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DIMENSION THEORY OF THE C^*-ALGEBRAS OF LIE GROUPS

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1.1. INTRODUCTION

M.A.Rieffel [R] introduced the notion of stable rank of C^*-algebras, i.e. non commutative complex dimension, and raised the problem such as describing stable rank of the C^*-algebras of Lie groups in terms of their geometry. First of all, A.J-L.Sheu [Sh] succeeded in the computation of stable rank of the C^*-algebras of certain simply-connected connected nilpotent Lie groups. By different methods, H. Takai and the author [ST1] showed that stable rank of the C^*-algebras of simply-connected connected nilpotent Lie groups is equal to complex dimension of the fixed point subspace of the real dual spaces of their Lie algebras under the coadjoint actions. This formula is not valid to the case of exponential Lie groups in general, for example $ax + b$-groups.

The first half of this talk is a joint reserch with H. Takai [ST2]. First of all, we analyze the spectrums of simply-connected connected solvable Lie groups of type I. This is crucial to the computaton of stable rank of their C^*-algebras. Next we show that stable rank of the C^*-algebras of simply-connected connected solvable Lie groups of type I is estimated by complex dimension of the fixed point subspaces of the real dual spaces of their Lie algebras under the coadjoint actions. This result generalizes the estimation in the case of simply-connected connected nilpotent Lie groups [ST1]. As corollaries, we show that the product formula of stable rank holds for the C^*-algebras of connected solvable Lie groups of type I, and estimate real rank in the case of simply-connected connected solvable Lie groups of type I.

In the second half, we consider non-amenable connected real Lie groups of type I [Su]. First of all, we show that stable rank of the reduced C^*-algebras of connected non compact real semi-simple Lie groups is estimated by real rank of these groups. This result is extended to the case of connected reductive Lie groups and partially even to the case of connected non-amenable real Lie groups of type I. As a corollary, we show that the product formula of stable rank holds for locally compact, σ-compact non-amenable groups of type I.

1.2. Spectrum of solvable Lie groups of type I

In this section we show that every irreducible representation of simply-connected connected solvable Lie groups of type I is either 1 or ∞ dimensional. This property is crucial to the estimation of stable rank of the C^*-algebras of those groups. Also we show that 1-dimensional representations of such groups correspond naturally to the fixed points of the real dual spaces of their Lie algebras under the coadjoint actions.

Let G be a connected Lie group and $\hat{G}$ its spectrum which consists of all continuous irreducible unitary representations of G up to equivalence equipped with hull-kernel topology. Let $C^*(G)$ be the C^*-algebra of G, which is generated by the image of the universal unitary representation of G. We identify the spectrum $C^*(G)^\wedge$ of $C^*(G)$ with $\hat{G}$. We denote by $\hat{G}_1$, $\hat{G}_\infty$ the set of all 1, ∞-dimensional representations of G respectively. We call $\hat{G}_1$ the character space of G, which is a topological group with the pointwise multiplication. Then we show $\hat{G} = \hat{G}_1 \cup \hat{G}_\infty$ if G is a simply-connected connected solvable Lie group of type I in what follows.

Let $\mathfrak{G}$ be the Lie algebra of G and $\mathfrak{G}^*$ the real dual space of $\mathfrak{G}$. We denote by Ad the adjoint action of G on $\mathfrak{G}$ and by Ad^* the coadjoint action of G on $\mathfrak{G}$ defined by $\mathrm{Ad}^*(g)\varphi(X) = \varphi(\mathrm{Ad}(g^{-1})(X))$ for g in G, X in $\mathfrak{G}$ and φ in $\mathfrak{G}^*$. We denote by $(\mathfrak{G}^*)^G$ the fixed point subspace of $\mathfrak{G}^*$ under Ad^*. Note that $\mathfrak{G}^*$ is isomorphic to a Euclidean space as a topological (vector) group. Then the following lemma holds:

Lemma 1.2.1. *Let G be a simply-connected connected Lie group. Then $\hat{G}_1$ is isomorphic to $(\mathfrak{G}^*)^G$ as a topological group.*

Sketch of proof. Let χ be an element of $\hat{G}_1$. Its differential $d\chi$ is a Lie homomorphism from $\mathfrak{G}$ to $i\mathbb{R}$ defined by $d\chi(X) = \frac{d}{dt}\chi(\exp tX)|_{t=0}$ for every X in $\mathfrak{G}$. Let Φ be the mapping from $\hat{G}_1$ to $(\mathfrak{G}^*)^G$ defined by $\Phi(\chi) = d\chi/2\pi i$ for χ in $\hat{G}_1$. In fact,

$$\begin{aligned}
\mathrm{Ad}^*(\exp(Y))(d\chi/2\pi i)(X) &= (d\chi/2\pi i)(\mathrm{Ad}(\exp(-Y))X) \\
&= \frac{d}{dt}(\chi/2\pi i)(\exp t(\mathrm{Ad}(\exp(-Y))X))|_{t=0} \\
&= \frac{d}{dt}(\chi/2\pi i)(\exp(-Y)\exp tX\exp(Y))|_{t=0} = (d\chi/2\pi i)(X)
\end{aligned}$$

for every X, Y in $\mathfrak{G}$. Then the injectiveness and surjectiveness of Φ depend on the connectedness and simply-connectedness of G respectively. $\square$

Remark 1.2.2. There exist some non simply-connected connected solvable Lie groups of type I, for which the above lemma is false. In fact, let G be the n-dimensional torus $\mathbb{T}^n$. Then $(\mathfrak{G}^*)^G = \mathbb{R}^n$. On the other hand, $\hat{G} = \mathbb{Z}^n$.

Lemma 1.2.3. *Let G be a connected Lie group. Then $\hat{G}_1$ is isomorphic to $(G/[G,G])^\wedge$ as a topological group where $[G,G]$ is the commutator subgroup of G.*

Sketch of proof. We consider the mapping Φ from $(G/[G,G])^\wedge$ to $\hat{G}_1$ defined by $\Phi(\chi) = \chi \circ q$ for χ in $\hat{G}_1$ where q is the quotient mapping from G to $G/[G,G]$. The surjectiveness of Φ follows from that $G/[G,G]$ is abelian. $\square$

Remark 1.2.4. Since $G/[G,G]$ is a connected commutative Lie group, it is isomorphic to $\mathbb{R}^k \times \mathbb{T}^{n-k}$ for some $k \geq 0$ where $n = \dim(G/[G,G])$. Thus, by Lemma 1.2.1,

$$\hat{G}_1 \cong (G/[G,G])^\wedge \cong \mathbb{R}^k \times \mathbb{Z}^{n-k}$$

as a topological group. If G is a simply-connected connected Lie group, then it follows from Lemma 1.2.1 and 1.2.3 that

$$(\mathfrak{G}^*)^G \cong \hat{G}_1 \cong (G/[G,G])^\wedge \cong \mathbb{R}^n$$

as a topological group.

Next we recall briefly the representation theory of simply-connected connnected solvable Lie groups by Auslander and Kostant [AK] in what follows:

Let G be a simply-connected connected solvable Lie group. Let $\mathfrak{G}_{\mathbb{C}}$ be the complexification of $\mathfrak{G}$ and $\mathfrak{G}_{\mathbb{C}}^*$ its dual space. Let φ be an element of $\mathfrak{G}^*$. We denote by G_φ

(resp. $[\varphi]$) the stabilizer (resp. the orbit) of φ with respect to the coadjoint action of G and by $\mathfrak{G}_\varphi$ its Lie algebra, which equals the radical of φ, i.e.

$$\{X \in \mathfrak{G} \mid \varphi([X,Y]) = 0 \text{ for every } Y \in \mathfrak{G}\}.$$

We extend φ to an element of $\mathfrak{G}_{\mathbb{C}}^*$ by $\varphi(X+iY) = \varphi(X)+i\varphi(Y)$ for $X+iY$ in $\mathfrak{G}_{\mathbb{C}}$. Let $\mathfrak{H}$ be a polarizaton for φ, which satisfies the following conditions:

(1) $\mathfrak{H}$ is a Lie subalgebra of $\mathfrak{G}_{\mathbb{C}}$,

(2) $\mathfrak{H}$ contains $\mathfrak{G}_\varphi$ and is stable under $\mathrm{Ad}(G_\varphi)$,

(3) $\varphi([\mathfrak{H},\mathfrak{H}]) = \{0\}$,

(4) $\dim_{\mathbb{C}}(\mathfrak{G}_{\mathbb{C}}/\mathfrak{H}) = \dim_{\mathbb{R}}[\varphi]$,

(5) $\mathfrak{H}+\bar{\mathfrak{H}}$ is a Lie subalgebra of $\mathfrak{G}_{\mathbb{C}}$,

where $\bar{\mathfrak{H}}$ is the conjugate space of $\mathfrak{H}$ in $\mathfrak{G}_{\mathbb{C}}$.

Put $\mathfrak{H}\cap\mathfrak{G} = \mathfrak{D}$ and $(\mathfrak{H}+\bar{\mathfrak{H}})\cap\mathfrak{G} = \mathcal{E}$. Then $\mathfrak{D}_{\mathbb{C}} = \mathfrak{H}\cap\bar{\mathfrak{H}}$ and $\mathcal{E}_{\mathbb{C}} = \mathfrak{H}+\bar{\mathfrak{H}}$. Let D_o and E_0 be the connected Lie subgroups of G corresponding to Lie algebras $\mathfrak{D}$ and $\mathcal{E}$ respectively. Put $D = G_\varphi D_0$ and $E = G_\varphi E_0$. Then it holds that $E = DE_0$. We have that $\mathrm{Ad}^*(D)\varphi$ is open in the affine subspace $\varphi+\mathcal{E}^\perp$ of $\mathfrak{G}^*$ where $\mathcal{E}^\perp$ is the annihilator of $\mathcal{E}$.

We define an alternating bilinear form $\bar{B}_\varphi$ on $\mathcal{E}/\mathfrak{D}$ by

$$\bar{B}_\varphi(\bar{X},\bar{Y}) = \varphi([Y,X])$$

for φ in $\mathfrak{G}^*$ and $\bar{X},\bar{Y}$ in $\mathcal{E}/\mathfrak{D}$. Then it is a non-singular alternating form on $\mathcal{E}/\mathfrak{D}$. $(\mathcal{E}/\mathfrak{D})_{\mathbb{C}}$ is identified with $\mathcal{E}_{\mathbb{C}}/\mathfrak{D}_{\mathbb{C}}$. Then $(\mathcal{E}/\mathfrak{D})_{\mathbb{C}} = \mathfrak{H}/\mathfrak{D}_{\mathbb{C}}\oplus\bar{\mathfrak{H}}/\mathfrak{D}_{\mathbb{C}}$ where $\oplus$ is the direct sum. Let J be a linear mapping of $(\mathcal{E}/\mathfrak{D})_{\mathbb{C}}$ defined by $J=-iI$ on $\mathfrak{H}/\mathfrak{D}_{\mathbb{C}}$ and $J = iI$ on $\bar{\mathfrak{H}}/\mathfrak{D}_{\mathbb{C}}$. Then J maps $\mathcal{E}/\mathfrak{D}$ onto itself, and $J^2 = -I$ on $\mathcal{E}/\mathfrak{D}$. Let S_φ be the bilinear form on $\mathcal{E}/\mathfrak{D}$ defined by

$$S_\varphi(\bar{X},\bar{Y}) = \bar{B}_\varphi(J\bar{X},\bar{Y}).$$

Then it is a non-singular symmetric bilinear form on $\mathcal{E}/\mathfrak{D}$. We say that a polarization $\mathfrak{H}$ for φ is positive if S_φ is positive definite.

