数理解析研究所講究録
957 巻 1996 年 167-185

Wick Product of White Noise Operators and Its Application to Quantum Stochastic Differential Equations

NOBUAKI OBATA

GRADUATE SCHOOL OF POLYMATHEMATICS
NAGOYA UNIVERSITY
NAGOYA, 464-01 JAPAN

Introduction

After the famous paper by Hudson–Parthasarathy [11] quantum stochastic processes on the (Boson) Fock space $\Gamma(L^2(\mathbb{R}))$ have been developed considerably by many authors, see the excellent books by Meyer [14] and by Parthasarathy [23] and references cited therein. In those works the annihilation process $\{A_t\}$, the creation process $\{A_t^*\}$ and the number process $\{\Lambda_t\}$ are considered as primary quantum noises and the bulk is devoted to establishing a quantum analogue of Itô theory, where the role of infinitesimal increment of the Brownian motion dB_t in the classical Itô theory is played by dA_t, dA_t^* and $d\Lambda_t$. Thus quantum stochastic differential equations to be discussed are typically of the form

$$dU=(L_1 d\Lambda+L_2 dA+L_3 dA^*+L_4 dt)U, \qquad U(0)=I. \tag{0.1}$$

Here, at the request of physical applications an initial Hilbert space or a system Hilbert space $\mathcal{H}$ being taken into account, L_i are operators acting on $\mathcal{H}$ and the solution U_t will be an operator process acting on $\mathcal{H}\otimes\Gamma(L^2(\mathbb{R}))$.

On the other hand, in view of the white noise approach to classical stochastic analysis (see, e.g., Kuo [13]), one expects that white noise distribution theory (WNDT) leads to a breakthrough in quantum stochastic analysis. In fact, during recent years white noise approach to quantum stochastic processes has initiated by a series of papers [18], [19], [20], [21], [22], etc., see also [9], [10]. The essence of this approach lies in the fact that every quantum stochastic process is expressible in terms of two quantum noises $\{a_t\}$ and $\{a_t^*\}$, which are time derivatives of the annihilation and the creation processes, that is, $dA_t=a_t dt$ and $dA_t^*=a_t^* dt$. From that viewpoint (0.1) is reduced to

$$\frac{dU}{dt}=(L_1 a_t^* a_t+L_2 a_t+L_3 a_t^*+L_4)U, \tag{0.2}$$

or in the normal form:

$$\frac{dU}{dt}=L_1 a_t^* U a_t+L_2 U a_t+L_3 a_t^* U+L_4 U. \tag{0.3}$$

Moreover, during the lectures of Accardi [2] a new type of a quantum stochastic differential equation such as

$$\frac{dU}{dt} = (M_1 a_t^{*2} + M_2 a_t^2)U \tag{0.4}$$

comes within our scope (though the above equation is understood just formally at the moment). Note that an equation as in (0.4) is highly singular from the usual aspect.

The main purpose of this paper is to give a first step toward a new theory of quantum stochastic differential equations on the basis of WNDT. We introduce the Wick product (or normal product) of operators by means of the characterization theorem of operator symbols. We then discuss existence and uniqueness of a solution of a certain class of quantum stochastic differential equations which possess fairly singular coefficients. It turns out that the refreshed WNDT due to Kuo [13], where the Hida–Kubo–Takenaka space is replaced with the Kondratiev–Streit space, is more suitable for our purpose. This generalization, however, causes no new difficulty since most basic results obtained so far for the Hida–Kubo–Takenaka space [15] admit straightforward generalizations to the Kondratiev–Streit space. We hope that our theory is also applied to some problems in quantum dissipation discussed by Accardi [1], [2], Arimitsu [4], Gardiner [8], Saito–Arimitsu [24], etc.

1 WNDT – White noise distribution theory

Let $H = L^2(\mathbb{R}, dt; \mathbb{R})$ be the real Hilbert space of $\mathbb{R}$-valued L^2-functions on $\mathbb{R}$. The norm and the inner product are denoted by $|\cdot|_0$ and $\langle\cdot,\cdot\rangle$, respectively. Then consider the real Gelfand triple

$$E = \mathcal{S}(\mathbb{R}) \subset H = L^2(\mathbb{R}, dt; \mathbb{R}) \subset E^* = \mathcal{S}'(\mathbb{R}).$$

Being a natural extension of the inner product of H, the canonical bilinear form on $E^* \times E$ is denoted by the same symbol $\langle\cdot,\cdot\rangle$. Let μ be the standard Gaussian measure on E^* and $L^2(E^*, \mu)$ the Hilbert space of $\mathbb{C}$-valued L^2-functions on E^*. The celebrated Wiener–Itô–Segal theorem says that $L^2(E^*, \mu)$ is unitarily isomorphic to the Boson Fock space $\Gamma(H_{\mathbb{C}})$, where $H_{\mathbb{C}}$ is the complexification of H. The isomorphism is a unique linear extension of the following correspondence between exponential functions and exponential vectors:

$$\phi_\xi(x) = e^{\langle x, \xi\rangle - \langle \xi, \xi\rangle/2} \quad \longleftrightarrow \quad \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots\right),$$

where ξ runs over $E_{\mathbb{C}}$. If $\phi \in L^2(E^*, \mu)$ and $(f_n)_{n=0}^\infty \in \Gamma(H_{\mathbb{C}})$ are related by the Wiener–Itô–Segal isomorphism, we write

$$\phi \sim (f_n)$$

for simplicity. It is then noted that

$$\| \phi \|_0^2 = \sum_{n=0}^\infty n! \, | f_n |_0^2, \tag{1.1}$$

where $\| \phi \|_0$ is the L^2-norm of $\phi \in L^2(E^*, \mu)$.

In order to introduce white noise distributions we need a particular family of seminorms defining the topology of $E = \mathcal{S}(\mathbb{R})$. By means of the differential operator $A = 1 + t^2 - d^2/dt^2$

we introduce a sequence of norms in $H_{\mathbb{C}}$ in such a way that $|\xi|_p = |A^p\xi|_0$. Let E_p be the Hilbert space obtained by completing E with respect to the norm $|\cdot|_p$. Then it is known that

$$E \cong \operatorname*{proj\,lim}_{p\to\infty} E_p, \qquad E^* \cong \operatorname*{ind\,lim}_{p\to\infty} E_{-p},$$

where the dual space carries the strong dual topology. The norms $|\cdot|_p$ are naturally extended to the tensor products $E^{\otimes n}$ and their complexification $E_{\mathbb{C}}^{\otimes n}$. The canonical bilinear form $\langle\cdot,\cdot\rangle$ is also extended to a complex bilinear form on $(E_{\mathbb{C}}^{\otimes n})^* \times E_{\mathbb{C}}^{\otimes n}$.

