

$$i_P(k,0) = i_P(k)\,,$$

is the Ehrhart polynomial.

Proposition 1.

$$i_P(k,m) = i_{g(P)}(k,m)\,,$$

for any g in $GL(d,\mathbb{Z})$.

Proof.
1) It is enough to consider two cases:
a) $g(x) = x + a$, $a \in \mathbb{Z}^d$, $Q = P + a$.
Then

$$\Pi_m(k\,P \cap \mathbb{Z}^d) = \Pi_m(k\,P \cap \mathbb{Z}^d) + \Pi_m(k\,a)\,.$$

b) $g = A \in \mathrm{GL}(d,\mathbb{Z})$.
A induces a bijection:

$$\tilde{A} : \mathbb{R}^d/m\mathbb{Z}^d \to \mathbb{R}^d/m\mathbb{Z}^d$$

sending $\Pi_m(k\,P \cap \mathbb{Z}^d)$ to $\Pi_m(k\,A(P) \cap \mathbb{Z}^d)$.

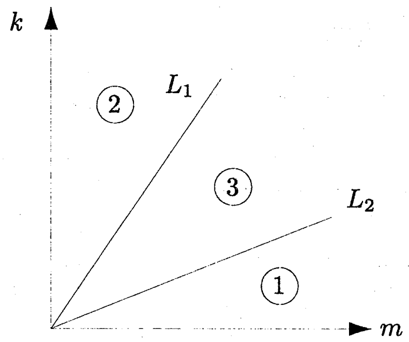
Proposition 2.
Suppose Π has an integral interior point. Then there exist two rationals α and β such that:

$$k > m\alpha \Longrightarrow i_P(k,m) = m^d$$

$$k < m\beta \Longrightarrow i_P(k,m) = i_P(k)$$

From this one deduces the existence of two critical lines in the plane (k,m) L_1, L_2 with the following properties:

in region 1, i_p is the Ehrhart polynomial
in region 2, i_p is m^d
in region 3, i_p is unknown



Proof:
1) Suppose the interior point is at the origin; let α be such that

$$\alpha > 0\,,$$

$$x = (x_i),\ 0 \le x_i < \alpha \Longrightarrow x \in P\,.$$

If $(m-1) < k\alpha$:

$$0 \le x_i \le m-1 \Longrightarrow x \in k\,P$$

$$k\,P \cap \mathbb{Z}^d \supseteq [0, m-1]^d$$

and this subset contains all equivalence classes modulo m.
2) We introduce discrete analogs of euclidean lengths:

Definition 5
1) If a and b are in $\mathbb{Z}^d$, the <u>discrete length</u> between a and b is

$$d(a,b) = \mathrm{Card}([a,b) \cap \mathbb{Z}^d] - 1$$

where $[a,b]$ is the segment joining them.

2) The discrete diameter of an integral polytope is

$$D(P) = \sup_{a,b\in P} d(a,b)\,.$$

The function d is not a distance!

Now notice that if two integral points x and x' satisfy

$$x - x' = mu,\ \ u \in \mathbb{Z}^d,\ \ u \neq 0$$

then:

$$x' + iu \in [x, x'],\ i = 0, \ldots, m\,,$$

(*)

$$d(x, x') > m\,.$$

In particular, if

$$m > D(P)\,,$$

(*) cannot be satisfied for points in P. Remark now that

$$d(ka, kb) \geq k\, d(a, b)\,,$$

and deduce that

$$\frac{m}{k} > D(P) \Longrightarrow i_P(k,m) = i_P(k)\,.$$

Valuations. Let A be any abelian group.

Definition 6. A map

$$\varphi : \mathcal{P}_d \to A$$

is said to be *additive*, or a *valuation* on $\mathcal{P}_d$, if whenever P, Q, $P \cup Q$, $P \cap Q$ are integral polytopes,

$$\varphi(P \cup Q) + \varphi(P \cap Q) = \varphi(P) + \varphi(Q)\,.$$

The following was proved in [2]:

Theorem 2. If φ is any valuation invariant under G_d, with values in A, then there exist unique elements α_i in A such that

$$\varphi(P) = \sum_{j=0}^{d} \alpha_j\, i_p(j)$$

where $i_p(j)$ are the values of the Ehrhart polynomial of P at integers j.