Let $\mathfrak{N}$ be the maximal nilpotent ideal of $\mathfrak{G}$. Since $\mathfrak{N}$ is stable under $\mathrm{Ad}(G)$, so is $\mathfrak{N}^*$ under $\mathrm{Ad}^*(G)$. A polarization $\mathfrak{H}$ for φ is called strongly admissible if $\mathfrak{H}\cap\mathfrak{N}_{\mathbb{C}}$ is a polarization for $\varphi|_{\mathfrak{N}}$ in $\mathfrak{N}^*$, which is stable under $G_{\varphi|_{\mathfrak{N}}}$ where $\varphi|_{\mathfrak{N}}$ is the restriction of φ to $\mathfrak{N}$.

We say that a polarization $\mathfrak{H}$ for φ satisfies Pukanszky condition if $\mathrm{Ad}^*(E)\varphi$ is closed in $\mathfrak{G}^*$. If this condition is satisfied, then $\mathrm{Ad}^*(D)\varphi = \varphi + \mathcal{E}^{\perp}$. Any strongly admissible positive polarization satisfies Pukanszky condition.

An element φ in $\mathfrak{G}^*$ is called integral if there exists a character η_φ of G_φ whose differential $d\eta_\varphi$ is equal to the restriction of $2\pi i\varphi$ to $\mathfrak{G}_\varphi$. More precisely, it is defined by $\eta_\varphi(\exp X) = e^{2\pi i\varphi(X)}$ for X in $\mathfrak{G}_\varphi$. If G is of type I, then every element φ in $\mathfrak{G}^*$ is integral. If a polarization $\mathfrak{H}$ for φ satisfies Pukanszky condition, then η_φ extends uniquely to a character χ_φ of D.

Let $L^2(E/D, \chi_\varphi)$ be a Hilbert space of all complex valued μ_E-measurable functions f on E satisfying

$$\chi_\varphi(d)^{-1} f(e) = f(ed)$$

for d in D and e in E, where μ_E is the Haar measure on E, and

$$\int_{E/D} |f(\bar{e})|^2 d\mu_{E/D}(\bar{e}) < \infty$$

where $\mu_{E/D}$ is the quotient measure of μ_E on E/D and $\bar{e} = eD$ in E/D. The inner product of $L^2(E/D, \chi_\varphi)$ is defined by

$$\langle f_1 | f_2 \rangle = \int_{E/D} f_1(\bar{e})\overline{f_2(\bar{e})} d\mu_{E/D}(\bar{e})$$

for f_1, f_2 in $L^2(E/D, \chi_\varphi)$. Then the induced representation $\mathrm{ind}_{D\uparrow E}\,\chi_\varphi$ of χ_φ to E on $L^2(E/D, \chi_\varphi)$ is defined by

$$(\operatorname*{ind}_{D\uparrow E} \chi_\varphi)(h) f(e) = f(h^{-1}e)$$

for e, h in E.

Let $\mathfrak{H}$ be a strongly admissible positive polarization for φ. Let $L^2(E/D, \chi_\varphi, \mathfrak{H})$ be the closed subspace of $L^2(E/D, \chi_\varphi)$ consisting of all smooth functions f on E with the property that

$$f \cdot Z = 2\pi i \varphi(Z) f$$

for every Z in $\mathfrak{H}$ where $Z = X + iY$ for X, Y in $\mathcal{E}$, $f \cdot Z = f \cdot X + if \cdot Y$ and

$$f \cdot X(e) = \frac{d}{dt} f(e \exp(-tX))|_{t=0}$$

for e in E. In fact, differentiating both sides of the following equation:

$$\chi_\varphi(\exp tX)^{-1} f(e) = f(e \exp tX)$$

at $t = 0$ for X in $\mathcal{D}$, we have that $f \cdot X = 2\pi i \varphi(X) f$. We denote by $\mathrm{ind}_{D\uparrow E}(\chi_\varphi, \mathfrak{H})$ the subrepresentation of $\mathrm{ind}_{D\uparrow E}\,\chi_\varphi$ corresponding to $L^2(E/D, \chi_\varphi, \mathfrak{H})$.

Let $L^2(G/E) \otimes L^2(E/D, \chi_\varphi, \mathfrak{H})$ be the Hilbert space of all $L^2(E/D, \chi_\varphi, \mathfrak{H})$-valued μ_G-measurable functions on G satisfying the similar conditions as above with respect to $\mathrm{ind}_{D\uparrow E}(\chi_\varphi, \mathfrak{H})$. We denote by $\mathrm{ind}_{E\uparrow G}(\mathrm{ind}_{D\uparrow E}(\chi_\varphi, \mathfrak{H}))$ the induced representation of $\mathrm{ind}_{D\uparrow E}(\chi_\varphi, \mathfrak{H})$ to G on $L^2(G/E) \otimes L^2(E/D, \chi_\varphi, \mathfrak{H})$. Let

$$\operatorname*{ind}_{D\uparrow G}(\chi_\varphi, \mathfrak{H}) = \operatorname*{ind}_{E\uparrow G}(\operatorname*{ind}_{D\uparrow E}(\chi_\varphi, \mathfrak{H})).$$

Then we know that if G is of type I, then every element π in $\hat{G}$ is equivalent to an induced representation $\mathrm{ind}_{D\uparrow G}(\chi_\varphi, \mathfrak{H})$ of G.

Note that E/D has a complex structure so that it is holomorphic to $\mathbb{C}^n$ for some $n \geq 0$. Let $\mathcal{A}(E/D)$ be the set of all holomorphic functions on E/D and $\tilde{\mathcal{A}}(E)$ the pull back of $\mathcal{A}(E/D)$ to E. We denote by $z_1^{k_1} \cdots z_n^{k_n}$ the functions of $\mathcal{A}(E/D)$ for $(k_1, \ldots, k_n)$ in $\mathbb{Z}_+^n$ with respect to a complex coordinates $(z_1, \ldots, z_n)$, where $\mathbb{Z}_+ = \{k \in \mathbb{Z} \,|\, k \geq 0\}$. Let $(z_1^{k_1} \cdots z_n^{k_n})^\sim$ be the pull back of $z_1^{k_1} \cdots z_n^{k_n}$ to E. Then there exists a nowhere vanishing smooth function f on E such that $\{(z_1^{k_1} \cdots z_n^{k_n})^\sim f\}$ for $\{(k_1, \ldots, k_n)\}$ in $\mathbb{Z}_+^n$ are in $L^2(E/D, \chi_\varphi, \mathfrak{H})$. Then

$$\langle (z_1^{k_1} \cdots z_n^{k_n})^\sim f | (l_1, \ldots, l_n)^\sim f \rangle = 0, \quad (k_1, \ldots, k_n) \neq (l_1, \ldots, l_n) \in \mathbb{Z}_+^n.$$

We now show the following lemma:

Lemma 1.2.5. *Let G be a simply-connected connected solvable Lie group of type I. Then $\hat{G} = \hat{G}_1 \cup \hat{G}_\infty$.*

Proof. We use the above observation. Let π be an element of $\hat{G}$, which is equivalent to some $\mathrm{ind}_{D\uparrow G}\chi_\varphi$. If $\mathfrak{D} = \mathfrak{G}$, then $\mathfrak{H} \cap \bar{\mathfrak{H}} = \mathfrak{G}_\mathbb{C}$. Hence $\mathfrak{H} = \mathfrak{G}_\mathbb{C}$. It implies that $\varphi([\mathfrak{G}, \mathfrak{G}]) = 0$. Thus φ is in $(\mathfrak{G}^*)^G$. Therefore $\pi = \chi_\varphi$.

Next suppose that $\mathfrak{D} \neq \mathfrak{G}$. If $\dim(E/D) > 0$, then $L^2(E/D, \chi_\varphi, \mathfrak{H})$ is infinite dimensional. If $\dim(E/D) = 0$, then $E_0 = \{1\}$, namely $E = D$. Since D_0 contains $(G_\varphi)_0$ which is the connected component of G_φ containing the unit,

$$D/D_0 = G_\varphi D_0/D_0 \cong G_\varphi/(D_0 \cap G_\varphi) = G_\varphi/(G_\varphi)_0.$$

Thus $\dim D = \dim D_0$, which implies $\dim(G/E) > 0$. Hence, $\mathrm{ind}_{D\uparrow G}\chi_\varphi$ is infinite dimensional. $\square$

Moreover, the following lemma holds:

Lemma 1.2.6. *Let G be a connected Lie group. Then $\hat{G}_1$ is closed in $\hat{G}$.*

Proof. Let π be in the closure of $\hat{G}_1$. Let $\varphi_{\pi,\xi}$ be the state of $C^*(G)$ defined by

$$\varphi_{\pi,\xi}(a) = \langle \pi(a)\xi | \xi \rangle$$

for a in $C^*(G)$ and ξ in the representaton space H_π of π with $\|\xi\| = 1$ where $\langle \cdot | \cdot \rangle$ means the inner product of H_π. By [D; Theorem 3.4.10], we have that

$$\varphi_{\pi,\xi}(a) = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \chi_i(a)$$

for $\{\chi_i\}$ in $\hat{G}_1$ and $\{\alpha_i\}$ in $\mathbb{C}$. It follows that $\varphi_{\pi,\xi}(ab) = \varphi_{\pi,\xi}(ba)$ for a, b in $C^*(G)$. Since ξ is arbitrary, $\pi(ab) = \pi(ba)$. From the irreducibility of π, it belongs to $\hat{G}_1$. Therefore $\hat{G}_1$ is closed in $\hat{G}$. $\square$

Remark 1.2.7. The similar result also holds for arbitrary C^*-algebras.

Combinig Lemma 1.2.5 and 1.2.6, we have the following:

Lemma 1.2.8. *Let G be a simply-connected connected solvable Lie group of type I and $C^*(G)$ its C^*-algebra. Let $\mathfrak{J}$ be the closed ideal of $C^*(G)$ corresponding to $\hat{G}_\infty$ and $C_0(\hat{G}_1)$ the C^*-algebra of all continuous functions on $\hat{G}_1$ vanishing at infinity. Then the following exact sequence is obtained:*

$$0 \to \mathfrak{J} \to C^*(G) \to C_0(\hat{G}_1) \to 0.$$

Remark 1.2.9. The similar result also holds for connected solvable Lie groups where $\hat{\mathfrak{J}}$ is $\hat{G} \setminus \hat{G}_1$.