Throughout the paper let β be a fixed number with $0 \le \beta < 1$. For $\phi \in L^2(E^*,\mu)$ we introduce a new norm

$$\|\phi\|_{p,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} |f_n|_p^2, \qquad \phi \sim (f_n). \tag{1.2}$$

Then $(E_p)_\beta = \{\phi\,;\, |\phi|_{p,\beta} < \infty\}$, $p \ge 0$, becomes a Hilbert space and

$$(E)_\beta = \operatorname*{proj\,lim}_{p\to\infty} (E_p)_\beta$$

a countable Hilbert nuclear space. Similarly,

$$\|\phi\|_{-p,-\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1-\beta} |f_n|_{-p}^2, \qquad \phi \sim (f_n), \tag{1.3}$$

defines a Hilbertian norm on $L^2(E^*,\mu)$ and we denote by $(E_{-p})_{-\beta}$ the completion. Then the dual space (with the strong dual topology as usual) of $(E)_\beta$ is obtained as

$$(E)_\beta^* \cong \operatorname*{ind\,lim}_{p\to\infty} (E_{-p})_{-\beta} = \bigcup_{p\ge 0} (E_{-p})_{-\beta}.$$

The resultant Gelfand triple

$$(E)_\beta \subset L^2(E^*,\mu) \subset (E)_\beta^* \tag{1.4}$$

is called the *Kondratiev–Streit* space. The canonical bilinear form on $(E)_\beta^* \times (E)_\beta$ will be denoted by $\langle\langle\cdot,\cdot\rangle\rangle$. Then

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n!\, \langle F_n, f_n \rangle, \qquad \Phi \sim (F_n) \in (E)_\beta^*, \quad \phi \sim (f_n) \in (E)_\beta. \tag{1.5}$$

We note that (1.1), (1.2), (1.3) and (1.5) are all compatible each other. The standard Hida–Kubo–Takenaka space is the case of $\beta = 0$ in (1.4). Moreover, there holds a natural inclusion relation:

$$(E)_\beta \subset (E)_0 = (E) \subset L^2(E^*,\mu) \subset (E)^* = (E)_0^* \subset (E)_\beta^*.$$

2 Operator symbols

The essence of white noise approach to Fock space operators consists of effective use of pointwisely defined annihilation and creation operators, integral kernel operators, Fock expansion and operator symbols. Observing Kuo's discussion [13] carefully, we are convinced that most results obtained for the Hida–Kubo–Takenaka space in [15] admit straightforward generalization to the case of Kondratiev–Streit space.

We first recall pointwisely defined annihilation and creation operators. For any $t \in \mathbb{R}$ there exists an operator $a_t \in \mathcal{L}((E)_\beta, (E)_\beta)$ uniquely determined by

$$a_t\phi_\xi = \xi(t)\phi_\xi, \qquad \xi \in E_{\mathbb{C}}.$$

The above a_t is called the *annihilation operator at a point* t and its adjoint $a_t^* \in \mathcal{L}((E)_\beta^*, (E)_\beta^*)$ the *creation operator at a point* t. It is easily seen (cf. [15, §4.1]) that

$$\| a_t\phi \|_{p,\beta} \le \left(\frac{(1-\beta)\rho^{-\frac{2q}{1-\beta}}}{-2qe\log\rho} \right)^{(1-\beta)/2} | \delta_t |_{-(p+q)} \| \phi \|_{p+q,\beta}, \quad \phi \in (E)_\beta, \quad p \in \mathbb{R},\ q \ge 0,$$

where $\rho = \| A^{-1} \|_{OP} = 1/2$.

Recall next operator symbols. Since the exponential vectors $\{\phi_\xi\,;\ \xi \in E_{\mathbb{C}}\}$ span a dense subspace of $(E)_\beta$, every continuous operator $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ is determined uniquely by its symbol

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi,\ \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}. \tag{2.1}$$

For instance, for an integral kernel operator $\Xi_{l,m}(\kappa)$, $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$, we have

$$\Xi_{l,m}(\kappa)^\frown(\xi, \eta) = \left\langle \kappa,\ \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle e^{\langle \xi, \eta \rangle}, \qquad \xi, \eta \in E_{\mathbb{C}}, \tag{2.2}$$

where an integral kernel operator $\Xi_{l,m}(\kappa)$ admits a formal integral expression:

$$\Xi_{l,m}(\kappa) = \int_{\mathbb{R}^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m,$$

for a rigorous definition see [15]. As a result, $\Xi_{l,m}(\kappa)$ is uniquely determined by (2.2).

We next need a stratification of the space of operators $\mathcal{L}((E)_\beta, (E)_\beta^*)$. By the kernel theorem there is a canonical isomorphism:

$$\mathcal{L}((E)_\beta, (E)_\beta^*) \cong ((E)_\beta \otimes (E)_\beta)^* = \bigcup_{p \ge 0} (E_{-p})_{-\beta} \otimes (E_{-p})_{-\beta}.$$

Let $\mathcal{L}_p((E)_\beta, (E)_\beta^*)$ denote the sapce of all operators $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ which correspond to elements (denoted by the same symbols) in $(E_{-p})_{-\beta} \otimes (E_{-p})_{-\beta}$. The norm is denoted by $\| \Xi \|_{-p,-\beta}$. Then, by definition we have

$$|\langle\langle \Xi\phi,\ \psi \rangle\rangle| = |\langle\langle \Xi,\ \phi \otimes \psi \rangle\rangle| \le \| \Xi \|_{-p,-\beta} \| \phi \|_{p,\beta} \| \psi \|_{p,\beta}, \qquad \phi, \psi \in (E)_\beta.$$

In particular, in view of

$$\| \phi_\xi \|_{p,\beta} \le 2^{\beta/2} \exp\left\{ (1-\beta) 2^{\frac{2\beta-1}{1-\beta}} | \xi |_p^{\frac{2}{1-\beta}} \right\}, \qquad \xi \in E_{\mathbb{C}}, \tag{2.3}$$

which is found in [13, §5.2], we have

$$|\langle\langle \Xi\phi_\xi,\ \phi_\eta \rangle\rangle| \le 2^\beta \| \Xi \|_{-p,-\beta} \exp\left\{ (1-\beta) 2^{\frac{2\beta-1}{1-\beta}} \left(| \xi |_p^{\frac{2}{1-\beta}} + | \eta |_p^{\frac{2}{1-\beta}} \right) \right\},$$

or equivalently,

$$|\widehat{\Xi}(\xi, \eta)| \le 2^\beta \| \Xi \|_{-p,-\beta} \exp\left\{ (1-\beta) 2^{\frac{2\beta-1}{1-\beta}} \left(| \xi |_p^{\frac{2}{1-\beta}} + | \eta |_p^{\frac{2}{1-\beta}} \right) \right\}. \tag{2.4}$$

Theorem 2.1 *For a $\mathbb{C}$-valued function $\Theta : E_{\mathbb{C}} \times E_{\mathbb{C}} \to \mathbb{C}$ to be the symbol of an operator $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ if and only if*
(O1) *for fixed $\xi, \xi_1, \eta, \eta_1 \in E_{\mathbb{C}}$ the complex function $(z, w) \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1)$ is entire holomorphic on $\mathbb{C} \times \mathbb{C}$;*
(O2) *there exist constant numbers $C \geq 0$, $K \geq 0$, $p \geq 0$ such that*

$$|\Theta(\xi, \eta)| \leq C \exp K \left(|\xi|_p^{\frac{2}{1-\beta}} + |\eta|_p^{\frac{2}{1-\beta}} \right), \qquad \xi, \eta \in E_{\mathbb{C}}.$$

The proof given in [15, §4.4] for the case of $\beta = 0$ is adjusted to the general case of $0 \leq \beta < 1$, see [13]. Note also that condition (O2) follows from (2.4).