The proof consists in studying the group

$$\Pi = \mathbb{Z}[\mathcal{P}_d]/\sim$$

where $\mathcal{P}_d$ is the set of all integral polytopes in $\mathbb{R}^d$, and Π is the quotient of the free abelian group on $\mathcal{P}_d$ by the equivalence relation generated by

$$(4)\qquad \begin{cases} [P] = (g(P)], \ g \in G_d \\[2ex] [P \cup Q] = [P] + [Q] - [P \cap Q] \qquad \text{if } P, Q, P \cup Q \in \mathcal{P}_d\,. \end{cases}$$

Remark. The functions $i_P(m,k)$ are not additive. Take for example $d = 1$, $m = 2$. For adjacent intervals I, and I_2 with at least two points

$$i_{I_1}(k,2) = i_{I_2}(k,2) = i_{I_1\cup I_2}(k,2) = 2$$

$$i_{I_1\cap I_2}(k,2) = 1\,.$$

I.4. Triangulations. The only triangulations we consider are triangulations by rational or integral simplices (the triangulations are then called rational or integral).

Definition 7. A triangulation $\mathcal{T}$ of the polytope P is called

primitive if all simplices are (integral) primitive simplices
minimal if all simplices are elementary.

It is easy to see that minimal triangulations are minimal with respect to the natural partial order on the set of integral triangulations.

Definition 8. If $\mathcal{T}$ is any triangulation, call f-vector of $\mathcal{T}$ the vector $f = (f_i)$, where f_i is the number of simplices of dimension i.

Lemma 4. If the integral polytope P has a primitive triangulation $\mathcal{T}$, the Ehrhart polynomial i_P is determined by the f-vector $f(\mathcal{T})$, and conversely.

Proof:
Consider P as the disjoint union of the relative interiors of simplices of dimension i (of number f_i in dimension i), and use the formula:

$$i_m(k) = \frac{(k+1)\dots(k+m)}{m!},$$

then:

$$i_P(k) = \sum f_j(-1)^j \, i_j(-k).$$

Proposition 3. Let P and Q be two integral polytopes. The following conditions are equivalent:

(i) P and Q have the same Ehrhart polynomial.

(ii) There exists a k such that kP and kQ have the same Ehrhart polynomial.

(iii) For all k, kP and kQ have the same Ehrhart polynomial.

Moreover, if P and Q have primitive triangulations $\mathcal{T}_P$ and $\mathcal{T}_Q$ the conditions above are also equivalent to

$$f(\mathcal{T}_P) = f(\mathcal{T}_Q)$$

$\mathcal{T}_P$ and $\mathcal{T}_Q$ are said to be **numerically equivalent.**
Proof:
The following is obvious:

$$i_{k\,P}(n) = i_P(k\,n), \qquad \text{for all } k \text{ and } n.$$

From this one deduces, using the polynomial character of i_P:

$$(i) \Longrightarrow (iii) \Longrightarrow (ii) \Longrightarrow (i)$$

I.5. Given a polytope P, it is a difficult question to decide whether there exist primitive triangulations of P.

Let us remark that the proof of theorem 2 (see [2]) shows that primitive triangulations exist stably, that is if replacing P by $P \cup Q$, for some Q.

1) Example: Order polytopes [29].
If $\mathcal{O}$ is a finite poset of d elements (partially ordered set):

$$\mathcal{O} = \{y_1, \dots, y_d\}$$

define $P(\mathcal{O})$ to be the set of all points in $\mathbb{R}^d$ such that

$$P(\mathcal{O}) = \left\{ x = (x_1, \dots, x_d);\ x \in \mathbb{R}^d \begin{Bmatrix} 0 \le x_i \le 1, \text{ if } i = 1, \dots, d, \\ x_i \le x_j \text{ if } y_i > y_j \text{ in } \mathcal{O}. \end{Bmatrix} \right\}$$

$P(\mathcal{O})$ is an integral convex polytope of dimension d, whose vertices correspond to the set $\mathcal{L}(P, 1)$ of maps $\sigma : \mathcal{O} \to \{0, 1\}$ such that

$$y_1 < y_2 \quad \text{in} \quad \mathcal{O} \Longrightarrow \sigma(y_1) \ge \sigma(y_2) .$$

Theorem 3. $P(\mathcal{O})$ has a canonical primitive triangulation.