1.3. Main theorems in the first half

First of all, we recall the definitions of stable rank and real rank respectively.

Let $\mathfrak{A}$ be a unital C^*-algebra. We denote by $\mathrm{sr}(\mathfrak{A})$ the stable rank of $\mathfrak{A}$. Then $\mathrm{sr}(\mathfrak{A}) \leq n$ if every element $(a_i)_{i=1}^n$ of the n-direct sum $\mathfrak{A}^n$ of $\mathfrak{A}$ can be approximated by the element $(b_i)_{i=1}^n$ of $\mathfrak{A}^n$ such that $\sum_{i=1}^n b_i^* b_i$ is invertible in $\mathfrak{A}$. If there exists no such n, then we let $\mathrm{sr}(\mathfrak{A}) = \infty$. If $\mathfrak{A}$ is non unital, then the stable rank of $\mathfrak{A}$ is defined by $\mathrm{sr}(\tilde{\mathfrak{A}})$ where $\tilde{\mathfrak{A}}$ means the unitization of $\mathfrak{A}$. We use the basic results of stable rank in [R] later.

Let $\mathfrak{A}_{sa}$ be the set of all self-adjoint elements of $\mathfrak{A}$. We denote by $\mathrm{rr}(\mathfrak{A})$ the real rank of $\mathfrak{A}$. Then $\mathrm{rr}(\mathfrak{A}) \leq n$ means that every element $(a_i)_{i=0}^n$ of $\mathfrak{A}_{sa}^{n+1}$ can be approximated by the element $(b_i)_{i=0}^n$ such that $\sum_{i=0}^n b_i^2$ is invertible in $\mathfrak{A}$. If there exists no such n, then we let $\mathrm{rr}(\mathfrak{A}) = \infty$. If $\mathfrak{A}$ is non unital, then real rank of $\mathfrak{A}$ is defined by $\mathrm{rr}(\tilde{\mathfrak{A}})$ (cf. [BP]).

Next result is useful for computation of stable rank, and related in a certain sense with the formula such that $\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) \leq 2$ for arbitrary C^*-algebra $\mathfrak{A}$ where $\mathbb{K}$ is the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space.

Proposition 1.3.1. *Let $\mathfrak{A}$ be a separable C^*-algebra of type I such that every element of $\hat{\mathfrak{A}}$ is infinite dimensional. Then* $\mathrm{sr}(\mathfrak{A}) \leq 2$.

Proof. Let $\{\mathcal{I}_n\}_{n=1}^\infty$ be a composition sereis of $\mathfrak{A}$ with $\mathcal{I}_0 = 0$ such that $\{\mathcal{I}_n/\mathcal{I}_{n-1}\}_{n=1}^\infty$ are of continuous trace. Consider the following exact sequences:

$$0 \to \mathcal{I}_n/\mathcal{I}_{n-1} \to (\mathcal{I}_n/\mathcal{I}_{n-1})^\sim \to \mathbb{C} \to 0$$

for every n. By Nistor's result [N; Lemma 2],

$$\mathrm{sr}((\mathcal{I}_n/\mathcal{I}_{n-1})^\sim) \leq 2 \vee \mathrm{sr}(\mathbb{C}) = 2,$$

where $\vee$ means maximum. Hence $\mathrm{sr}(\mathcal{I}_n/\mathcal{I}_{n-1}) \leq 2$ for every n. Next consider the following exact sequences:

$$0 \to \mathcal{I}_k/\mathcal{I}_{k-1} \to \mathcal{I}_n/\mathcal{I}_{k-1} \to \mathcal{I}_n/\mathcal{I}_k \to 0$$

for $1 \leq k \leq n-1$. Again by Nistor's result,

$$\mathrm{sr}(\mathcal{I}_n/\mathcal{I}_{k-1}) \leq 2 \vee \mathrm{sr}(\mathcal{I}_n/\mathcal{I}_k)$$

for $1 \leq k \leq n-1$. It follows that $\mathrm{sr}(\mathcal{I}_n) \leq 2$ for every n. By the density of $\cup_{n=1}^{\infty}\mathcal{I}_n$ in $\mathfrak{A}$, we conclude that $\mathrm{sr}(\mathfrak{A}) \leq 2$. $\square$

As a first step of the computation of stable rank of the C^*-algebras of simply-connected connected solvable Lie groups of type I, we have the following:

Lemma 1.3.2. *Let G be a simply-connected connected solvable Lie group of type I, $\hat{G}_1$ its character space and $C^*(G)$ its C^*-algebra. Then*

$$\mathrm{sr}(C^*(G))\begin{cases} \leq 2 & \text{if } \dim \hat{G}_1 = 1, \\ = \dim_{\mathbb{C}}(\hat{G}_1) & \text{if } \dim \hat{G}_1 \geq 2 \end{cases}$$

where $\dim_{\mathbb{C}}(\cdot) = [\dim(\cdot)/2] + 1$ *and* $[\cdot]$ *is Gauss symbol.*

Proof. Put $\mathfrak{A} = C^*(G)$. Let $\{\mathfrak{I}_k\}_{k=1}^{\infty}$ be a composition series of $\mathfrak{A}$ with $\mathfrak{I}_0 = \{0\}$ such that $\{\mathfrak{I}_k/\mathfrak{I}_{k-1}\}_{k=1}^{\infty}$ are of continuous trace. We consider the following exact sequences:

$$0 \to \mathfrak{I} \cap \mathfrak{I}_k \to \mathfrak{I}_k \to C_0(\hat{G}_1 \cap (\hat{\mathfrak{I}_k} \setminus (\mathfrak{I} \cap \mathfrak{I}_k)^\wedge)) \to 0$$

for every k, where $\mathfrak{I}$ is the closed ideal of $\mathfrak{A}$ as in Lemma 1.2.8. Then $\{\mathfrak{I} \cap \mathfrak{I}_s\}_{s=1}^{k}$ is the finite composition series of $\mathfrak{I} \cap \mathfrak{I}_k$. Put $\mathfrak{D}_s = \mathfrak{I} \cap \mathfrak{I}_s$ for $1 \leq s \leq k$ with $\mathfrak{D}_0 = \{0\}$. Next we consider the following exact sequences:

$$0 \to \mathfrak{D}_s/\mathfrak{D}_{s-1} \to \mathfrak{I}_k/\mathfrak{D}_{s-1} \to \mathfrak{I}_k/\mathfrak{D}_s \to 0$$

for $1 \leq s \leq k$. Note that $\{\mathfrak{D}_s/\mathfrak{D}_{s-1}\}_{s=1}^{k}$ are of continuous trace, and every element of $(\mathfrak{D}_s/\mathfrak{D}_{s-1})^\wedge$ is infinite dimensional. Then applying Nistor's result [N; Lemma 2],

$$\mathrm{sr}(\mathfrak{I}_k/\mathfrak{D}_{s-1}) \leq 2 \vee \mathrm{sr}(\mathfrak{I}_k/\mathfrak{D}_s)$$

for $1 \leq s \leq k$. By repetition, $\mathrm{sr}(\mathfrak{I}_k) \leq 2 \vee \mathrm{sr}(C_0(\hat{G}_1 \cap (\hat{\mathfrak{I}_k} \setminus (\mathfrak{I} \cap \mathfrak{I}_k)^\wedge)))$. Hence, we obtain $\mathrm{sr}(\mathfrak{I}_k) \leq 2 \vee \dim_{\mathbb{C}}(\hat{G}_1)$ for every k.

Now put $m = 2 \vee \dim_{\mathbb{C}}(\hat{G}_1)$. Let $(a_i)_{i=1}^{m}$ be an arbitrary element of $\mathfrak{A}^m$. Then for a large enough $n \geq 1$, there exists an element $(b_i)_{i=1}^{m}$ of $\mathfrak{I}_n^m$ such that $\|a_i - b_i\| < \varepsilon/2$ for

$1 \leq i \leq m$. Since $\mathrm{sr}(\mathfrak{J}_n) \leq m$, there exists an element $(c_i)_{i=1}^m$ of $\mathfrak{J}_n^m$ such that $\sum_{i=1}^m c_i^* c_i$ is invertible in $\mathfrak{J}_n$ and $\|b_i - c_i\| < \varepsilon/2$ for $1 \leq i \leq m$. Thus $\|a_i - c_i\| < \varepsilon$ for $1 \leq i \leq m$ and $\sum_{i=1}^m c_i^* c_i$ is invertible in $\mathfrak{A}$. Therefore $\mathrm{sr}(\mathfrak{A}) \leq m$. □

We now show that Lemma 1.3.2 extends to the case of connected solvable Lie groups of type I.

Proposition 1.3.3. *Let G be a connected solvable Lie group of type I, $\hat{G}_1$ its character space and $C^*(G)$ its C^*- algebra. Then*

$$\mathrm{sr}(C^*(G)) \begin{cases} \leq 2 & \text{if } \dim \hat{G}_1 = 0 \text{ or } 1, \\ = \dim_{\mathbb{C}} \hat{G}_1 & \text{if } \dim \hat{G}_1 \geq 2. \end{cases}$$

Proof. Let G be a connected Lie group of type I and $\tilde{G}$ its universal covering group. We denote by q the quotient map from $\tilde{G}$ to G and by Γ the kernel of q. Then we define the map Φ from $\hat{G}$ to $(\tilde{G})^\wedge$ by $\Phi(\pi)(g) = \pi(g\Gamma)$ for π in $\hat{G}$ and g in $\tilde{G}$. It follows from Lemma 1.2.5 that $\hat{G} = \hat{G}_\infty \cup \hat{G}_1$. Therefore Lemma 1.3.2 holds for connected solvable Lie groups of type I. □

Remark 1.3.4. This result suggests that stable rank of $C^*(G)$ is controlled by the character space $\hat{G}_1$ of G. By Remark 1.2.2, $\hat{G}_1$ is not replaced by $(\mathfrak{G}^*)^G$ in general.

We give the application of Proposition 1.3.3 to show the product formula of stable rank in the case of the C^*-algebras of connected solvable Lie groups of type I as follows:

Corollary 1.3.5. *Let G, H be two connected solvable Lie groups of type I, and $C^*(G)$, $C^*(H)$ their C^*-algebras respectively. Then*

$$\mathrm{sr}(C^*(G) \otimes C^*(H)) \leq \mathrm{sr}(C^*(G)) + \mathrm{sr}(C^*(H)).$$

Sketch of proof. First of all, note that $C^*(G) \otimes C^*(H)$ is isomorphic to $C^*(G \times H)$. We also have that

$$\dim_{\mathbb{C}} (G \times H)_1^\wedge \leq \dim_{\mathbb{C}} \hat{G}_1 + \dim_{\mathbb{C}} \hat{H}_1, \quad \mathrm{sr}(C^*(G)) + \mathrm{sr}(C^*(H)) \geq 2.$$

By Proposition 1.3.3, the proof is complete. □

Remark 1.3.6. The above product formula gives an affirmative answer to a question raised by M. A. Rieffel [R], whether for any two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$,

$$\mathrm{sr}(\mathfrak{A} \otimes \mathfrak{B}) \leq \mathrm{sr}(\mathfrak{A}) + \mathrm{sr}(\mathfrak{B}).$$

We proceed to refine Lemma 1.3.2. Next lemma is useful in computation of stable rank. To prove it we use the basic results of K-theory and a generalized index theory (refer to [W]).