Theorem 2.2 *Let T be a locally compact space satisfying the first axiom of countability and let $t_0 \in T$ be a fixed point. Then for the map $t \mapsto \Xi_t \in \mathcal{L}((E)_\beta, (E)_\beta^*)$, $t \in T$, the following three conditions are equivalent:*
(i) *$t \mapsto \Xi_t$ is continuous at $t = t_0$;*
(ii) *there exist $p \geq 0$ and an open neighborhood U of t_0 such that*

$$\{\Xi_t \,;\, t \in U\} \subset \mathcal{L}_p((E)_\beta, (E)_\beta^*) \quad \text{and} \quad \lim_{t \to t_0} \| \Xi_t - \Xi_{t_0} \|_{-p,-\beta} = 0.$$

(iii) *there exist $C \geq 0$, $K \geq 0$, $p \geq 0$ and an open neighborhood U of t_0 such that*

$$\left| \widehat{\Xi}_t(\xi, \eta) \right| \leq C \exp K \left(|\xi|_p^{\frac{2}{1-\beta}} + |\eta|_p^{\frac{2}{1-\beta}} \right), \qquad \xi, \eta \in E_{\mathbb{C}}, \quad t \in U, \tag{2.5}$$

and

$$\lim_{t \to t_0} \widehat{\Xi}_t(\xi, \eta) = \widehat{\Xi}_{t_0}(\xi, \eta), \qquad \xi, \eta \in E_{\mathbb{C}}.$$

PROOF. (i) $\Longleftrightarrow$ (ii) follows from the general result in Appendix.
(ii) $\Longrightarrow$ (iii) In view of (2.4) we have

$$\left| \widehat{\Xi}_t(\xi, \eta) - \widehat{\Xi}_{t_0}(\xi, \eta) \right| \leq 2^\beta \| \Xi_t - \Xi_{t_0} \|_{-p,-\beta} \exp \left\{ (1-\beta) 2^{\frac{2\beta-1}{1-\beta}} \left(|\xi|_p^{\frac{2}{1-\beta}} + |\eta|_p^{\frac{2}{1-\beta}} \right) \right\},$$

from which the assertion is clear.
(iii) $\Longrightarrow$ (i) By a similar argument as in [15, §4.4] there exist $q \geq 0$ and $M = M(K, p, q) \geq 0$ such that

$$\| \Xi_t \phi \|_{-(p+q+1),-\beta} \leq CM \| \phi \|_{p+q+1,\beta}, \qquad \phi \in (E)_\beta, \quad t \in U,$$

and hence

$$\| \Xi_t \|_{-(p+q+2),-\beta} \leq CM \| \Gamma(A)^{-1} \|_{HS}^2, \qquad t \in U.$$

By assumption

$$\langle\!\langle \Xi_t - \Xi_{t_0}, \phi_\xi \otimes \phi_\eta \rangle\!\rangle = \langle\!\langle (\Xi_t - \Xi_{t_0}) \phi_\xi, \phi_\eta \rangle\!\rangle \to 0, \qquad t \to t_0.$$

Since the exponential vectors span a dense subspace of $(E)_\beta$, for any $\omega \in (E)_\beta \otimes (E)_\beta$ and $\epsilon > 0$ there exists a linear combination of exponential vectors $\omega' = \sum_i \phi_{\xi_i} \otimes \phi_{\eta_i}$ such that $\| \omega - \omega' \|_{p+q+2,\beta} < \epsilon$. By the triangle inequality

$$
\begin{aligned}
&|\langle\langle \Xi_t - \Xi_{t_0}, \omega \rangle\rangle| \\
&\quad \le \quad |\langle\langle \Xi_t - \Xi_{t_0}, \omega - \omega' \rangle\rangle| + |\langle\langle \Xi_t - \Xi_{t_0}, \omega' \rangle\rangle| \\
&\quad \le \quad \| \Xi_t - \Xi_{t_0} \|_{-(p+q+2),-\beta} \| \omega - \omega' \|_{p+q+2,\beta} + \left| \sum_i \langle\langle \Xi_t - \Xi_{t_0}, \phi_{\xi_i} \otimes \phi_{\eta_i} \rangle\rangle \right| \\
&\quad \le \quad \epsilon \left(\| \Xi_t \|_{-(p+q+2),-\beta} + \| \Xi_{t_0} \|_{-(p+q+2),-\beta} \right) + \left| \sum_i \langle\langle \Xi_t - \Xi_{t_0}, \phi_{\xi_i} \otimes \phi_{\eta_i} \rangle\rangle \right| \\
&\quad \longrightarrow \quad 2\epsilon C M \| \Gamma(A)^{-1} \|_{HS}^2, \qquad t \to t_0.
\end{aligned}
$$

Therefore Ξ_t converges to Ξ_{t_0} as $t \to t_0$ with respect to the weak topology of $((E)_\beta \otimes (E)_\beta)^*$, and hence with respect to the strong topology due to the first countability of T. Since $((E)_\beta \otimes (E)_\beta)^* \cong \mathcal{L}((E)_\beta, (E)_\beta^*)$ with respect to the strong topology, it follows that $t \mapsto \Xi_t \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ is continuous at $t = t_0$. qed

Theorem 2.3 *Let Θ_n be a sequence of $\mathbb{C}$-valued functions defined on $E_\mathbb{C} \times E_\mathbb{C}$ satisfying the following two conditions:*

(i) *for fixed $\xi, \xi_1, \eta, \eta_1 \in E_\mathbb{C}$ the complex function $(z, w) \mapsto \Theta_n(z\xi + \xi_1, w\eta + \eta_1)$ is entire holomorphic on $\mathbb{C} \times \mathbb{C}$;*

(ii) *there exist $C \ge 0$, $K \ge 0$ and $p \ge 0$ such that*

$$
|\Theta_n(\xi, \eta)| \le C \exp K \left(| \xi |_p^{\frac{2}{1-\beta}} + | \eta |_p^{\frac{2}{1-\beta}} \right), \qquad \xi, \eta \in E_\mathbb{C}, \quad n = 1, 2, \cdots. \tag{2.6}
$$

If for any $\xi, \eta \in E_\mathbb{C}$ the limit

$$
\Theta(\xi, \eta) \equiv \lim_{n \to \infty} \Theta_n(\xi, \eta)
$$

exists, then there exists $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ such that $\widehat{\Xi} = \Theta$. In that case, denoting by $\Xi_n \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ an operator of which symbol is Θ_n, the sequence Ξ_n converges to Ξ in $\mathcal{L}((E)_\beta, (E)_\beta^*)$.

PROOF. Let $\xi, \xi_1, \eta \in E_\mathbb{C}$ be fixed. For simplicity we put

$$
g_n(z) = \Theta_n(z\xi + \xi_1, \eta), \qquad g(z) = \Theta(z\xi + \xi_1, \eta), \qquad z \in \mathbb{C}.
$$

We shall prove that $g(z)$ is holomorphic on $\mathbb{C}$. Suppose that γ is a smooth closed curve in $\mathbb{C}$. Since $g_n(z)$ is holomorphic by (i),

$$
\int_\gamma g_n(z) dz = 0.
$$

On the other hand, since γ is a compact set, by assumption (ii) there exists some $M > 0$ such that

$$
|g_n(z)| \le C \exp \left(| z\xi + \xi_1 |_p^{\frac{2}{1-\beta}} + | \eta |_p^{\frac{2}{1-\beta}} \right) \le M, \qquad z \in \gamma, \quad n = 1, 2, \cdots.
$$

It then follows from the bounded convergence theorem that

$$0 = \lim_{n\to\infty} \int_\gamma g_n(z)dz = \int_\gamma g(z)dz.$$

Therefore $g(z)$ is holomorphic by Morera's theorem. It is then clear that Θ satisfies the same conditions (i) and (ii), and therefore by Theorem 2.1 there exists $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ such that $\widehat{\Xi} = \Theta$. Thus condition (iii) in Theorem 2.2 is satisfied, and consequently, Ξ_n converges in Ξ in $\mathcal{L}((E)_\beta, (E)_\beta^*)$. qed

Remark For an operator $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ the function

$$\widetilde{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle\, e^{-\langle \xi, \eta\rangle}, \qquad \xi, \eta \in E_{\mathbb{C}},$$

is called the *Wick symbol*, see [5], [6]. The symbol and Wick symbol are related in an obvious manner:

$$\widetilde{\Xi}(\xi, \eta) = \widehat{\Xi}(\xi, \eta) e^{-\langle \xi, \eta\rangle}.$$

It is then easy to see that the above mentioned statements are also valid when the "symbol" is replaced with "Wick symbol."