The primitive simplices of top dimension are given by the maximal chains:

$$x_{i_1} \ge x_{i_d} \ge \dots \ge x_{i_d}$$

associated with the poset.

See [6] for another construction of triangulations of $\mathcal{O}(P)$ giving explicitly the number of simplices in all dimensions.

2) The following result is proved in [16]:

Theorem 6. For any integral polytope P, there exists an integer k such that kP posesses a primitive triangulation.

Let $t(P)$ be the minimal integer with this property.

Let us recall [12] that to P is associated a fan $\sum$ and a toric variety X_Σ equipped with an ample line bundle $\mathcal{L}$.

Let $k_{\min}$ be the minimal integer k such that $\mathcal{L}^k$ is very ample.

Conjecture 1. $t(P) = k_{\min}$.

It was noticed by B. Sturmfels (unpublished) that an example from [11] shows that

$$t(P) \ge \dim P - 1$$

in general.

I.6. Ehrhart polynomial and triangulations: the main conjecture. Let P be an integral polytope in $\mathbb{R}^d$. The Ehrhart polynomial i_P is clearly invariant by the group G_d.

Let $\mathcal{G}_d$ be the pseudogroup associated to G_d and P and Q two integral polytopes.

Definition 7. A map $\varphi : P \to Q$ belongs to $\mathcal{G}_d$ (or "is locally in G_d") if

- φ is a homeomorphism
- there exists a rational triangulation $\mathcal{T}$ of P (resp. $\mathcal{T}'$ of Q) such that on the interior of each simplex σ of top dimension of $\mathcal{T}$, φ coincides with an element of G_d, and

$$\varphi(\sigma) \in \mathcal{T}' .$$

Proposition 4. The Ehrhart polynomial is invariant with respect to the pseudogroup $\mathcal{G}_d$.

Proof:

Let φ be as above. The homeomorphism φ preserves $\mathbb{Z}^d$: this is clear for an integral point a which is interior to a simplex of top dimension, because φ coincides there with an element of G_d. If a belongs to a face of such a simplex, $\varphi(a)$ can be expressed by continuity via an element of G_d and so is still in $\mathbb{Z}^d$.

The same argument applies to the lattices $\frac{1}{k}\mathbb{Z}^d$, and shows that they are preserved by φ. This allows to extend φ as

$$\varphi_k : k\,P \mapsto k\,Q$$

in a compatible manner with φ and with

$$\varphi_k(k\,P\cap\mathbb{Z}^d)\subseteq k\,Q\cap\mathbb{Z}^d\,.$$

Applying the same argument to the inverse of φ shows that

$$\mathrm{Card}(k\,P\cap\mathbb{Z}^d)=\mathrm{Card}(k\,Q\cap\mathbb{Z}^d)\,.$$

In dimension two, Peter Greenberg proved the following [14]:

Proposition 8. Let P and Q be two integral polytopes in $\mathbb{R}^2$. They have the same Ehrhart polynomial if and only if they are equivalent with respect to $\mathcal{G}_2$.

The considerations above, and some computations with the counting functions $i_P(k,m)$, led us to the following

Conjecture 2. Let P and Q be integral polytopes in $\mathbb{R}^d$ such that

$$i_P(k,m)=i_Q(k,m)\,,\quad \text{for all } k \text{ and } m.$$

Then there is a linear unimodular map sending P to Q.

(Main) Conjecture 3. In dimension three and above, it is **not** true in general that if P and Q are integral polytopes with the same Ehrhart polynomial, they are equivalent by the pseudogroup $\mathcal{G}_d$.

Remark. In view of Proposition 3, this can be considered as a kind of "Arithmetical Hauptvermutung": the problem is to find P and Q with *numerically equivalent* primitive triangulations $\mathcal{T}$ and $\mathcal{T}'$, such that $\mathcal{T}$ and $\mathcal{T}'$ cannot be refined to rational triangulations $\mathcal{T}_1$, $\mathcal{T}_1'$, *combinatorially* equivalent and with all simplices of top dimensions having same volumes (for both).