Lemma 1.3.7. *Let G be a simply-connected connected solvable Lie group and $C^*(G)$ its C^*-algebra. Then $\mathrm{sr}(C^*(G)) = 1$ if and only if $G \cong \mathbb{R}$.*

Proof. If $G \cong \mathbb{R}$, then by Fourier transform, $C^*(G) \cong C_0(\mathbb{R})$. Hence $\mathrm{sr}(C^*(G)) = 1$.

Conversely, let $\dim G = m + 1 \geq 2$. Then G is considered as a semi-direct product $N \rtimes \mathbb{R}$ where N is a simply-connected connected solvable Lie subgroup of G and $\dim N = m$. By Lemma 1.2.8, the following exact sequence is obtained:

$$0 \to \mathfrak{I}_N \to C^*(N) \to C_0(\hat{N}_1) \to 0$$

where $\mathfrak{I}_N$ is the ideal corresponding to an open subset $\hat{N} \setminus \hat{N}_1$ of $\hat{N}$. Moreover, since $\hat{N}_1$ is $\mathbb{R}$-invariant closed, the following exact sequence is obtained:

$$0 \to \mathfrak{I}_N \rtimes \mathbb{R} \to C^*(N) \rtimes \mathbb{R} \to C_0(\hat{N}_1) \rtimes \mathbb{R} \to 0.$$

Note that $\hat{N}_1$ is homeomorphic to a Euclidean space $\mathbb{R}^n$ for $n = \dim(\hat{N}_1) \geq 1$.

Denote by $\mathbb{R}^n_1$ the set of all φ in $\mathbb{R}^n$ such that $\mathbb{R}_\varphi = \mathbb{R}$ where $\mathbb{R}_\varphi$ means the stabilizer of φ under the coadjoint action of $\mathbb{R}$. Since $\mathbb{R}^n_1$ is $\mathbb{R}$-invariant, we have the following exact sequence:

$$0 \to C_0(\mathbb{R}^n \setminus \mathbb{R}^n_1) \rtimes \mathbb{R} \to C_0(\mathbb{R}^n) \rtimes \mathbb{R} \to C_0(\mathbb{R}^n_1 \times \mathbb{R}) \to 0.$$

If $\mathbb{R}^n_1 \neq \{0\}$, then $\mathrm{sr}(C_0(\mathbb{R}^n_1 \times \mathbb{R})) \geq 2$. It implies that $\mathrm{sr}(C^*(G)) \geq 2$.

Next consider the case $\mathbb{R}^n_1 = \{0\}$. Then we have the following six-term exact sequence:

$$\begin{array}{ccccc}
K_0(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}) & \longrightarrow & K_0((C_0(\mathbb{R}^n) \rtimes \mathbb{R})) & \longrightarrow & K_0(C_0(\mathbb{R})) \\
{\scriptstyle \delta}\big\uparrow & & & & \big\downarrow \\
K_1(C_0(\mathbb{R})) & \longleftarrow & K_1((C_0(\mathbb{R}^n) \rtimes \mathbb{R})) & \longleftarrow & K_1(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R})
\end{array}$$

Using the Connes'Thom isomorphism,

$$K_i((C_0(\mathbb{R}^n) \rtimes \mathbb{R})) \cong K_{i+1}(C_0(\mathbb{R}^n)) \cong K_{i+1+n}(\mathbb{C})$$

for $i = 0$ or 1. If n is even, say $n = 2m \geq 2$, then

$$K_i((C_0(\mathbb{R}^{2m}) \rtimes \mathbb{R})) \cong \begin{cases} K_{1+2m}(\mathbb{C}) = 0 & \text{if } i = 0 \\ K_{2+2m}(\mathbb{C}) = \mathbb{Z} & \text{if } i = 1. \end{cases}$$

If n is odd, say $n = 2m + 1 \geq 1$, then

$$K_i((C_0(\mathbb{R}^{2m+1}) \rtimes \mathbb{R})) \cong \begin{cases} K_{1+2m+1}(\mathbb{C}) = \mathbb{Z} & \text{if } i = 0 \\ K_{2+2m+1}(\mathbb{C}) = 0 & \text{if } i = 1. \end{cases}$$

Again, using the Connes'Thom isomorphism,

$$K_i(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}) \cong K_{i+1}(C_0(\mathbb{R}^n \setminus \{0\})) \cong K_{i+1}(C(S^{n-1}) \otimes C_0(\mathbb{R}_+)).$$

Then using the Künneth formula, $K_{i+1}(C(S^{n-1}) \otimes C_0(\mathbb{R}_+)) \cong$

$$\begin{cases} (K_0(C(S^{n-1})) \otimes K_1(C_0(\mathbb{R}))) \oplus (K_1(C(S^{n-1})) \otimes K_0(C_0(\mathbb{R}))) & \text{if } i = 0 \\ (K_0(C(S^{n-1})) \otimes K_0(C_0(\mathbb{R}))) \oplus (K_1(C(S^{n-1})) \otimes K_1(C_0(\mathbb{R}))) & \text{if } i = 1. \end{cases}$$

Note that $K_i(C(S^{n-1})) \cong K_i(C_0(\mathbb{R}^{n-1}) \oplus \mathbb{C}) \cong$

$$\begin{cases} K_0(C_0(\mathbb{R}^{n-1})) \oplus \mathbb{Z} \cong K_{n-1}(\mathbb{C}) \oplus \mathbb{Z} & \text{if } i = 0 \\ K_1(C_0(\mathbb{R}^{n-1})) \cong K_n(\mathbb{C}) & \text{if } i = 1. \end{cases}$$

Hence, if $n = 2m \geq 2$, then $K_i(C_0(\mathbb{R}^{2m} \setminus \{0\}) \rtimes \mathbb{R}) \cong$

$$\begin{cases} ((K_{2m-1}(\mathbb{C}) \oplus \mathbb{Z}) \otimes \mathbb{Z}) \oplus (K_{2m}(\mathbb{C}) \otimes 0) \cong \mathbb{Z} & \text{if } i = 0 \\ ((K_{2m-1}(\mathbb{C}) \oplus \mathbb{Z}) \otimes 0) \oplus (K_{1+2m-1}(\mathbb{C}) \otimes \mathbb{Z}) \cong \mathbb{Z} & \text{if } i = 1. \end{cases}$$

If $n = 2m + 1 \geq 1$, then $K_i(C_0(\mathbb{R}^{2m+1} \setminus \{0\}) \rtimes \mathbb{R}) \cong$

$$\begin{cases} ((K_{2m}(\mathbb{C}) \oplus \mathbb{Z}) \otimes \mathbb{Z}) \oplus (K_{2m+1}(\mathbb{C}) \otimes 0) \cong \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 0 \\ ((K_{2m}(\mathbb{C}) \oplus \mathbb{Z}) \otimes 0) \oplus (K_{2m+1}(\mathbb{C}) \otimes \mathbb{Z}) \cong 0 & \text{if } i = 1. \end{cases}$$

Thus, the above six-term exact sequence is equal to the following diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \\ \mathbb{Z} \arrow[u, "\delta"] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l] \end{tikzcd} \qquad \text{if } n \text{ is even,}$$

$$\begin{array}{ccccc} \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ {\scriptstyle\delta}\uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \end{array} \quad \text{if } n \text{ is odd.}$$

Note that the index map δ from $K_1(C_0(\mathbb{R}))(\cong K_1(C(S^1)))$ to $K_0(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R})$ is non zero in both cases.

Putting $\mathfrak{J} = (C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}) \otimes \mathbb{K}$, we have the following exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{J} & \longrightarrow & ((C_0(\mathbb{R}^n) \rtimes \mathbb{R})^{\sim}) \otimes \mathbb{K} & \xrightarrow{\sigma} & C(S^1) \otimes \mathbb{K} & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\mu}\downarrow & & {\scriptstyle\tau}\downarrow & & \\ 0 & \longrightarrow & \mathfrak{J} & \longrightarrow & M(\mathfrak{J}) & \xrightarrow{q} & M(\mathfrak{J})/\mathfrak{J} & \longrightarrow & 0 \end{array}$$

where $M(\mathfrak{J})$ is the multiplier algebra of $\mathfrak{J}$. Then the following six-term exact sequence is obtained:

$$\begin{array}{ccccc} K_0(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}) & \longrightarrow & K_0(M(\mathfrak{J})) & \longrightarrow & K_0(M(\mathfrak{J})/\mathfrak{J}) \\ {\scriptstyle\eta}\uparrow & & & & \downarrow \\ K_1(M(\mathfrak{J})/\mathfrak{J}) & \longleftarrow & K_1(M(\mathfrak{J})) & \longleftarrow & K_1(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}) \end{array}$$

From the fact that $K_i(M(\mathfrak{A} \otimes \mathbb{K}) \otimes \mathfrak{B}) = 0$ for $i = 0, 1$ where $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras and $\mathfrak{B}$ is unital [W; Theorem 10.2], we have $K_i(M(\mathfrak{J})) = 0$ for $i = 0, 1$. Thus,

$$K_i(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}) \cong K_{i+1}(M(\mathfrak{J})/\mathfrak{J}), \quad \text{for } i = 0, 1 \,(\mathrm{mod}\, 2).$$

Then the above six-term exact sequence is equal to the following diagram:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\ {\scriptstyle\eta}\uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} \end{array} \quad \text{if } n \text{ is even,}$$

$$\begin{array}{ccccc} \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\ {\scriptstyle\eta}\uparrow & & & & \downarrow \\ \mathbb{Z}\oplus\mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \end{array} \quad \text{if } n \text{ is odd.}$$