3 Wick product of operators

We start with the following

Lemma 3.1 *For two operators $\Xi_1, \Xi_2 \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ there exists $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ such that*

$$\widehat{\Xi}(\xi, \eta) = \widehat{\Xi}_1(\xi, \eta)\widehat{\Xi}_2(\xi, \eta) e^{-\langle \xi, \eta\rangle}, \qquad \xi, \eta \in E_{\mathbb{C}}, \tag{3.1}$$

PROOF. We apply Theorem 2.1. For simplicity we put

$$\Theta(\xi, \eta) = \widehat{\Xi}_1(\xi, \eta)\widehat{\Xi}_2(\xi, \eta) e^{-\langle \xi, \eta\rangle}, \qquad \xi, \eta \in E_{\mathbb{C}}.$$

Obviously, condition (O1) in Theorem 2.1 is fulfilled. By assumption, we have

$$\left|\widehat{\Xi}_i(\xi, \eta)\right| \le 2^\beta \, \| \Xi_i \|_{-p,-\beta} \exp\left\{(1-\beta)2^{\frac{2\beta-1}{1-\beta}} \left(|\,\xi\,|_p^{\frac{2}{1-\beta}} + |\,\eta\,|_p^{\frac{2}{1-\beta}}\right)\right\}, \quad i = 1, 2,$$

for some $p \ge 0$, see (2.4). On the other hand, in view of an obvious inequality $a^2 \le 1 + a^{2/(1-\beta)}$ we have

$$|e^{-\langle \xi, \eta\rangle}| \le \exp \frac{\rho^{2p}}{2}\left(|\,\xi\,|_p^2 + |\,\eta\,|_p^2\right) \le e^{\rho^{2p}} \exp \frac{\rho^{2p}}{2}\left(|\,\xi\,|_p^{\frac{2}{1-\beta}} + |\,\eta\,|_p^{\frac{2}{1-\beta}}\right).$$

Then,

$$\begin{aligned}
|\Theta(\xi, \eta)| &\le 2^{2\beta} \| \Xi_1 \|_{-p,-\beta} \| \Xi_2 \|_{-p,-\beta} \\
&\qquad \times \exp\left\{2(1-\beta)2^{\frac{2\beta-1}{1-\beta}} \left(|\,\xi\,|_p^{\frac{2}{1-\beta}} + |\,\eta\,|_p^{\frac{2}{1-\beta}}\right)\right\} \\
&\qquad \times e^{\rho^{2p}} \exp \frac{\rho^{2p}}{2}\left(|\,\xi\,|_p^{\frac{2}{1-\beta}} + |\,\eta\,|_p^{\frac{2}{1-\beta}}\right) \\
&= 2^{2\beta} e^{\rho^{2p}} \| \Xi_1 \|_{-p,-\beta} \| \Xi_2 \|_{-p,-\beta} \\
&\qquad \times \exp\left\{\left((1-\beta)2^{\frac{\beta}{1-\beta}} + \frac{\rho^{2p}}{2}\right)\left(|\,\xi\,|_p^{\frac{2}{1-\beta}} + |\,\eta\,|_p^{\frac{2}{1-\beta}}\right)\right\}.
\end{aligned} \tag{3.2}$$

Thus Θ satisfies condition (O2) in Theorem 2.1, and hence there exists $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ whose symbol is Θ. qed

The operator Ξ defined in Lemma 3.1 above is denoted as

$$\Xi = \Xi_1 \diamond \Xi_2$$

and is called the *Wick product.* By definition

$$\langle\langle (\Xi_1 \diamond \Xi_2)\phi_\xi,\, \phi_\eta \rangle\rangle = \langle\langle \Xi_1\phi_\xi,\, \phi_\eta \rangle\rangle \,\langle\langle \Xi_2\phi_\xi,\, \phi_\eta \rangle\rangle \, e^{-\langle \xi, \eta\rangle}. \tag{3.3}$$

Remark In terms of Wick symbols one has

$$(\Xi_1 \diamond \Xi_2)\tilde{}\,(\xi, \eta) = \tilde{\Xi}_1(\xi, \eta)\tilde{\Xi}_2(\xi, \eta),$$

which is slightly simpler than (3.1). However, to avoid confusion we use hereafter only operator symbols.

Here are some algebraic properties of the Wick product. The proofs follow directly from (3.3).

$$\Xi \diamond I = \Xi \tag{3.4}$$
$$\Xi_1 \diamond \Xi_2 = \Xi_2 \diamond \Xi_1 \tag{3.5}$$
$$(\Xi_1 \diamond \Xi_2) \diamond \Xi_3 = \Xi_1 \diamond (\Xi_2 \diamond \Xi_3) \tag{3.6}$$
$$(\Xi_1 \diamond \Xi_2)^* = \Xi_1^* \diamond \Xi_2^* \tag{3.7}$$

Proposition 3.2 *The Wick product is a separately continuous bilinear map from* $\mathcal{L}((E)_\beta, (E)_\beta^*) \times \mathcal{L}((E)_\beta, (E)_\beta^*)$ *into* $\mathcal{L}((E)_\beta, (E)_\beta^*)$.

PROOF. Suppose $\Xi_1, \Xi_2 \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ and put $\Xi = \Xi_1 \diamond \Xi_2$. It follows from (3.2) that

$$|\widehat{\Xi}(\xi, \eta)| \le C \,\|\, \Xi_1 \,\|_{-p,-\beta} \,\|\, \Xi_2 \,\|_{-p,-\beta} \exp K \left(|\, \xi \,|_p^{\frac{2}{1-\beta}} + |\, \eta \,|_p^{\frac{2}{1-\beta}} \right)$$

for some $C \ge 0$ and $K \ge 0$. Then, observing the proof of Theorem 2.2 carefully, we see that for any $p \ge 0$ there exist $C' \ge 0$ and $q \ge 0$ such that

$$\|\, \Xi_1 \diamond \Xi_2 \,\|_{-(p+q)} \le C' \,\|\, \Xi_1 \,\|_{-p} \,\|\, \Xi_2 \,\|_{-p}\,, \qquad \Xi_1, \Xi_2 \in \mathcal{L}_p((E)_\beta, (E)_\beta^*). \tag{3.8}$$

Suppose Ξ_2 is fixed. Then (3.8) means that $\Xi_1 \mapsto \Xi_1 \diamond \Xi_2$ is a continuous linear map from $\mathcal{L}_p((E)_\beta, (E)_\beta^*)$ into $\mathcal{L}_{p+q}((E)_\beta, (E)_\beta^*)$, and hence into $\mathcal{L}((E)_\beta, (E)_\beta^*)$. Since

$$\mathcal{L}((E)_\beta, (E)_\beta^*) \cong \operatorname*{ind\,lim}_{p\to\infty} \mathcal{L}_p((E)_\beta, (E)_\beta^*),$$

$\Xi_1 \mapsto \Xi_1 \diamond \Xi_2$ is a continuous linear map from $\mathcal{L}((E)_\beta, (E)_\beta^*)$ into itself. qed

Proposition 3.3 *For an operator* $\Omega \in \mathcal{L}((E)_\beta, (E)_\beta^*)$ *the following conditions are equivalent:*

(i) $\Xi \diamond \Omega = \Xi\Omega$ *for any* $\Xi \in \mathcal{L}((E)_\beta^*, (E)_\beta^*)$;

(ii) $\Omega^* \diamond \Xi = \Omega^* \Xi$ *for any* $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta)$*;*
(iii) *the Fock expansion of* Ω *contains only annihilation operators, i.e., is of the form:*

$$\Omega = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m}).$$

In that case, if $\Omega \in \mathcal{L}((E)_\beta, (E)_\beta)$ *in addition, then* $\Xi \diamond \Omega = \Xi\Omega$ *for any* $\Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*)$.

