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Mathematical Theory of State Reduction in Quantum Mechanics

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Abstract

A state reduction is the state change caused by a measurement on a quantum system conditional upon the outcome. A rigorous theory of the state reduction is developed with mathematical formalism, physical interpretation, and models. A special emphasis is on the pure state reduction which transforms a pure prior state to the pure posterior state for every outcome. Mathematical structure of general pure state reductions is discussed and it is proved that every pure state reduction is decomposed into just two types, called the von Neumann-Davies (ND) type and the Gordon-Louisell (GL) type; a state reduction $\psi \mapsto \psi_x$ is of the ND type if the mapping $\psi \mapsto P(x|\psi)^{1/2}\psi_x$ is linear, where $P(x|\psi)$ is the probability density of the outcome x, and of the GL type if ψ_x depends only on the outcome x (independent of the prior state ψ).

1. Introduction

From a statistical point of view, a quantum measurement is completely specified by the following two elements: the probability distribution $P(dx|\rho)$ of the outcome x depending on the initial state ρ and the state reduction from a prior state (represented by a density operator) ρ to the posterior state ρ_x conditional upon the outcome x. If two measurements on a system share the same outcome probability distribution and the same state reduction, they are said to be statistically equivalent. The problem of mathematical characterizations and realizations of all the possible quantum measurements in the standard formulation of quantum mechanics [Yue87] has considerable potential importance in engineering [YL73, Hel76, Oza80, Hol82] and precision measurement experiments [BV74, CTD*80, Yue83, Oza88]. As a general solution to this problem, it is proved in our previous work [Oza84] that a measurement is realizable in the standard formulation if and only if there is a normalized completely positive (CP) map valued measure $\mathbf{X}$ such that $\mathbf{X}(dx)\rho = \rho_x P(dx|\rho)$ where the CP maps $\mathbf{X}(\Delta)$ is defined on the space of trace class operators for all Borel subsets Δ of the space of outcomes. The statistical equivalence classes of measurements are thus characterized as the normalized CP map valued measures.

In this paper, we shall develop the quantum theory of measurement based on the above characterization. We shall investigate further the structure of a class of measurements which are important from both foundational and experimental points of view. A measurement is said to be pure if it reduces pure prior states (represented by vectors) ψ to pure posterior states ψ_x with probability one. It is proved in [Oza86] that such measurements are characterized by the property that the state reduction decreases the entropy in average. The state reductions caused by typical examples of pure measurements fall into the following two characteristic types. Those of one type, called the von Neumann-Davies type, are characterized by the property that the mapping $W : \psi \mapsto P(x|\psi)^{1/2}\psi_x$ is a linear isometry from $\mathcal{H}$ to $L^2(\Lambda, \mu, \mathcal{H})$, where $\mathcal{H}$ is the Hilbert space of the object and μ is a measure on the space Λ of outcomes such that $P(x|\psi)\mu(dx) = P(dx|\psi)$. Those of the other type, called the Gordon-Louisell type, are characterized by the property that the posterior state ψ_x depends only on the outcome x (independent of the intial state ψ). We shall prove that the state reduction of a general pure measurement is decomposed into the above two types in the sense that the space Λ of outcomes has such a decompositon $\Lambda = \Lambda_I \cup \Lambda_{II}$ that the state reduction is of the von Neumann-Davies type on Λ_I and of the Gordon-Lousell type on Λ_{II}.

Throughout this paper, any quantum system is a system with finite degrees of freedom without any superselection rules and every Hilbert space is supposed to be separable so that the states of the system are described by density operators on a Hilbert space and that the observables by self-adjoint operators (densely defined) on the same Hilbert space. We shall denote by E^A the spectral measure corresponding to a self-adjoint operator A. A *standard Borel space* is a Borel space Λ endowed with a σ-field $\mathcal{B}(\Lambda)$ of subsets of Λ which is Borel isomorphic to the Borel space associated with a Borel subset of a complete separable metric space; it is well-known that two standard Borel spaces are Borel isomorphic if and only if they have the same cardinal number and that the only infinite cardinals possible are $\aleph_0$ and $2^{\aleph_0}$ [Mac57].

2. Measurement models

In the physics literature [vN55, AK65, Cav85, Oza88, Oza90] models of measurement are described as experiments consisting of the following processes: the preparation of the probe, the interaction between the object and the probe, the measurement for the probe, and the data processing. In what follows we shall give a mathematical formulation for general features of such models of measurement.

Let $\mathcal{H}$ be a Hilbert space which describes a quantum system $\mathbf{S}$, and Λ a standard Borel space which describes the space of possible outcomes of a measurement. A *measurement model* for $(\Lambda, \mathcal{H})$ is a 5-tuple $\mathbf{M} = [\mathcal{K}, \sigma, H, \langle M_1, \ldots, M_n \rangle, f]$ consisting of a Hilbert space $\mathcal{K}$, a density operator σ on $\mathcal{K}$, a self-adjoint operator H on $\mathcal{H} \otimes \mathcal{K}$, a finite sequence $\langle M_1, \ldots, M_n \rangle$ of self-adjoint operators on $\mathcal{K}$, and a Borel function f from $\mathbf{R}^n$ to Λ.