Let D be an element of $(C_0(\mathbb{R}^n) \rtimes \mathbb{R})^{\sim}$ such that $\sigma(D) = \mathrm{id}$ where $\mathrm{id}(z) = z$ for z in S^1, which is identified with a diagonal matrix in $M_\infty((C_0(\mathbb{R}^n) \rtimes \mathbb{R})^{\sim})$ having the diagonal entries $(D, 0, \dots)$. Then the class $[\sigma(D)]$ in $K_1(C(S^1))$ is a genarater. By generalized index theory, the index of $\mu(D)$ is defined by

$$\mathrm{index}(\mu(D)) = \eta([q(\mu(D))]) \quad \text{in } K_0(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R})$$

where $[q(\mu(D))]$ is in $K_1(M(\mathfrak{J})/\mathfrak{J})$. We take a unitary w in $M_2((C_0(\mathbb{R}^n) \rtimes \mathbb{R})^\sim)$ such that $\sigma(D) \oplus \sigma(D)^* = \sigma(w)$. Then $\tau(\sigma(D)) \oplus \tau(\sigma(D)^*) = \tau(\sigma(w))$. It follows that $q(\mu(D)) \oplus q(\mu(D))^* = q(\mu(w))$ and $\mu(w)$ is a unitary in $M_2(M(\mathfrak{J}))$. By the definition of the index map,

$$\delta([\sigma(D)]) = [wp_2w^*] - [p_2] \neq 0$$

where p_2 is a rank 2 projection in $(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R})^\sim \otimes \mathbb{K}$, which is identified with a diagonal matrix in $M_\infty((C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R})^\sim)$ having the diagonal entries $(1, 1, 0, \dots)$. On the other hand,

$$\eta([q(\mu(D))]) = [\mu(w)p_2\mu(w)^*] - [p_2] = [wp_2w^*] - [p_2].$$

If $\mathrm{sr}(C^*(G)) = 1$, then $\mathrm{sr}(C_0(\mathbb{R}^n) \rtimes \mathbb{R}) = 1$. Hence, $\mathrm{sr}((C_0(\mathbb{R}^n) \rtimes \mathbb{R})^\sim \otimes \mathbb{K}) = 1$. It follows that invertible elements of $M(\mathfrak{J})$ are dense in $\mu((C_0(\mathbb{R}^n) \rtimes \mathbb{R})^\sim \otimes \mathbb{K})$. By the property of the generalized index, we deduce that $\mathrm{index}(\mu(D)) = 0$ which is a contradiction. Therefore $\mathrm{sr}(C^*(G)) \geq 2$. $\square$

Remark 1.3.8. Let G be as in Lemma 1.3.7. If $\dim G = 2$, then $\mathrm{sr}(C^*(G)) = 2$. In fact, it is known that G is isomorphic to $\mathbb{R}^2$ or the real $ax + b$-group which is treated in Example 1.4.1 later. Thus $\mathrm{sr}(C^*(G)) = 2$. However, the converse of the implication is false in general. For example, $\mathbb{R}^3$ is a counter example.

Combining Lemma 1.2.1, 1.3.2 and 1.3.7, we obtain the following main result in the first half:

Theorem 1.3.9. *Let G be a simply-connected connected solvable Lie group of type I, $C^*(G)$ its C^*-algebra and $(\mathfrak{G}^*)^G$ the fixed point subspace under its coadjoint action. Then*

$$\mathrm{sr}(C^*(G)) = (\dim_{\mathbb{C}}(\mathfrak{G}^*)^G \vee 2) \wedge \dim G.$$

Proof. By Lemma 1.3.7, we know that $\mathrm{sr}(C^*(G)) = 1$ if and only if $\dim G = 1$. By Lemma 1.2.1, we replace $\hat{G}_1$ in Lemma 1.3.2 with $(\mathfrak{G}^*)^G$. By Lemma 1.3.2 and 1.3.7, if $\dim G \geq 2$ and $\dim(\mathfrak{G}^*)^G = 1$, then

$$\mathrm{sr}(C^*(G)) = 2 = (\dim_{\mathbb{C}}(\mathfrak{G}^*)^G \vee 2) \wedge \dim G.$$

By Lemma 1.3.7, if $\dim G \geq 2$ and $\dim(\mathfrak{G}^*)^G \geq 2$, then

$$\mathrm{sr}(C^*(G)) = \dim_{\mathbb{C}}(\mathfrak{G}^*)^G = (\dim_{\mathbb{C}}(\mathfrak{G}^*)^G \vee 2) \wedge \dim G. \quad \square$$

Remark 1.3.10. This result extends our estimation in the case that G is a simply-connected connected nilpotent Lie group. It also suggests that stable rank of $C^*(G)$ is controlled by the geometrical structure of G. If G is abelian, then $C^*(G) \cong C_0(\hat{G})$. Thus $\mathrm{sr}(C^*(G)) = \dim_{\mathbb{C}} \hat{G}$. By Lemma 1.2.3, the formula in Theorem 1.3.9 is replaced by

$$\mathrm{sr}(C^*(G)) = (\dim_{\mathbb{C}}(G/[G,G])^{\wedge} \vee 2) \wedge \dim G.$$

Therefore, Theorem 1.3.9 extends naturally the abelian case.

Next, we apply Theorem 1.3.9 to compute real rank as follows:

Corollary 1.3.11. *Let G be a simply-connected connected solvable Lie group of type I, $C^*(G)$ its C^*-algebra and $(\mathfrak{G}^*)^G$ the fixed point subspace under its coadjoint action. Then*

$$\mathrm{rr}(C^*(G)) = 1 \quad \text{if } \dim G = 1,$$

$$\dim(\mathfrak{G}^*)^G \leq \mathrm{rr}(C^*(G)) \leq \begin{cases} \dim(\mathfrak{G}^*)^G + 1 & \text{if } \dim(\mathfrak{G}^*)^G \text{ is even,} \\ \dim(\mathfrak{G}^*)^G \vee 3 & \text{if } \dim(\mathfrak{G}^*)^G \text{ is odd.} \end{cases} \quad \text{if } \dim G \geq 2,$$

Sketch of proof. We use the following inequality:

$$\mathrm{rr}(C_0((\mathfrak{G}^*)^G)) \leq \mathrm{rr}(C^*(G)) \leq 2\,\mathrm{sr}(C^*(G)) - 1.$$

See [BP] for the second inequality. Applying Theorem 1.3.9, the proof is complete. $\square$

1.4. Examples

In this section we give several examples which support Theorem 1.3.9 in what follows:

Example 1.4.1. Let G be the extended real $ax+b$-group, i.e. the semi-direct product $\mathbb{R}^n \rtimes \mathbb{R}$ defined by all $(n+1) \times (n+1)$ matrices of the following form:

$$g = \begin{pmatrix} \alpha(t) & a \\ 0 & 1 \end{pmatrix}, \quad \alpha(t) = \begin{pmatrix} e^t & & 0 \\ & \ddots & \\ 0 & & e^t \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

for each $t, a_1, \dots a_n$ in $\mathbb{R}$. Put $g = (t, a_1, \dots a_n)$. If $n = 1$, then G is the real $ax+b$-group. The Lie algebra $\mathfrak{G}$ of G is defined by all $(n+1) \times (n+1)$ matrices of the following form:

$$X = \begin{pmatrix} tI_n & x \\ 0 & 0 \end{pmatrix}, \quad I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

for each $t, x_1, \dots x_n$ in $\mathbb{R}$. The real dual space $\mathfrak{G}^*$ of $\mathfrak{G}$ is defined by all $(n+1) \times (n+1)$ matrices of the following form:

$$\varphi = \begin{pmatrix} lI_n & 0 \\ m & 0 \end{pmatrix}, \quad m = \begin{pmatrix} m_1 & \cdots & m_n \end{pmatrix}$$

for each $l, m_1, \dots m_n$ in $\mathbb{R}$. We let $\varphi = (l, m_1, \dots m_n)$. The duality is defined by $\varphi(X) = \mathrm{tr}(X\varphi)$ for X in $\mathfrak{G}$ and φ in $\mathfrak{G}^*$ where tr is the natural trace of $M_{n+1}(\mathbb{R})$. Then the coadjoint action of G is given by

$$\mathrm{Ad}^*(\exp X)\varphi = (l - (nt)^{-1}(e^{-t} - 1)\sum_{i=1}^{n} x_i m_i, e^{-t}m_1, \dots, e^{-t}m_n).$$

Thus $(\mathfrak{G}^*)^G$ consists of all matrices of the form $(l, 0, \dots, 0)$. Hence $\dim_{\mathbb{C}}(\mathfrak{G}^*)^G = 1$. By Theorem 1.3.9, we conclude that $\mathrm{sr}(C^*(G)) = 2$.

On the other hand, let $g = (t, a_1, \dots, a_n), h = (s, b_1, \dots, b_n)$ be in G. Then

$$ghg^{-1}h^{-1} = (0, (1 - e^s)a_1 + (1 - e^t)b_1, \cdots, (1 - e^s)a_n + (1 - e^t)b_n).$$

It follows that $[G, G]$ contains all matrices of the form $(0, a_1, \dots, a_n)$. Thus we see $G/[G, G] \cong \mathbb{R}$. Hence $(G/[G, G])^\wedge \cong \mathbb{R}$.

Next we consider the structure of $C^*(G)$. Then the following exact sequence is obtained:

$$0 \to C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R} \to C^*(G) \to C_0(\mathbb{R}) \to 0.$$

Then $C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R} \cong C(S^{n-1}) \otimes \mathbb{K}$ where S^{n-1} is the $(n-1)$-dimensional sphere and $S^0 = \{-1, +1\}$. If $n \geq 3$, then $\mathrm{sr}(C^*(G)) = 2$. In the case of $n = 1$ or 2, we have $\mathrm{sr}(C^*(G))$ is either 1 or 2. By Theorem 1.3.9, we conclude that $\mathrm{sr}(C^*(G)) = 2$.

From the above observation,

$$\dim_{\mathbb{C}}(\mathfrak{G}^* \oplus \mathfrak{G}^*)^{G \times G} = 2, \quad (G \times G/[G \times G, G \times G])^\wedge \cong \mathbb{R}^2.$$

Applying Theorem 1.3.9, we obtain $\mathrm{sr}(C^*(G \times G)) = 2$.

Example 1.4.2. Let G be the split oscillator group, i.e. the semi-direct product $H \rtimes \mathbb{R}$ defined by all 3×3 matrices of the following form:

$$g = \begin{pmatrix} 1 & a & b \\ 0 & e^t & c \\ 0 & 0 & 1 \end{pmatrix}$$

for t, a, b, c in $\mathbb{R}$ where H is the 3-dimensional Heisenberg group. Put $g = (t, a, b, c)$. Then G is a simply-connected connected exponential solvable Lie group. The Lie algebra $\mathfrak{G}$ of G is defined by all 3×3 matrices of the following form:

$$X = \begin{pmatrix} 0 & x & y \\ 0 & t & z \\ 0 & 0 & 0 \end{pmatrix}$$

for t, x, y, z in $\mathbb{R}$. The real dual space $\mathfrak{G}^*$ of $\mathfrak{G}$ is defined by all 3×3 matrices of the following form:

$$\varphi = \begin{pmatrix} 0 & 0 & 0 \\ l & u & 0 \\ m & n & 0 \end{pmatrix}$$

for u, l, m, n in $\mathbb{R}$. We let $\varphi = (u, l, m, n)$. The duality is the same as in Example 1.4.1. Then the coadjoint action of G is given by

$$\mathrm{Ad}^*(\exp X)\varphi = (u', e^t l + t^{-1}(e^t - 1)zm, m, e^{-t}n + t^{-1}(e^{-t} - 1)xm)$$

where $u' = t^{-1}(e^t - 1)xl - t^{-1}(e^{-t} - 1)zn - 2xym + u$. Thus $(\mathfrak{G}^*)^G$ consists of all matrices of the form $(u, 0, 0, 0)$. Hence $\dim_{\mathbb{C}}(\mathfrak{G}^*)^G = 1$. By Theorem 1.3.9, we conclude that $\mathrm{sr}(C^*(G)) = 2$.