PROOF. (i) $\iff$ (ii) is obvious because these are obtained by duality from each other.
(i) $\Longrightarrow$ (iii) Put

$$\Omega = \Xi_{0,1}(\zeta) = \int_{\mathbb{R}} \zeta(t) a_t \, dt, \qquad \zeta \in E_{\mathbb{C}}.$$

Then

$$(\Xi_{0,1}(\zeta) \diamond \Xi)\widehat{\ }(\xi, \eta) = \widehat{\Xi}_{0,1}(\zeta)(\xi,\eta)\widehat{\Xi}(\xi,\eta) e^{-\langle \xi, \eta \rangle} = \langle \zeta, \xi \rangle \, \widehat{\Xi}(\xi,\eta) = \langle\!\langle \Xi \Xi_{0,1}(\zeta)\phi_\xi, \phi_\eta \rangle\!\rangle .$$

Since $\Xi \diamond \Xi_{0,1}(\zeta) = \Xi\Xi_{0,1}(\zeta)$ by assumption, we obtain

$$\Xi_{0,1}(\zeta)\Xi = \Xi\Xi_{0,1}(\zeta), \qquad \zeta \in E_{\mathbb{C}}.$$

It is proved [17] that any operator commuting with $\Xi_{0,1}(\zeta)$ contains no creation operators in its Fock expansion.
(iii) $\Longrightarrow$ (i) Assume that $\Xi = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m})$. Then for $\Omega \in \mathcal{L}((E)_\beta^*, (E)_\beta^*)$,

$$\begin{aligned}
(\Omega\Xi)\widehat{\ }(\xi,\eta) &= \sum_{m=0}^{\infty} \langle\!\langle \Omega \Xi_{0,m}(\kappa_{0,m})\phi_\xi, \phi_\eta \rangle\!\rangle \\
&= \sum_{m=0}^{\infty} \left\langle \kappa_{0,m}, \xi^{\otimes m} \right\rangle \langle\!\langle \Omega\phi_\xi, \phi_\eta \rangle\!\rangle \\
&= \sum_{m=0}^{\infty} \langle\!\langle \Xi_{0,m}(\kappa_{0,m})\phi_\xi, \phi_\eta \rangle\!\rangle \, e^{-\langle \xi, \eta \rangle} \langle\!\langle \Omega\phi_\xi, \phi_\eta \rangle\!\rangle .
\end{aligned}$$

This implies that $\Omega\Xi = \Omega \diamond \Xi$.

Finally, assume that $\Omega = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m}) \in \mathcal{L}((E)_\beta, (E)_\beta)$. Then, since the series converges in $\mathcal{L}((E)_\beta, (E)_\beta)$, we have

$$\begin{aligned}
(\Xi\Omega)\widehat{\ }(\xi,\eta) &= \langle\!\langle \Xi\Omega\phi_\xi, \phi_\eta \rangle\!\rangle \\
&= \sum_{m=0}^{\infty} \langle\!\langle \Xi\Xi_{0,m}(\kappa_{0,m})\phi_\xi, \phi_\eta \rangle\!\rangle \\
&= \sum_{m=0}^{\infty} \left\langle \kappa_{0,m}, \xi^{\otimes m} \right\rangle \langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle \\
&= \widehat{\Xi}(\xi,\eta) e^{-\langle \xi, \eta \rangle} \widehat{\Omega}(\xi,\eta).
\end{aligned}$$

Consequently, $\Xi \diamond \Omega = \Xi\Omega$. qed

A similar assertion as above has appeared in Huang–Luo [10].

Corollary 3.4 *For any $\Xi \in \mathcal{L}((E)_\beta, (E)^*_\beta)$ it holds that*

$$a^*_{s_1} \cdots a^*_{s_l} \Xi a_{t_1} \cdots a_{t_m} = \Xi \diamond (a^*_{s_1} \cdots a^*_{s_l} a_{t_1} \cdots a_{t_m}).$$

In particular,

$$a^*_s \Xi = \Xi \diamond a^*_s, \qquad \Xi a_t = \Xi \diamond a_t,$$

and

$$a_s \diamond a_t = a_s a_t, \qquad a^*_s \diamond a_t = a^*_s a_t, \qquad a_s \diamond a^*_t = a^*_t a_s, \qquad a^*_s \diamond a^*_t = a^*_s a^*_t.$$

4 Wick exponential function

Given $\Xi \in \mathcal{L}((E)_\beta, (E)^*_\beta)$ with Fock expansion

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}),$$

we put $\deg \Xi = \sup\{l+m\,;\, \kappa_{l,m} \neq 0\}$. It can happen that $\deg \Xi = \infty$. For simplicity we put

$$\Xi^{\diamond n} = \underbrace{\Xi \diamond \cdots \diamond \Xi}_{n \text{ times}}, \qquad \Xi^{\diamond 0} = I.$$

Theorem 4.1 *Let $\Xi \in \mathcal{L}((E), (E)^*)$. Then*

$$\sum_{n=0}^{\infty} \frac{1}{n!} \Xi^{\diamond n} \tag{4.1}$$

converges in $\mathcal{L}((E)_\beta, (E)^*_\beta)$ if and only if $\deg \Xi \le 2/(1-\beta)$. *In particular, (4.1) converges in $\mathcal{L}((E), (E)^*)$ if and only if* $\deg \Xi \le 2$.

PROOF. Given $\Xi \in \mathcal{L}((E), (E)^*)$ we consider the partial sum:

$$S_N = \sum_{n=0}^{N} \frac{1}{n!} \Xi^{\diamond n}.$$

In view of a general formula:

$$(\Xi_1 \diamond \cdots \diamond \Xi_n)^{\widehat{}}(\xi, \eta) = \widehat{\Xi}_1(\xi, \eta) \cdots \widehat{\Xi}_n(\xi, \eta) e^{-(n-1)\langle \xi, \eta \rangle},$$

we have

$$\widehat{S}_N(\xi, \eta) = \sum_{n=0}^{N} \frac{1}{n!} \left(\widehat{\Xi}(\xi, \eta)\right)^n e^{-(n-1)\langle \xi, \eta \rangle} = \sum_{n=0}^{N} \frac{1}{n!} \left(\widehat{\Xi}(\xi, \eta) e^{-\langle \xi, \eta \rangle}\right)^n e^{\langle \xi, \eta \rangle},$$

and hence

$$\lim_{N \to \infty} \widehat{S}_N(\xi, \eta) = \exp(\widehat{\Xi}(\xi, \eta) e^{-\langle \xi, \eta \rangle} + \langle \xi, \eta \rangle), \qquad \xi, \eta \in E_{\mathbb{C}}.$$

Then by Theorem 2.3, S_N converges in $\mathcal{L}((E)_\beta, (E)^*_\beta)$ if and only if there exist some constant numbers $C \geq 0$, $K \geq 0$ and $p \geq 0$ such that

$$\left|\exp(\widehat{\Xi}(\xi,\eta)e^{-\langle\xi,\eta\rangle} + \langle\xi,\eta\rangle)\right| \leq C\exp K\left(|\xi|_p^{\frac{2}{1-\beta}} + |\eta|_p^{\frac{2}{1-\beta}}\right)$$

or equivalently, such that

$$\left|\exp(\widehat{\Xi}(\xi,\eta)e^{-\langle\xi,\eta\rangle})\right| \leq C\exp K\left(|\xi|_p^{\frac{2}{1-\beta}} + |\eta|_p^{\frac{2}{1-\beta}}\right). \tag{4.2}$$