According to the following physical interpretation of the measurement model $\mathbf{M}$, the Hilbert space $\mathcal{K}$ describes the probe, σ describes the preparation of the probe, H describes the interaction between the object and the probe, $\langle M_1, \ldots, M_n \rangle$ describes the measurement for the probe, and f describes the data processing.

The measurement model $\mathbf{M}$ represents the mathematical features of the following physical description of a model of measurement. The *probe* $\mathbf{P}$ is a microscopic part of the measuring apparatus which directly interacts with the object $\mathbf{S}$. The probe $\mathbf{P}$ is described by the Hilbert space $\mathcal{K}$. The probe $\mathbf{P}$ is coupled to $\mathbf{S}$ during finite time interval from time t to $t + \Delta t$. The time t is called the *time of measurement* and the time $t + \Delta t$ is called the *time just after measurement*. The system $\mathbf{S}$ is free from the measuring apparatus after $t + \Delta t$. The state ρ of $\mathbf{S}$ at the time of measurement is called the *prior state*. In order to assure the reproducibility of this experiment, the probe $\mathbf{P}$ is always prepared in a fixed state σ, called the *probe preparation*, at the time of measurement. The composite system $\mathbf{S} + \mathbf{P}$ is thus in the state $\rho \otimes \sigma$ at the time of measurement.

Let $H_{\mathbf{S}}$ and $H_{\mathbf{P}}$ be the free Hamiltonians of $\mathbf{S}$ and $\mathbf{P}$, respectively. The total Hamiltonian of the composite system $\mathbf{S} + \mathbf{P}$ is taken to be

$$H_{\mathbf{S}+\mathbf{P}} = H_{\mathbf{S}} \otimes 1 + 1 \otimes H_{\mathbf{P}} + KH \tag{1}$$

where H represents the interaction and K the coupling constant. The coupling is assumed for simplicity so strong $(1 \ll K)$ that the free Hamiltonians $H_{\mathbf{S}}$ and $H_{\mathbf{P}}$ can be neglected. The duration Δt of the coupling is assumed so small $(0 < \Delta t \ll 1)$ that we can choose the units such that $K\Delta t = 1 \sim \hbar$. Thus the unitary operator

U, called the *time evolution operator*, on $\mathcal{H} \otimes \mathcal{K}$ representing the time evolution of the composite system $\mathbf{S} + \mathbf{P}$ from time t to $t + \Delta t$ is given by

$$U = \exp(-\frac{i}{\hbar}H). \tag{2}$$

At the time just after measurement the composite system $\mathbf{S} + \mathbf{P}$ is in the state $U(\rho \otimes \sigma)U^{\dagger}$.

Note that, even in the case where the above assumptions on K and Δt cannot apply, if the interaction H is perturbed as

$$H \mapsto H - \frac{1}{K}(H_{\mathbf{S}} \otimes 1 + 1 \otimes H_{\mathbf{P}}) \tag{3}$$

then Eq. (2) may give the time evolution of $\mathbf{S} + \mathbf{P}$ in the units with $K\Delta t = 1$; see [vN55, pages 352–357] for the discussion on the time of measurement and the perturbations of measuring interactions.

At the time just after measurement, the systems $\mathbf{S}$ and $\mathbf{P}$ have no interaction, and in order to obtain the outcome of this experiment a finite sequence $\langle M_1, \ldots, M_n \rangle$ of compatible observables, called the *probe observables*, of the system $\mathbf{P}$ is measured by the subsequent macroscopic stages of the measuring apparatus. By this process the probe observables $M_1, \ldots, M_n$ are transduced to the macroscopic *meter variables* $\mathbf{m}_1, \ldots, \mathbf{m}_n$ so that the joint probability distribution of the meter variables in the prior state ρ obeys the Born statistical formula for the joint probability distribution of $M_1, \ldots, M_n$ in the state $U(\rho \otimes \sigma)U^{\dagger}$, i.e.,

$$\Pr[\mathbf{m}_1 \in \Delta_1, \ldots, \mathbf{m}_n \in \Delta_n \| \rho] = \mathrm{Tr}\{[1 \otimes E^{M_1}(\Delta_1) \cdots E^{M_n}(\Delta_n)]U(\rho \otimes \sigma)U^{\dagger}\} \tag{4}$$

for all $\Delta_1, \ldots, \Delta_n \in \mathcal{B}(\mathbf{R})$, where $\mathcal{B}(\mathbf{R})$ stands for the Borel σ-field of the real line $\mathbf{R}$. After reading the meter variables, the observer obtains the outcome of this measurement by the *data processing* represented by a Borel function f from $\mathbf{R}^n$ to a standard Borel space Λ, called the *outcome space*, so that the *outcome variable* $\mathbf{x}$ of this measurement is obtained by the relation

$$\mathbf{x} = f(\mathbf{m}_1, \ldots, \mathbf{m}_n). \tag{5}$$

3. Outcome distribution

The *outcome distribution* of the measurement model $\mathbf{M}$ is the probability distribution $\Pr[\mathbf{x} \in \Delta \| \rho]$ of the outcome variable $\mathbf{x}$ in the prior state ρ, where $\Delta \in \mathcal{B}(\Lambda)$.