On the other hand, let $g = (t, a_1, 0, 0), h = (s, a_2, 0, 0)$ be in G. Then

$$ghg^{-1}h^{-1} = (t, e^{-t}(1 - e^{-s})a_1 + e^{-s}(e^{-t} - 1)a_2, 0, 0).$$

Let $g = (t, 0, 0, c_1), h = (s, 0, 0, c_2)$ be in G. Then

$$ghg^{-1}h^{-1} = (0, 0, 0, (1 - e^s)c_1 + (e^t - 1)c_2).$$

It follows that $[G, G]$ contains all matrices of the form $(0, a, 0, c)$. Let $g = (0, a_1, b_1, c_1)$, $h = (0, a_2, b_2, c_2)$ be in G. Then

$$ghg^{-1}h^{-1} = (0, 0, a_1c_2 - a_2c_1, 0).$$

Note that $(0, a, b, c) = (0, a, 0, c)(0, 0, b, 0)$. Since $[G, G]$ is a subgroup of G, it contains all matrices of the form $(0, a, b, c)$. It follows that $[G, G] \cong H$. Thus $G/[G, G] \cong \mathbb{R}$. Hence $(G/[G, G])^{\wedge} \cong \mathbb{R}$.

From the above observation,

$$\dim_{\mathbb{C}}(\mathfrak{G}^* \oplus \mathfrak{G}^*)^{G \times G} = 2, \quad (G \times G/[G \times G, G \times G])^{\wedge} \cong \mathbb{R}^2.$$

Applying Theorem 1.3.9, we obtain $\mathrm{sr}(C^*(G \times G)) = 2$.

Example 1.4.3. Let G be the semi-direct product $\mathbb{R}^2 \rtimes \mathbb{R}$ defined by all 3×3 matrices of the following form:

$$g = \begin{pmatrix} \alpha(t) & a \\ 0 & 1 \end{pmatrix}, \quad \alpha(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

for each t, a_1, a_2 in $\mathbb{R}$. Put $g = (t, a_1, a_2)$. Then G is the only non exponential simply-connected connected solvable Lie group with dimensions ≤ 3 up to isomorphisms (cf. [LL]). Actually, the Lie algebra $\mathfrak{G}$ of G is defined by all 3×3 matrices of the following form:

$$X = \begin{pmatrix} 0 & -t & x_1 \\ t & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}$$

The real dual space $\mathfrak{G}^*$ of $\mathfrak{G}$ is defined by all 3×3 matrices of the following form:

$$\varphi = \begin{pmatrix} 0 & m & 0 \\ -m & 0 & 0 \\ l_1 & l_2 & 0 \end{pmatrix}$$

Put $\varphi = (m, l_1, l_2)$. The duality is the same as in Example 1.4.1. Then the coadjoint action of G is given by

$$\mathrm{Ad}^*(\exp X) = (m', l_1 \cos(-t) + l_2 \sin(-t), -l_1 \sin(-t) + l_2 \cos(-t))$$

where $m' = m + (2t)^{-1}(\sin(-t)(x_2 l_1 - x_1 l_2) + (1 - \cos(-t))(x_1 l_1 + x_2 l_2))$. Note that $G_\varphi = \mathbb{R}^2 \rtimes \mathbb{Z}$ for $\varphi = (0, l_1, l_2)$ with non zero l_1, l_2. It is known that if G is an exponential Lie group, then G_φ is connected for every φ in $\mathfrak{G}^*$ (cf. [LL]). Thus G is non exponential. Then $(\mathfrak{G}^*)^G$ consists of all matrices of the form $(m, 0, 0)$. Hence $\dim_{\mathbb{C}}(\mathfrak{G}^*)^G = 1$. By Theorem 1.3.9, we conclude that $\mathrm{sr}(C^*(G)) = 2$.

On the other hand, let $g = (t, a_1, a_2), h = (s, b_1, b_2)$ be in G. Then

$$ghg^{-1}h^{-1} = \begin{pmatrix} \alpha(0) & (1_2 - \alpha(s))a + (\alpha(t) - 1_2)b \\ 0 & 1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus $[G, G]$ consists of all matrices of the form $(0, a_1, a_2)$. Hence $G/[G, G] \cong \mathbb{R}$. Thus $(G/[G, G])^\wedge \cong \mathbb{R}$.

From the above observation,

$$\dim_{\mathbb{C}}(\mathfrak{G}^* \oplus \mathfrak{G}^*)^{G\times G} = 2, \quad (G \times G/[G \times G, G \times G])^\wedge \cong \mathbb{R}^2.$$

Applying Theorem 1.3.9, we obtain $\mathrm{sr}(C^*(G \times G)) = 2$.

2.1. Introduction of the second half

In the first half, stable rank of the C^*-algebras of the radical part of simply-connected connected Lie groups of type I has been computed. In the second half, we first focus our attention on the non radical part of connected Lie groups, i.e. connected non compact real semi-simple Lie groups. They are non-amenable so that we only consider their reduced C^*-algebras. We show that stable rank of these algebras is handled by real rank of those groups. This result extends to the case of connected reductive Lie groups and partially even to the case of connected non-amenable Lie groups of type I. As a corollary, we show that the product formula of stable rank holds for the reduced C^*-algebras of locally compact, σ-compact non-amenable groups of type I.

Let G be a locally compact group and $\hat{G}_r$ its reduced dual which is the support of the regular representaton of G. Let $C_r^*(G)$ be the reduced C^*-algebra of G, which is generated by the image of the regular representaton of G. We identify the spectrum $C_r^*(G)^\wedge$ of $C_r^*(G)$ with $\hat{G}_r$.

2.2. The case of semi-simple Lie groups

First of all, we give some basic properties of connected non compact real semi-simple Lie groups (refer to [Kn]).

Let G be a connected non compact real semi-simple Lie group with its Lie algebra $\mathfrak{g}$. Let θ be a Cartan involution of G, which is an automorphism of G such that $\theta^2 = 1$.

Let $K = \{g \in G \mid \theta(g) = g\}$ be the maximal compact subgroup of G corresponding to θ. Let $d\theta$ be the differential of θ. Since $(d\theta)^2 = 1$, we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ where $\mathfrak{k}, \mathfrak{p}$ are $+1, -1$ eigenspaces of $\mathfrak{g}$ under $d\theta$ respectively.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ its real dual space. We identify $\mathfrak{a}^*$ with a Euclidean space. For every φ in $\mathfrak{a}^*$, let $\mathfrak{g}_\varphi$ be its root space defined by

$$\{X \in \mathfrak{g} \mid [Y, X] = \varphi(Y)X \text{ for every } Y \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\varphi \neq \{0\}$, we call φ a root of $\mathfrak{g}$. Let Δ be the set of all roots of $\mathfrak{g}$. Fix a basis $\{\varphi_i\}_{i=1}^n$ of $\mathfrak{a}^*$. We call φ positive if $\varphi = \sum_{i=1}^n x_i\varphi_i$ with $x_i = 0 (1 \leq i \leq k)$ and $x_{k+1} > 0$ for some $k \geq 0$. Let Δ^+ be the set of all positive roots of $\mathfrak{g}$. Put $\mathfrak{n} = \sum_{\varphi \in \Delta^+} \mathfrak{g}_\varphi$ which is a nilpotent Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}$ decomposes into the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Let K, A and N be the Lie subgroups of G corresponding to $\mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}$ respectively. Then G has an Iwasawa decomposition $G = KAN$. Define by $\mathrm{rr}(G)$ the dimension of A, i.e. real rank of G. Let $M = Z_K(\mathfrak{a})$ which is defined by

$$\{g \in K \mid \mathrm{Ad}(g)X = X \text{ for every } X \in \mathfrak{a}\}.$$

It is a compact subgroup of G with its Lie algebra $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ which is defined by

$$\{X \in \mathfrak{k} \mid [Y, X] = 0 \text{ for every } Y \in \mathfrak{a}\}.$$

Then $P = MAN$ is a Lie subgroup of G, which is called a minimal parabolic subgroup of G determined uniquely up to conjugacy.

Let W be the Weyl group defined by the quotient $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ where $N_K(\mathfrak{a})$ is defined by $\{g \in K \mid \mathrm{Ad}(g)\mathfrak{a} = \mathfrak{a}\}$. Then W acts on $\hat{M} \times \hat{A}$ as follows:

$$w \cdot \sigma(m) = \sigma(u^{-1}mu) \quad \sigma \in \hat{M}, m \in M, \quad w \cdot \chi_s(a) = \chi_s(u^{-1}au) \quad a \in A$$

where u is any representative of w in W, s is in $\mathfrak{a}^*$ and $\chi_s(\exp X) = e^{is(X)}$ for X in $\mathfrak{a}$. We identify χ_s in $\hat{A}$ with s in $\mathfrak{a}^*$. Let (σ, s) be an element of $\hat{M} \times \hat{A}$. We denote by $[(\sigma, s)]$ the orbit of (σ, s) under W and by $(\hat{M} \times \hat{A})/W$ the orbit space of $\hat{M} \times \hat{A}$.

Then the induced representations $\mathrm{ind}_{P\uparrow G}(\sigma \otimes \chi_s)$ of $\sigma \otimes \chi_s$ to G are in $\hat{G}_r$ where $\sigma \otimes \chi_s$ are the unitary representations of P defined by $\sigma \otimes \chi_s(man) = \sigma(m)\chi_s(a)$ for m in M,

a in A and n in N. Put $\pi(\sigma, s) = \mathrm{ind}_{P\uparrow G}(\sigma \otimes \chi_s)$. Then $\pi(\sigma, s)$ is unitarily equivalent to $\pi(\sigma', s')$ if and only if there exists an element w of W such that $w \cdot (\sigma, s) = (\sigma', s')$. Thus we denote by $\pi([(\sigma, s)])$ the equivalence class of $\pi(\sigma, s)$.

We refer to [L] for a topology on $\hat{G}_r$. Then the following lemma is obtained:

Lemma 2.2.1. *Let G be a connected non compact real semi-simple Lie group and $C_r^*(G)$ its reduced C^*-algebra. If* $\mathrm{rr}(G) \geq 2$, *then* $\mathrm{sr}(C_r^*(G)) \geq 2$.