First we assume that $d \equiv \deg \Xi \leq 2/(1-\beta)$. Choose $p \geq 0$ such that

$$K' = \max_{l+m\leq d} |\kappa_{l,m}|_{-p} < \infty.$$

Since the symbol of Ξ is of the form:

$$\widehat{\Xi}(\xi,\eta) = \sum_{l+m\leq d} \left\langle \kappa_{l,m},\, \eta^{\otimes l}\otimes\xi^{\otimes m}\right\rangle e^{\langle\xi,\eta\rangle},$$

we have

$$\begin{aligned}
\left|\exp(\widehat{\Xi}(\xi,\eta)e^{-\langle\xi,\eta\rangle})\right| &\leq \exp\left\{\sum_{l+m\leq d}\left|\left\langle\kappa_{l,m},\, \eta^{\otimes l}\otimes\xi^{\otimes m}\right\rangle\right|\right\} \\
&\leq \exp\left\{\sum_{l+m\leq d}|\kappa_{l,m}|_{-p}|\eta|_p^l|\xi|_p^m\right\} \\
&\leq \exp\left\{K'\sum_{l+m\leq d}|\eta|_p^l|\xi|_p^m\right\}.
\end{aligned} \tag{4.3}$$

Using an obvious inequality $a^l b^m \leq a^{l+m} + b^{l+m}$, $a, b \geq 0$, we have

$$\sum_{l+m=k}|\eta|_p^l|\xi|_p^m \leq \sum_{l+m=k}\left(|\eta|_p^{l+m} + |\xi|_p^{l+m}\right) = (k+1)(|\eta|_p^k + |\xi|_p^k).$$

Then (4.3) becomes

$$\begin{aligned}
\left|\exp(\widehat{\Xi}(\xi,\eta)e^{-\langle\xi,\eta\rangle})\right| &\leq \exp\left\{K'\sum_{k=0}^d\sum_{l+m=k}|\eta|_p^l|\xi|_p^m\right\} \\
&\leq \exp\left\{K'\sum_{k=0}^d(k+1)(|\eta|_p^k + |\xi|_p^k)\right\} \\
&\leq \exp\left\{K'(d+1)\sum_{k=0}^d(|\eta|_p^k + |\xi|_p^k)\right\}.
\end{aligned} \tag{4.4}$$

In view of an inequality $1 + a + a^2 + \cdots + a^d \leq 1 + d + da^d$, $a \geq 0$, (4.4) becomes

$$\begin{aligned}
&\leq \exp\left\{K'(d+1)(1+d+d|\eta|_p^d + 1 + d + d|\xi|_p^d)\right\} \\
&= \exp\left\{2K'(d+1)^2 + K'(d+1)d(|\eta|_p^d + |\xi|_p^d)\right\}
\end{aligned} \tag{4.5}$$

We put

$$C' = \exp\left(2K'(d+1)^2\right).$$

Since $d \le 2/(1-\beta)$, we have $|\eta|_p^d \le 1 + |\eta|_p^{2/(1-\beta)}$. Hence (4.5) becomes

$$\le C' \exp\left\{K'(d+1)d(2 + |\eta|_p^{\frac{2}{1-\beta}} + |\xi|_p^{\frac{2}{1-\beta}}\right\}.$$

Finally we put

$$C = C' \exp(2d(d+1)K'), \qquad K = K'(d+1)d.$$

We obtain

$$\left|\exp(\widehat{\Xi}(\xi,\eta)e^{-\langle\xi,\eta\rangle})\right| \le C \exp K\left(|\eta|_p^{\frac{2}{1-\beta}} + |\xi|_p^{\frac{2}{1-\beta}}\right).$$

Hence (4.2) is fulfilled.

Conversely we assume (4.2). For simplicity we put

$$\theta(z) = \widehat{\Xi}(z\xi,\eta)e^{-z\langle\xi,\eta\rangle}, \qquad z \in \mathbb{C}.$$

Then $F(z) = e^{\theta(z)}$ becomes an entire holomorphic function without zeroes of order $\le 2/(1-\beta)$. It then follows from Lemma 4.2 below that $\theta(z)$ is a polynomial of degree $\le 2/(1-\beta)$, i.e., $\Xi_{l,m}(\kappa_{l,m}) = 0$ whenever $l > 2/(1-\beta)$. Similarly, we see that $\Xi_{l,m}(\kappa_{l,m}) = 0$ whenever $m > 2/(1-\beta)$, and hence $d \equiv \deg \Xi < \infty$. We shall show that $d \le 2/(1-\beta)$. By definition $\kappa_{l,m} \ne 0$ for some l, m with $l + m = d$. Hence there exist $\xi, \eta \in E_{\mathbb{C}}$ such that

$$\omega \equiv \sum_{l+m=d} \left\langle \kappa_{l,m},\, \eta^{\otimes l} \otimes \xi^{\otimes m}\right\rangle \ne 0.$$

We may assume without loss of generality that $\omega > 0$. In that case (4.2) implies that

$$\left|\exp\left\{\sum_{l+m\le d} \left\langle \kappa_{l,m},\, \eta^{\otimes l} \otimes \xi^{\otimes m}\right\rangle\right\}\right| \le C \exp K\left(|\eta|_p^{\frac{2}{1-\beta}} + |\xi|_p^{\frac{2}{1-\beta}}\right).$$

Hence for any $z \in \mathbb{C}$ we have

$$\left|\exp\left\{\sum_{l+m\le d} \left\langle \kappa_{l,m},\, \eta^{\otimes l} \otimes \xi^{\otimes m}\right\rangle z^{l+m}\right\}\right| \le C \exp K|z|^{\frac{2}{1-\beta}}\left(|\eta|_p^{\frac{2}{1-\beta}} + |\xi|_p^{\frac{2}{1-\beta}}\right),$$

namely,

$$\left|\exp\left\{\omega z^d + P_{d-1}(z)\right\}\right| \le C \exp\left(\omega'|z|^{\frac{2}{1-\beta}}\right), \tag{4.6}$$

where

$$\omega' = K\left(|\eta|_p^{\frac{2}{1-\beta}} + |\xi|_p^{\frac{2}{1-\beta}}\right) > 0$$

and $P_{d-1}(z)$ is a polynomial in z of degree at most $d-1$. Then (4.6) becomes

$$\left|\exp\left\{\omega z^d + P_{d-1}(z) - \omega'|z|^{\frac{2}{1-\beta}}\right\}\right| \le C. \tag{4.7}$$

Inequality (4.7) holds for any $z \in \mathbb{C}$ and hence for any $z = t > 0$. Obviously this can happen only when $d \le 2/(1-\beta)$. qed

Lemma 4.2 *Let $F(z)$ be an entire holomorphic function with no zeroes in $\mathbb{C}$ of finite order $\alpha \geq 0$, where*

$$\alpha = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}, \qquad M(r) = \max_{|z|=r} |F(z)|.$$

Then there exists a polynomial $P(z)$ of degree $\leq \alpha$ such that $F(z) = e^{P(z)}$.

PROOF. This is a simple consequence from Hadamard's factorization theorem for entire holomorphic functions, see e.g., Ahlfors [3]. qed

The convergent series introduced in Theorem 4.1 is called the *Wick exponential function* of Ξ and is denoted by

$$\operatorname{wexp} \Xi = \sum_{n=0}^{\infty} \frac{1}{n!} \Xi^{\diamond n}.$$

Note that the Wick exponential is defined only for $\Xi \in \mathcal{L}((E),(E)^*)$ with finite degree, or equivalently, only for finite sums of integral kernel operators.