We will study in detail the arithmetical specificity in dimension 3 in the next paragraph.

II. Hilbert's third problem for rational polytopes.

Problem. Let P and Q be two integral (resp. rational) polytopes. Under which condition are they *equidecomposable*? *equicomplementable*?

As in usual scissor xcongruence framework [24], P and Q are equidecomposable if there exist subdivisions:

$$P=\bigcup_{i\in I}P_i\qquad Q=\bigcup_{i\in I}Q_i$$

with P_i and Q_i integral, (resp. rational) polytopes with disjoint interior, such that

$$P_i=g_i(Q_i)\quad,\quad g_i\in G_d\quad i\in I.$$

Equicomplementability is defined in a similar way, allowing addition of other polytopes (loc.cit.). This is completely analogous to the usual framework of scissor congruence in the classical sense, but the group of motions considered is here the group of *unimodular* mappings.

Proposition 9. P and Q rational polytopes are *equicomplementable* over $\mathbb{Q}$ if and only if they are *equidecomposable.*

The proof of Zylev [28] adapts without any difficulty to this situation.

Remark: Scissor congruence as defined above does not preserve counting points. A continuity condition is needed, as for maps of $\mathcal{G}_d$.

III. The case of dimension 3: Arithmetical Euler-Poincaré formula.

III.1. We will use the classification of elementary simplices modulo G_3 [21,27].

Proposition 10. Let $T_{p,q}$ be the simplex with vertices the origin and the points $A(1,0,0)$, $B(0,1,0)$, $C(1,p,q)$ in $\mathbb{R}^3$, with

$$1 \leq p < q \ , \ (p,q) = 1 \, .$$

1) $T_{p,q}$ is an elementary simplex of volume q/6.

2) Any elementary simplex of $\mathbb{R}^3$ is equivalent to some $T(p,q)$;
$T(p,q)$ and $T(p',q')$ are equivalent if and only if

$$q = q' \ ; \ p = \pm p' \ (\mathrm{mod}\, q) \, .$$

The main point in the proof consists in proving that any elementary simplex σ in dimension 3 has width equal one, where the width is defined by

$$w(\sigma) = \inf_{u \in \mathbb{Z}^{3*}} D[u(\sigma)] \, .$$

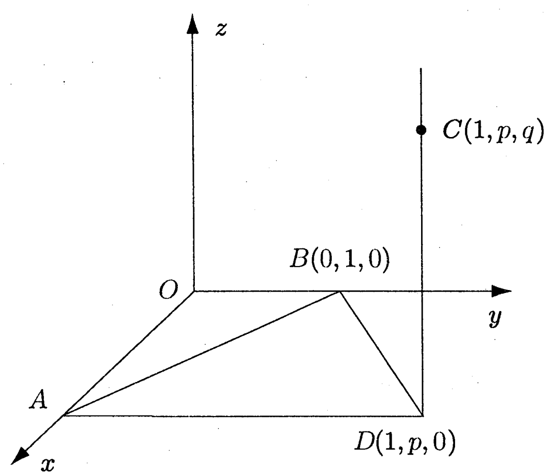
Proposition 11. If $\sigma = T(p,q)$ is as above, its Ehrhart polynomial is

$$i_\sigma(k) = 1 + (2 - q/6)k + k^2 + \frac{q}{6} k^3 \, .$$

Proof:
The two coefficients of top degree are easy to compute from the properties of $i_\sigma(k)$; the coefficient a_1 is determined by writing

$$i_\sigma(1) = 4 \, .$$

Geometric interpretation. Consider the basic triangle $0AB$ in $\mathbb{R}^2$, and add the point $D(1,p,0)$:



$T(p,q)$ is the pyramid over $0AB$ with vertex $C(1,p,q)$.

Consider the cone $T'(p,q)$ of vertex C with basis BDA. By subdividing the trapeze $0BDA$ using $0D$ instead of AB, one gets two different simplices T_1 and T_2. It is easy to show that all T_1, T_2 and $T'(p,q)$ are G_3-equivalent to the simplex

$$T(q) = [0, A, B, E(0,0,q)] \, .$$

Denoting by the same symbol $T_1, T_2, T'(p,q)$ by $T(q)$ one gets

$$T(p,q) \uplus T(q) = T(q) \uplus T(q)$$

($\uplus$: union with no common interior points) which implies by additivity (the intersection of the simplices are primitive triangles) that the Ehrhart polynomial of $T(p,q)$ is equal to the Ehrhart polynomial of $T(q)$.