Proof. It is known that $\pi([(1_M, s)])$ is irreducible for every s in $\hat{A}$ where 1_M is the trivial representation of M [Ko]. Since $\{1_M\} \times \hat{A}$ is W-invariant clopen subset of $\hat{M} \times \hat{A}$, we see that $(\{1_M\} \times \hat{A})/W = \hat{A}/W$ is clopen in $(\hat{M} \times \hat{A})/W$. Thus there exist the direct summands $\mathfrak{J}$ and $\mathfrak{K}$ of $C_r^*(G)$ such that $C_r^*(G) = \mathfrak{J} \oplus \mathfrak{K}$, $\hat{\mathfrak{J}} = \hat{A}/W$ and $\hat{\mathfrak{K}}$ is the complement of $\hat{A}/W$ in $(\hat{M} \times \hat{A})/W$. Since $C_r^*(G)$ is liminal, so is $\mathfrak{J}$. As $\hat{A}/W$ is a locally compact T_2-space, $\mathfrak{J}$ is isomorphic to the C^*-algebra associated with the continuous fields on $\hat{\mathfrak{J}}$ [D; Theorem 10.5.4.]. We take a closed ideal $\mathfrak{L}$ of $\mathfrak{J}$, which is of continuous trace. It is also isomorphic to the C^*-algebra associated with the continuous fields on $\hat{\mathfrak{L}}$. By its local triviality [D; Theorem 10.9.5.], there exists a closed ideal $\mathcal{E}$ of $\mathfrak{L}$, which is isomorphic to $C_0(\hat{\mathcal{E}}) \otimes \mathbb{K}$. Since $\dim \hat{A} \geq 2$ and W is finite, we see $\dim(\hat{A}/W) \geq 2$ so that $\dim \hat{\mathcal{E}} \geq 2$. Thus $\mathrm{sr}(\mathcal{E}) = 2$. Therefore $\mathrm{sr}(C_r^*(G)) \geq 2$. $\square$

We refer to [BM] for a topology on $\hat{G}_r$ in the case $\mathrm{rr}(G) = 1$ Then we have the following lemma:

Lemma 2.2.2. *Let G be a connected non compact real semi-simple Lie group and $C_r^*(G)$ its reduced C^*-algebra. If* $\mathrm{rr}(G) = 1$, *then* $\mathrm{sr}(C_r^*(G)) = 1$.

Proof. If $\mathrm{rr}(G) = 1$, then $\hat{A} \cong \mathbb{R}$ and $W = \{1, w\}$ where w is the unique non trivial element of W. It acts on $\hat{M} \times \hat{A}$ as follows:

$$1 \cdot (\sigma, s) = (\sigma, s), \quad w \cdot (\sigma, s) = (w \cdot \sigma, -s) \quad (\sigma, s) \in \hat{M} \times \hat{A}.$$

Then $(\hat{M} \times \hat{A})/W$ is a locally compact T_2-space. Let $F = \{\sigma \in \hat{M} \mid w \cdot \sigma = \sigma\}$. Then $(F \times \hat{A})/W = F \times [0, \infty)$. Then $F \times (0, \infty)$ is embedded in $\hat{G}_r$. Each point $(\sigma, 0)$ of $F \times \{0\}$ corresponds to two irreducible representatons $\{\pi_\sigma^+, \pi_\sigma^-\}$ of G. The topology on

$(\{\sigma\} \times (0,\infty)) \cup \{\pi_\sigma^+, \pi_\sigma^-\}$ is the usual topology except that $\{(\sigma, s)\}$ converges to $\pi_\sigma^+, \pi_\sigma^-$ as s tends to 0.

Let C be the complement of F in $\hat{M}$. Let (σ, s) be in $C \times \hat{A}$. Then we have that $(\{\sigma\} \times \hat{A} \sqcup \{w \cdot \sigma\} \times \hat{A})/W = \mathbb{R}$. It follows that $(C \times \hat{A})/W = \sqcup_{C/W}\mathbb{R}$. Then $\hat{G}_r$ decomposes into the following fashion:

$$\hat{G}_r = \hat{G}_p \cup \hat{G}_l \cup \hat{G}_d, \quad \hat{G}_p = (F \times (0,\infty)) \sqcup (\sqcup_{C/W}\mathbb{R}), \quad \hat{G}_l = \sqcup_{\sigma \in F}\{\pi_\sigma^+, \pi_\sigma^-\},$$

and $\hat{G}_d$ is the discrete series of G.

We construct a finite composition series $\{\mathfrak{I}_k\}_{k=1}^3$ of $C_r^*(G)$ with $\mathfrak{I}_0 = \{0\}$ and $\mathfrak{I}_3 = C_r^*(G)$ as follows: $\hat{\mathfrak{I}}_1 = \hat{G}_p$, $(\mathfrak{I}_2/\mathfrak{I}_1)^\wedge = \hat{G}_l$ and $(\mathfrak{I}_3/\mathfrak{I}_2)^\wedge = \hat{G}_d$. Then

$$\mathfrak{I}_1 \cong (\oplus_{C/W} C_0(\mathbb{R}) \otimes \mathbb{K}) \oplus (\oplus_F C_0((0,\infty)) \otimes \mathbb{K}),$$
$$\mathfrak{I}_2/\mathfrak{I}_1 \cong \oplus_F(\mathbb{K} \oplus \mathbb{K}), \quad \mathfrak{I}_3/\mathfrak{I}_2 \cong \oplus_{\hat{G}_d}\mathbb{K}.$$

Then $\{\mathfrak{I}_k/\mathfrak{I}_{k-1}\}_{k=1}^3$ have stable rank 1 and $\{\mathfrak{I}_k/\mathfrak{I}_{k-1}\}_{k=2}^3$ have connected stable rank 1 (cf. [R]). Therefore $\mathrm{sr}(C_r^*(G)) = 1$. $\square$

Next result is useful in the computation of stable rank.

Proposition 2.2.3. *Let G be a locally compact, σ-compact non-amenable group of type I and $C_r^*(G)$ its reduced C^*-algebra. Then* $\mathrm{sr}(C_r^*(G)) \leq 2$.

Proof. It is known that if $\hat{G} \neq \hat{G}_r$, then every element of $\hat{G}_r$ is infinite dimensional [F]. By Proposition 1.3.1, the proof is complete. $\square$

We give an application of Proposition 2.2.3 to show the product formula of stable rank in the case of the reduced C^*-algebras of locally compact, σ-compact non-amenable groups of type I as follows:

Corollary 2.2.4. *Let G, H be two connected locally compact, σ-compact non-amenable groups of type I, and $C_r^*(G)$, $C_r^*(H)$ their reduced C^*-algebras respectively. Then*

$$\mathrm{sr}(C_r^*(G) \otimes C_r^*(H)) \leq \mathrm{sr}(C_r^*(G)) + \mathrm{sr}(C_r^*(H)).$$

Proof. Let e_G, e_H and $e_{G\times H}$ be the units of G, H and $G \times H$ respectively. Let $1_G, 1_H$ and $1_{G\times H}$ be their trivial representations, and λ_G, λ_H and $\lambda_{G\times H}$ their regular representations respectively. Then by [FD; Corollary 12.18, 13.6],

$$\lambda_{G\times H} \simeq \operatorname*{ind}_{\{e_{G\times H}\}\uparrow G\times H} 1_{G\times H} \simeq (\operatorname*{ind}_{\{e_G\}\uparrow G} 1_G) \otimes (\operatorname*{ind}_{\{e_H\}\uparrow H} 1_H) \simeq \lambda_G \otimes \lambda_H$$

where $\simeq$ is unitary equivalence. Thus $C_r^*(G \times H)$ is isomorphic to $C_r^*(G) \otimes C_r^*(H)$. By Proposition 2.2.3, $\mathrm{sr}(C_r^*(G) \otimes C_r^*(H)) \leq 2$. Therefore the proof is complete. $\square$

Combining Lemma 2.2.1, 2.2.2 and Proposition 2.2.3, we have the following theorem:

Theorem 2.2.5. *Let G be a connected non compact real semi-simple Lie group and $C_r^*(G)$ its reduced C^*-algebra. Then*

$$\mathrm{sr}(C_r^*(G)) = \mathrm{rr}(G) \wedge 2$$

where $\wedge$ means the minimum.

Remark 2.2.6. This result suggests that stable rank of the reduced C^*-algebras of connected non compact real semi-simple Lie groups is controlled by the real rank (i.e. the geometrical structure) of G. Note that $\mathrm{rr}(G) = 0$ if and only if G is compact. Then $\hat{G}$ is discrete. Thus $C_r^*(G)$ is isomorphic to $\oplus_{\lambda\in\hat{G}} M_{n_\lambda}(\mathbb{C})$ where $M_{n_\lambda}(\mathbb{C})$ is the C^*-algebra of all $n_\lambda \times n_\lambda$ complex matrices. Hence $\mathrm{sr}(C_r^*(G)) = 1$.

We give some examples which support Theorem 2.2.5 in what follows:

Example 2.2.7. Let G be a connected real semi-simple Lie group with $\mathrm{rr}(G) = 1$. Then it is known that G is locally isomorphic to one of the following groups (cf. [HV]):

$$\begin{aligned} &SO_0(n,1), \quad SU(n,1), \\ &Sp(n,1), \quad F_{4(-20)}, \quad (n \geq 2). \end{aligned}$$

Thus their reduced C^*-algebras have stable rank 1.

Example 2.2.8. Let $G = SL_n(\mathbb{R})$ for $n \geq 2$. Its Iwasawa decomposition is obtained as follows: Then $K = SO_n(\mathbb{R})$. A consists of all diagonal matrices such that

$$\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

where $a_i > 0$ $(1 \leq i \leq n)$ and $\Pi_{i=1}^n a_i = 1$. It is isomorphic to $(\mathbb{R}_+^*)^{n-1}$ where $\mathbb{R}_+^*$ is the multiplicative group of positive real numbers. N consists of all upper trianguler matrices such that

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Thus $\mathrm{rr}(G) = 1$ if and only if $n = 2$. Therefore we obtain that

$$\mathrm{sr}(C_r^*(SL_n(\mathbb{R}))) = \begin{cases} 1 & \text{if } n = 2 \\ 2 & \text{if } n \geq 3. \end{cases}$$

Example 2.2.9. Let $G = \widetilde{SL_n(\mathbb{R})}$ be the universal covering group of $SL_n(\mathbb{R})$ for $n \geq 2$. It is known that G is a non linear semi-simple Lie group. Since the fundamental group of $SL_n(\mathbb{R})$ is equal to $\mathbb{Z}$ $(n = 2)$ and $\mathbb{Z}_2$ $(n \geq 3)$, we have that $G/\mathbb{Z} \cong SL_2(\mathbb{R})$ and $G/\mathbb{Z}_2 \cong SL_n(\mathbb{R})$ $(n \geq 3)$ respectively. Since $\mathbb{Z}$ and $\mathbb{Z}_2$ are amenable closed normal subgroups of G, we know that $C_r^*(SL_n(\mathbb{R}))$ is the quotient of $C_r^*(G)$ (cf. [Ka; p.1349]). By Example 2.2.8, $\mathrm{sr}(C_r^*(G)) \geq 2$ if $n \geq 3$. If $n = 2$, then $\mathrm{rr}(G) = 1$. Therefore we obtain that

$$\mathrm{sr}(C_r^*(\widetilde{SL_n(\mathbb{R})})) = \begin{cases} 1 & \text{if } n = 2 \\ 2 & \text{if } n \geq 3. \end{cases}$$

2.3. The case of reductive Lie groups

In this section, we show that Theorem 2.2.5 extends to the case of connected reductive Lie groups. First of all, we examine the structure of these groups.