Lemma 4.3 *Let $\Xi_i \in \mathcal{L}((E),(E)^*)$ with $\deg \Xi_i < \infty$, $i = 1, 2$. Then*

$$(\operatorname{wexp} \Xi_1) \diamond (\operatorname{wexp} \Xi_2) = \operatorname{wexp}(\Xi_1 + \Xi_2). \tag{4.8}$$

In particular,

$$\operatorname{wexp} \Xi \diamond \operatorname{wexp}(-\Xi) = I.$$

PROOF. In view of definition we observe that

$$\begin{aligned}
&((\operatorname{wexp} \Xi_1) \diamond (\operatorname{wexp} \Xi_2))\widehat{\ }(\xi,\eta) = \\
&\quad = (\operatorname{wexp} \Xi_1)\widehat{\ }(\xi,\eta) \cdot (\operatorname{wexp} \Xi_1)\widehat{\ }(\xi,\eta) \cdot e^{-\langle \xi,\eta\rangle} \\
&\quad = e^{\langle \xi,\eta\rangle} \exp\left(\widehat{\Xi}_1(\xi,\eta) e^{-\langle \xi,\eta\rangle}\right) \cdot e^{\langle \xi,\eta\rangle} \exp\left(\widehat{\Xi}_2(\xi,\eta) e^{-\langle \xi,\eta\rangle}\right) \cdot e^{-\langle \xi,\eta\rangle} \\
&\quad = e^{\langle \xi,\eta\rangle} \exp\left((\Xi_1(\xi,\eta) + \Xi_2(\xi,\eta))\widehat{\ }(\xi,\eta) e^{-\langle \xi,\eta\rangle}\right) \\
&\quad = (\operatorname{wexp}(\Xi_1 + \Xi_2))\widehat{\ }(\xi,\eta).
\end{aligned}$$

Then (4.8) follows. qed

Lemma 4.4 *Assume that $\Xi \in \mathcal{L}((E),(E)^*)$ is of finite degree $\leq 2/(1-\beta)$. Then $z \mapsto \operatorname{wexp}(z\Xi) \in \mathcal{L}((E)_\beta,(E)^*_\beta)$ is an entire holomorphic and*

$$\frac{d}{dz} \operatorname{wexp}(z\Xi) = \Xi \diamond \operatorname{wexp}(z\Xi)$$

holds in $\mathcal{L}((E)_\beta,(E)^*_\beta)$.

Lemma 4.5 *Assume that $\Xi \in \mathcal{L}((E),(E)^*)$ is of finite degree $\leq 2/(1-\beta)$. Let $t \mapsto \Xi_t \in \mathcal{L}((E),(E)^*)$ be a continuous map defined on an interval $T \subset \mathbb{R}$. Then $t \mapsto \operatorname{wexp}(\Xi_t) \in \mathcal{L}((E)_\beta,(E)^*_\beta)$ is continuous. If in addition $t \mapsto \Xi_t \in \mathcal{L}((E),(E)^*)$ is differentiable,*

$$\frac{d}{dt} \operatorname{wexp}(\Xi_t) = \frac{d\Xi}{dt} \diamond \operatorname{wexp}(z\Xi)$$

holds in $\mathcal{L}((E)_\beta,(E)^*_\beta)$.

The above lemmas are proved with the help of Theorem 2.2 (iii) by studying the symbol of a wick exponential function:

$$(\mathrm{wexp}\ \Xi)^{\wedge}(\xi,\eta) = \exp\left(\langle \xi, \eta\rangle + \widehat{\Xi}(\xi,\eta)e^{-\langle \xi,\eta\rangle}\right).$$

Remark Note that $\Xi \mapsto \mathrm{wexp}\ \Xi$ is not continuous. In fact, the Wick exponential is defined only for Ξ with finite degree and such operators do not constitute an open set in $\mathcal{L}((E),(E)^*)$.

Remark In the recent paper Cochran–Kuo–Sengupta [7] they introduced a further generalization of white noise functions. It is plausible that the Wick exponential wexp Ξ converges for *any* $\Xi \in \mathcal{L}((E),(E)^*)$ in a suitably extended space of operators. A further detailed study in this connection will appear elsewhere.

5 Quantum stochastic differential equations

Lemma 5.1 *Let $\{L_t\} \subset \mathcal{L}((E),(E)^*)$ be a quantum stochastic process, i.e., $t \mapsto L_t$ is continuous for $t \in T$, where T is a time interval. Then the quantum stochastic integral defined by*

$$M_t = \int_a^t L_s\, ds$$

is also a quantum stochastic process with $\deg M_t \le \deg L_t$. *Moreover,*

$$\frac{dM_t}{dt} = L_t$$

holds in $\mathcal{L}((E),(E)^*)$.

PROOF. That M_t is a quantum stochastic process satisfying $dM_t/dt = L_t$ is known [18]. Let the Fock expansion of L_t is given as

$$L_t = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}(t)).$$

It is known that the map $t \mapsto \kappa_{l,m}(t)$ is continuous for any l, m. Then, obviously

$$\int_a^t \Xi_{l,m}(\kappa_{l,m}(s))ds = \Xi_{l,m}(\lambda_{l,m}(t)), \qquad \lambda_{l,m}(t) = \int_a^t \kappa_{l,m}(s)ds.$$

Therefore

$$M_t = \int_a^t L_s\, ds = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\lambda_{l,m}(t)),$$

which proves the assertion. qed

Theorem 5.2 *Let* $\{L_t\}_{t\in T} \subset \mathcal{L}((E),(E)^*)$ *be a quantum stochastic process, where* $T \subset \mathbb{R}$ *is an interval containing* 0. *Assume that there exists a number* β *with* $0 \le \beta < 1$ *such that* $\deg L_t \le 2/(1-\beta)$, $t \in T$. *Then the initial value problem*

$$\begin{cases} \dfrac{d\Xi_t}{dt} = \Xi_t \diamond L_t \\ \Xi\Big|_{t=0} = \Xi_0 \in \mathcal{L}((E),(E)^*) \end{cases} \tag{5.1}$$

has a unique solution in $\mathcal{L}((E)_\beta,(E)_\beta^*)$ *which is expressed in the form:*

$$\Xi_t = \Xi_0 \diamond \mathrm{wexp} \int_0^t L_s\, ds.$$

PROOF. The assertion follows by combining Lemmas 4.5 and 5.1. qed

Here are a few examples, some of which have appeared in Huang–Luo [10] taking no notice of convergence of wick products or existence of solutions.

Example 1 Let $\{L_t\} \in \mathcal{L}((E),(E)^*)$ be a quantum stochastic process. Assume that $\deg L_t \le 2/(1-\beta)$ and that the expansion of L_t involves no creation operators. (In that case $L_t \in \mathcal{L}((E),(E))$ follows automatically.) Consider the quantum stochastic differential equation:

$$\frac{d\Xi_t}{dt} = \Xi_t L_t, \tag{5.2}$$

where the right hand side is a usual product. Taking $\Xi_t L_t = \Xi \diamond L_t$ into account, we apply Theorem 5.2. There exists a unique solution in $\mathcal{L}((E)_\beta,(E)_\beta^*)$ which is given by

$$\Xi_t = \Xi_0 \diamond \mathrm{wexp} \int_0^t L_s\, ds = \Xi_0 \diamond \exp \int_0^t L_s\, ds.$$

By a more precise argument in terms of an *equicontinuous generator* one sees that the solution lives in $\mathcal{L}((E)_\beta,(E)_\beta)$. A similar argument is applied to

$$\frac{d\Xi_t}{dt} = L_t^* \Xi_t$$

which is dual to (5.2).

Example 2 As a particular case of Example 1 one may consider

$$\frac{d\Xi_t}{dt} = \Xi_t a_t, \qquad \frac{d\Xi_t}{dt} = a_t^* \Xi_t,$$

and their linear combination:

$$\frac{d\Xi_t}{dt} = \omega_1 \Xi_t a_t + \omega_2 a_t^* \Xi_t, \qquad \omega_1,\omega_2 \in \mathbb{C}. \tag{5.3}$$

Equation (5.3) appears in a problem of quantum stochastic limit of an interacting quantum system [1]. Since

$$\omega_1 \Xi_t a_t + \omega_2 a_t^* \Xi_t = \Xi_t \diamond (\omega_1 a_t + \omega_2 a_t^*)$$

and $\deg(\omega_1 a_t + \omega_2 a_t^*) \leq 1$, it follows from Theorem 5.2 that equation (5.3) has a unique solution in $\mathcal{L}((E),(E)^*)$.