Other remarkable relations between the $T(p,q)$'s can be obtained. For example consider the famous decomposition of Euclid of a prism as a union of three simplices [4]. Begin with a simplex $T(p,q)$ and construct a prism by adding two simplices like in Euclid. One gets

$$\mathcal{P} = I \times \sigma \qquad I = [0, C(1,p,q)]$$
$$\sigma = \{0, A(1,0,0), B(0,1,0)\}$$

$$\mathcal{P} = T_{p,q} \uplus T' \uplus T''$$

where T' is G_3-equivalent to $T_{p,q}$ and T'' is G_3-equivalent to $T_{q+1-p,q}$

$$T_2 = T(\beta, q-p) \qquad \text{modulo } G_3$$

with

$$\alpha q = 1 \pmod p$$
$$\beta = q \pmod{q-p}$$

and σ and σ' are primitive simplices.

Other decompositions. Another relation can be obtained by adding a point exterior to $T(p,q)$ (as in [2]). One gets

$$T(p,q) \uplus \sigma = \sigma' \uplus T_1 \uplus T_2$$

where T_1 and T_2 can be explicitly described.

All these relations suggest that there should be some arithmetical invariants of an elementary triangulation (apart from the sum of volumes of the various simplices).

III.2. Minimal triangulations in dimension 3. Let $\mathcal{T}$ be a minimal triangulation of the polytope P in $\mathbb{R}^3$, and f the f-vector of $\mathcal{T}$.

Theorem 7. (Arithmetic Euler-Poincaré formula).

The Ehrhart polynomial of P is

$$i_P(k) = 1 + a_1(P)k + (\frac{f_2}{2} - f_3)\,k^2 + V\,k^3$$

where V is the volume of P and

$$a_1(P) = f_1 - \frac{3}{2}f_2 + 2f_3 - V\,.$$

Proof:

Results from the proof of lemma 4 and proposition 11 in dimension 3.

Remarks. a) The Ehrhart polynomial does <u>not</u> depend on the various volumes of the simplices of dimension 3 which occur in the minimal triangulation.
b) Consider the tetrahedron $\mathbf{T}$ in $\mathbb{R}^3$ with vertices the origin and

$$(a,0,0); \ (0,b,0); \ (0,0,c);$$

The only known formula for the number of integral points in $\mathbf{T}$ involves Dedekind sums; [17.24] here, for small values of the integers a, b, c theorem 7, allows to compute this number through triangulations.

IV. Dimension four and above: Convex triangulations.

IV.1 The following is proved in [26]:
Theorem If σ is an elementary simplex of dimension 4, σ has a primitive facet (face of codimension 1).

This means there exists a basis of $\mathbb{Z}^4$ such that σ can be written as the convex envelope of the following vectors

$$\begin{pmatrix} 0 & 1 & 0 & 0 & a_1 \\ 0 & 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 0 & a_4 \end{pmatrix}$$

with g.c.d.$(a_1, a_2, a_3, a_4) = 1$

$$0 \leq a_i < a_4 \quad i = 1, \ldots, 3 .$$

The classification of such simplices is still unknown. In particular:

Conjecture 4.
Elementary simplices in dimension 4 have width less or equal to two.

Definition 8. A triangulation $\mathcal{T}$ of the integral polytope P is said to be *convex* (*projective* in [16]) if the maximal simplices of $\mathcal{T}$ correspond to the domains of linearity of a convex function on P.

The set of convex integral triangulations of P can be identified with a finite set of points in $\mathbb{R}^N$, and the secondary polytope $Q(P)$ is defined as the convex hull of this set. From [13], we know that the *edges* of $Q(P)$ correspond exactly to elementary transformations (called flips or modifications). We have

Proposition 13. Two minimal regular triangulations can be connected by elementary transformations.

This result allows to study problems mentioned above using secondary polytopes. In general elementary transforms of elementary simplices can be elementary or not.

We hope to come back to this.

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