Let G be a connected real reductive Lie group with its Lie algebra $\mathfrak{g}$ and $\tilde{G}$ its universal covering group. Then $\mathfrak{g}$ has Levi decomposition $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$. It is known that any two simply-connected Lie groups with the same Lie algebras are isomorphic (cf. [Kn; Appendix A.114]). Thus, $\tilde{G}$ is isomorphic to the direct product $Z \times S$ where Z is the Lie subgroup of $\tilde{G}$ with its Lie algebra $\mathfrak{z}$ and S is the semi-simple Lie subgroup of $\tilde{G}$ with its Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Then the center $Z_{\tilde{G}}$ of $\tilde{G}$ is of the form $Z \times Z_S$ where Z_S is the center of S. Let Γ be a discrete subgroup of $\tilde{G}$ contained in $Z_{\tilde{G}}$ such that $G = (Z \times S)/\Gamma$. Then Γ is isomorphic to the direct product $\Gamma_Z \times \Gamma_S$ where Γ_Z and Γ_S are discrete subgroups of Z and Z_S respectively. Thus we have $G = (Z/\Gamma_Z) \times (S/\Gamma_S)$.

Let $G_a = Z/\Gamma_Z$ be the abelian direct factor of G and $G_s = S/\Gamma_S$ the semi-simple one of G. Note that G_s is equal to the commutater subgroup $[G, G]$ of G. By the same reason in Corollary 2.2.4, $C_r^*(G)$ is isomorphic to $C_r^*(G_a) \otimes C_r^*(G_s)$. Thus $\hat{G}_r = \hat{G}_a \times (\hat{G}_s)_r$. Hence $\hat{G} \neq \hat{G}_r$ if and only if $\mathrm{rr}(G_s) \geq 1$. Since G_a is isomorphic to $\mathbb{R}^k \times \mathbb{T}^{n-k}$ for some $k \geq 0$ and $n = \dim Z \geq 0$, $C_r^*(G_a)$ is isomorphic to $C_0(\mathbb{R}^k \times \mathbb{Z}^{n-k})$.

We denote by Z_G the center of G. Then $Z_G = G_a \times Z_{G_s}$ where Z_{G_s} is the at most countable center of G_s.

Then we have the following theorem:

Theorem 2.3.1. *Let G be a connected non-amenable real reductive Lie group with its center Z_G and $C_r^*(G)$ its reduced C^*-algebra. Then*

$$\mathrm{sr}(C_r^*(G)) = (\mathrm{rr}([G, G]) \vee (\dim(Z_G)^\wedge + 1)) \wedge 2$$

where $\vee$ means the maximum.

Proof. If G_a is compact, and $\mathrm{rr}(G_s) = 1$, then $C_r^*(G)$ is isomorphic to $C_0(\mathbb{Z}^n) \otimes C_r^*(G_s)$ for $n = \dim(G_a)$. Using the structure of $C_r^*(G_s)$ in Lemma 2.2.2, and tensoring $C_0(\mathbb{Z}^n)$ with $C_r^*(G_s)$, we conclude that $\mathrm{sr}(C_r^*(G)) = 1$. On the other hand, since $\dim(Z_G)^\wedge = 0$, we have that $(\mathrm{rr}([G, G]) \vee (\dim(Z_G)^\wedge + 1)) \wedge 2 = 1$.

Next, by the methods of Lemma 2.2.1, $C_r^*(G_s)$ has a closed ideal $\mathfrak{J}$ which is isomorphic to $C_0(\hat{\mathfrak{J}}) \otimes \mathbb{K}$ where $\dim(\hat{\mathfrak{J}}) = \mathrm{rr}(G_s)$. Then $C_0((G_a)^\wedge) \otimes \mathfrak{J}$ is a closed ideal of $C_r^*(G)$, which is isomorphic to $C_0((G_a)^\wedge \times \hat{\mathfrak{J}}) \otimes \mathbb{K}$. If G_a is non compact and $\mathrm{rr}(G_s) = 1$, or $\mathrm{rr}(G_s) \geq 2$, then $\dim((G_a)^\wedge \times \hat{\mathfrak{J}}) \geq 2$. Thus $\mathrm{sr}(C_0((G_a)^\wedge \times \hat{\mathfrak{J}}) \otimes \mathbb{K}) = 2$. Hence $\mathrm{sr}(C_r^*(G)) \geq 2$. By Proposition 2.2.3, we see $\mathrm{sr}(C_r^*(G)) = 2$. On the other hand, since $\dim(Z_G)^\wedge \geq 1$ or $\mathrm{rr}([G, G]) \geq 2$, we have $(\mathrm{rr}([G, G]) \vee (\dim(Z_G)^\wedge + 1)) \wedge 2 = 2$. □

Remark 2.3.2. We consider the case that G is amenable. If G_a is compact, and $\mathrm{rr}(G_s) = 0$, then G is compact. It follows that $\mathrm{sr}(C^*(G)) = 1$.

If G_a is non compact, and $\mathrm{rr}(G_s) = 0$, then G is of the from $\mathbb{R}^k \times \mathbb{T}^{n-k} \times G_s$ for $k \geq 1$ and $n = \dim G_a$, and G_s is compact. Then

$$\begin{aligned} C^*(G) &\cong C_0(\mathbb{R}^k) \otimes C_0(\mathbb{Z}^{n-k}) \otimes C^*(G_s) \\ &\cong (\oplus_{\mathbb{Z}^{n-k}} C_0(\mathbb{R}^k)) \otimes (\oplus_{\lambda \in \hat{G}_s} M_{n_\lambda}(\mathbb{C})) \cong \oplus_{\mathbb{Z}^{n-k}, \lambda \in \hat{G}_s} (C_0(\mathbb{R}^k) \otimes M_{n_\lambda}(\mathbb{C})). \end{aligned}$$

Thus we obtain that

$$\begin{aligned}\mathrm{sr}(C^*(G)) &= \sup_{\lambda\in\hat{G}_s} \mathrm{sr}(C_0(\mathbb{R}^k)\otimes M_{n_\lambda}(\mathbb{C}))\\ &= \sup_{\lambda\in\hat{G}_s} (\lceil(\mathrm{sr}(C_0(\mathbb{R}^k))-1)/n_\lambda\rceil+1) = \sup_{\lambda\in\hat{G}_s} (\lceil([k/2])/n_\lambda\rceil+1)\end{aligned}$$

where $[\cdot]$ is Gauss symbol, $\lceil x\rceil = [x]+1$ for x in $\mathbb{R}\setminus\mathbb{Z}$ and $\lceil x\rceil = x$ for x in $\mathbb{Z}$ (cf. [R]).

Next we give an example which support Theorem 2.3.1 as follows:

Example 2.3.3. Let $G = GL_n(\mathbb{R})_0$ be the connected component of $GL_n(\mathbb{R})$ containing the unit of G for $n\geq 2$, which consists of all invertible matrices with positive determinant. We consider the mapping Φ from G to $\mathbb{R}_+^*\times SL_n(\mathbb{R})$ defined by $\Phi(g) = (\det(g), g/\det(g))$ for g in G. It is clear that Φ is a Lie group isomorphism. Since G_a is non compact, we conclude that

$$\mathrm{sr}(C_r^*(GL_n(\mathbb{R})_0)) = 2 \quad \text{for}\, n\geq 2.$$

2.4. The case of non-amenable Lie groups of type I

In this section, we show that Theorem 2.3.1 extends partially to the case of connected real Lie groups of type I.

Let G be a connected real Lie group of type I and R its radical, which is the maximal connected solvable normal Lie subgroup of G. It is known that if G/R is compact, then $\hat{G} = \hat{G}_r$ [D; Proposition 18.3.9]. Thus, if $\hat{G}\neq\hat{G}_r$, then G/R is non compact. We only consider this case. Since R is amenable, we know that $C_r^*(G/R)$ is the quotient of $C_r^*(G)$ (cf. [Ka; p.1349]). Then we have the following result:

Theorem 2.4.1. *Let G be a connected non-amenable real Lie group of type I with its radical R and $C_r^*(G)$ its reduced C^*-algebra. Then*

$$\mathrm{sr}(C_r^*(G)) = \begin{cases} 1 \text{ or } 2 & \text{if } \mathrm{rr}(G/R)=1,\\ 2 & \text{if } \mathrm{rr}(G/R)\geq 2.\end{cases}$$

Proof. By Proposition 2.2.3, we know $\mathrm{sr}(C_r^*(G))\leq 2$. By Lemma 2.2.1, if $\mathrm{rr}(G/R)\geq 2$, then $\mathrm{sr}(C_r^*(G/R))\geq 2$. Thus $\mathrm{sr}(C_r^*(G))\geq 2$. Therefore, we obtain $\mathrm{sr}(C_r^*(G)) = 2$. $\square$

Remark 2.4.2. The above formula is the best inequalitity. For example, let G be the direct product $\mathbb{T} \times S$ where S is a connected real semi-simple Lie group with $\mathrm{rr}(S) = 1$. By Theorem 2.3.1, we know that $\mathrm{sr}(C_r^*(G)) = 1$. On the other hand, let G be the direct product $\mathbb{R} \times S$ where S is the same as before. By Theorem 2.3.1, we have that $\mathrm{sr}(C_r^*(G)) = 2$.

Finally, we give an example which support Theorem 2.4.1 as follows:

Example 2.4.3. Let G be the direct product $H \times SL_n(\mathbb{R})$ for $n \geq 2$ where H is the real 3-dimensional Heisenberg group. Then G is a connected real non reductive Lie group of type I. If $n \geq 3$, then $\mathrm{rr}(G/H) \geq 2$. By Theorem 2.4.1, we have $\mathrm{sr}(C_r^*(G)) = 2$. Next we consider the case $n = 2$. Then $\mathrm{rr}(SL_2(\mathbb{R})) = 1$. Note that $\hat{G} = \hat{H} \times (SL_2(\mathbb{R}))^\wedge$. Thus $\hat{G}_r = \hat{H} \times (SL_2(\mathbb{R}))_r^\wedge$. It follows that $C_r^*(G) \cong C^*(H) \otimes C_r^*(SL_2(\mathbb{R}))$. It is known that $C^*(H)$ decomposes into the following exact sequence:

$$0 \to C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \to C^*(H) \to C_0(\mathbb{R}^2) \to 0.$$

Tensoring $C_r^*(SL_2(\mathbb{R}))$ with this sequence, we have that

$$0 \to C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \otimes C_r^*(SL_2(\mathbb{R})) \to C_r^*(G) \to C_0(\mathbb{R}^2) \otimes C_r^*(SL_2(\mathbb{R})) \to 0.$$

Using the structure in Lemma 2.2.2, we know that $C_r^*(SL_2(\mathbb{R}))$ has $\mathbb{K}$ as a quotient. Thus $C_r^*(G)$ has $C_0(\mathbb{R}^2) \otimes \mathbb{K}$ as a quotient. Hence $\mathrm{sr}(C_r^*(G)) \geq 2$. Therefore we have that

$$\mathrm{sr}(C_r^*(H \times SL_n(\mathbb{R}))) = 2 \quad \text{if } n \geq 2.$$

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