Example 3 Consider

$$\frac{d\Xi_t}{dt} = a_t^* \Xi_t a_t. \tag{5.4}$$

In terms of Wick product we have

$$\frac{d\Xi_t}{dt} = \Xi_t \diamond (a_t^* a_t),$$

hence the solution to (5.3) is given as

$$\Xi_t = \Xi_0 \diamond \text{wexp} \int_0^t a_s^* a_s \, ds.$$

Here

$$\Lambda_t = \int_0^t a_s^* a_s \, ds$$

is called the *number process* or the *gauge process.* Consequently, the solution becomes

$$\Xi_t = \Xi_0 \diamond \text{wexp}\, \Lambda_t$$

and lives in $\mathcal{L}((E),(E)^*)$.

Example 4 There is no difficulty of discussing

$$\frac{d\Xi_t}{dt} = \Xi_t a_t^2 + a_t^{*2} \Xi_t. \tag{5.5}$$

In fact, since

$$\Xi a_t^2 + a_t^{*2} \Xi_t = \Xi \diamond (a_t^2 + a_t^{*2})$$

and $\deg(a_t^2 + a_t^{*2}) = 2$, equation (5.5) has a unique solution in $\mathcal{L}((E),(E)^*)$ and is given by

$$\Xi_t = \Xi_0 \diamond \text{wexp} \int_0^t (a_s^2 + a_s^{*2}) ds.$$

Example 5 Let L_t and M_t be quantum stochastic processes in $\mathcal{L}((E),(E)^*)$ and consider

$$\frac{d\Xi}{dt} = \Xi \diamond L_t + M_t. \tag{5.6}$$

If $\deg L_t \leq 2/(1-\beta)$, the solution to (5.6) lies in $\mathcal{L}((E)_\beta,(E)_\beta^*)$ and given as

$$\Xi_t = \left(\int_0^t M_s \diamond \Omega_s^{\diamond(-1)} \, ds + \Xi_0 \right) \diamond \Omega_t,$$

where

$$\Omega_t = \text{wexp} \int_0^t L_s ds, \qquad \Omega_t^{\diamond(-1)} = \text{wexp} \left(- \int_0^t L_s ds \right).$$

Appendix

Let $\mathfrak{X}$ be a countable Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. Then there exists a sequence of Hilbert spaces $\{H_p\}_{p=-\infty}^{\infty}$ such that

$$\cdots \subset H_{p+1} \subset H_p \subset \cdots \subset H_0 \subset \cdots \subset H_{-p} \subset H_{-(p+1)} \subset \cdots$$

and

$$\mathfrak{X} \cong \operatorname{proj}\lim_{p\to\infty} H_p, \qquad \mathfrak{X}^* \cong \operatorname{ind}\lim_{p\to\infty} H_{-p}.$$

If $\mathfrak{X}$ is a nuclear space, we may assume without loss of generality that the natural injection $H_{p+1} \to H_p$ is of Hilbert–Schmidt type for any $p \geq 0$. We denote by $|\cdot|_p$ the norm of H_p.

Proposition A.1 *Let $\mathfrak{X}$ be a countable Hilbert nuclear space and H_p the same as above. Let Ω be a locally compact space. Then for a map $f : \Omega \to \mathfrak{X}^*$ the following two conditions are equivalent:*
(i) *f is continuous;*
(ii) *for each $\omega_0 \in \Omega$ there exists $p \geq 0$ such that $f(\omega_0) \in H_{-p}$ and*

$$\lim_{\omega\to\omega_0} |f(\omega) - f(\omega_0)|_{-p} = 0.$$

In that case for any compact subset $\Omega_0 \subset \Omega$ there exists $p \geq 0$ such that $f : \Omega_0 \to H_{-p}$ is continuous.

Proof. (i) $\Longrightarrow$ (ii) Let $V \subset \Omega$ be an open neighborhood of ω_0 with compact closure. Since f is cotinuous, $f(\overline{V}) \subset \mathfrak{X}^*$ is compact and hence bounded. Then $f(\overline{V}) \subset H_{-p}$ is bounded for some p. In other words, there exists $M \geq 0$ such that

$$|f(\omega)|_{-p} \leq M, \qquad \omega \in V.$$

Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal basis of H_{p+1}. Then by definition,

$$|f(\omega) - f(\omega_0)|_{-(p+1)}^2 = \sum_{j=1}^{\infty} \langle f(\omega) - f(\omega_0), e_j\rangle^2.$$

We note that

$$\langle f(\omega) - f(\omega_0), e_j\rangle^2 \leq |f(\omega) - f(\omega_0)|_{-p}^2 \, |e_j|_p^2 \leq 4M^2 |e_j|_p^2, \qquad \omega \in V.$$

Given $\epsilon > 0$ we choose N such that

$$4M^2 \sum_{j>N} |e_j|_p^2 < \frac{\epsilon}{2},$$

which is possible since $H_{p+1} \to H_p$ is of Hilbert–Schmidt type and hence $\sum_{j=1}^{\infty} |e_j|_p^2 < \infty$. On the other hand, $\omega \mapsto \langle f(\omega), e_j\rangle$ is continuous by assumption. Then for each $j = 1, \cdots, N$ one may find an open neighborhood $U_j \subset \Omega$ of ω_0 such that

$$|\langle f(\omega), e_j\rangle - \langle f(\omega_0), e_j\rangle| < \sqrt{\frac{\epsilon}{2N}}, \qquad \omega \in U_j.$$

Put $U = V \cap U_1 \cap \cdots \cap U_N$. Then

$$\begin{aligned}
|\, f(\omega) - f(\omega_0)\,|^2_{-(p+1)} &= \sum_{j=1}^{N} \langle f(\omega) - f(\omega_0),\, e_j \rangle^2 + \sum_{j>N} \langle f(\omega) - f(\omega_0),\, e_j \rangle^2 \\
&\le \sum_{j=1}^{N} \frac{\epsilon}{2N} + 4M^2 \sum_{j>N} |\, e_j\,|_p^2 \\
&< N \times \frac{\epsilon}{2N} + \frac{\epsilon}{2} = \epsilon, \qquad \omega \in U.
\end{aligned}$$

This is the assertion of (ii).

(ii) $\Longrightarrow$ (i) The topology of $\mathfrak{X}^*$ is defined by the seminorms

$$\|\, f\,\|_B = \sup_{\omega \in B} |\, \langle f,\, \omega \rangle\,|, \qquad f \in \mathfrak{X}^*,$$

where B runs over the bounded subsets of $\mathfrak{X}$. Then for any B we have

$$\begin{aligned}
\|\, f(\omega) - f(\omega_0)\,\|_B &\le \sup_{\omega \in B} |\, f(\omega) - f(\omega_0)\,|_{-p} \,|\, \omega\,|_p \\
&= |\, B\,|_p \,|\, f(\omega) - f(\omega_0)\,|_{-p} \longrightarrow 0, \qquad \omega \longrightarrow \omega_0,
\end{aligned}$$

by assumption, which shows that f is continuous at ω_0.

The rest of the statement is already clear. qed

Corollary A.2 *Let $\{x_n\}$ be a sequence in $\mathfrak{X}^*$ and let $x \in \mathfrak{X}^*$. Then x_n converges to x in $\mathfrak{X}^*$ if and only if there exists $p \ge 0$ such that* $\lim_{n\to\infty} |\, x_n - x\,|_{-p} = 0$.

PROOF. Consider $\Omega = \{0, 1, 1/2, 1/3, \cdots\}$ equipped with the relative topology induced from $[0, 1]$. Set $f(1/n) = x_n$ and $f(0) = x$ and apply Proposition A.1. qed

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