In order to obtain the outcome distribution, let $E^{(M_1,\dots,M_n)}:\mathcal{B}(\mathbf{R}^n)\to\mathcal{L}(\mathcal{K})$ be the joint spectral measure of $M_1,\dots,M_n$, i.e.,

$$E^{(M_1,\dots,M_n)}(\Delta_1\times\cdots\times\Delta_n)=E^{M_1}(\Delta_1)\cdots E^{M_n}(\Delta_n) \tag{6}$$

for all $\Delta_1,\dots,\Delta_n\in\mathcal{B}(\mathbf{R})$, and $E^{f(M_1,\dots,M_n)}:\mathcal{B}(\Lambda)\to\mathcal{L}(\mathcal{K})$ the spectral measure defined by

$$E^{f(M_1,\dots,M_n)}(\Delta)=E^{(M_1,\dots,M_n)}(f^{-1}(\Delta)) \tag{7}$$

for all $\Delta\in\mathcal{B}(\Lambda)$. From Eqs. (4), (5), the outcome distribution is given by

$$\Pr[\mathbf{x}\in\Delta\|\rho]=\mathrm{Tr}\{[1\otimes E^{f(M_1,\dots,M_n)}(\Delta)]U(\rho\otimes\sigma)U^\dagger\} \tag{8}$$

for all $\Delta\in\mathcal{B}(\Lambda)$.

Denote by $\mathcal{L}(\mathcal{H})^+$ the space of positive linear operators on $\mathcal{H}$. A *probability operator-valued measure (POM) for* $(\Lambda,\mathcal{H})$ is a map $F:\mathcal{B}(\Lambda)\to\mathcal{L}(\mathcal{H})^+$ which satisfies the following two conditions:

(1) For any disjoint sequence $\Delta_1,\Delta_2,\ldots\in\mathcal{B}(\Lambda)$,

$$F(\bigcup_{i=1}^{\infty}\Delta_i)=\sum_{i=1}^{\infty}F(\Delta_i)$$

where the sum is convergent in the weak operator topology.

(2) $F(\Lambda)=1$.

It is easy to see that for any POM F and density operator ρ the function $\Delta\mapsto\mathrm{Tr}[F(\Delta)\rho]$ is a probability measure on $\mathcal{B}(\Lambda)$. Obviously, a spectral measure is a POM which is projection-valued.

For any $\Delta\in\mathcal{B}(\Lambda)$, let $F^{\mathbf{x}}(\Delta)$ be defined by

$$F^{\mathbf{x}}(\Delta)=\mathrm{Tr}_{\mathcal{K}}\{U^\dagger[1\otimes E^{f(M_1,\dots,M_n)}(\Delta)]U(1\otimes\sigma)\} \tag{9}$$

where $\mathrm{Tr}_{\mathcal{K}}$ stands for the partial trace over $\mathcal{K}$. Then the map $F^{\mathbf{x}}:\Delta\mapsto F^{\mathbf{x}}(\Delta)$ is a POM for $(\Lambda,\mathcal{H})$. By Eq. (8) we have

$$\Pr[\mathbf{x}\in\Delta\|\rho]=\mathrm{Tr}[F^{\mathbf{x}}(\Delta)\rho] \tag{10}$$

for any prior state ρ where $\Delta\in\mathcal{L}(\Lambda)$. The above POM F is called the *POM of* $\mathbf{M}$.

A measurement which is described by the measurement model $\mathbf{M}$ with the outcome variable $\mathbf{x}$ is called an $\mathbf{x}$*-measurement.* An $\mathbf{x}$-measurement is called a *measurement of an observable* A if $\Lambda=\mathbf{R}$ and $F^{\mathbf{x}}=E^A$. Let $A_1,\dots,A_m$ be mutually commutable observables of $\mathbf{S}$. An $\mathbf{x}$-measurement is called a *simultaneous measurement of* $A_1, \ldots, A_m$ if $\Lambda = \mathbf{R}^m$ and $F^{\mathbf{x}} = E^{(A_1,\ldots,A_m)}$. In general, for any Borel function $g : \mathbf{R}^m \to \Lambda$ an $\mathbf{x}$-measurement is called a *measurement of an observable* $g(A_1, \ldots, A_m)$ if $F^{\mathbf{x}} = E^{g(A_1,\ldots,A_m)}$.

4. State reduction

The *state reduction* of the measurement model $\mathbf{M}$ is the state transformation $\rho \mapsto \rho_{\{\mathbf{x}=x\}}$ where $x \in \Lambda$ which maps the prior state ρ to the state $\rho_{\{\mathbf{x}=x\}}$ of $\mathbf{S}$ at the time just after measurement provided that the measurement leads to the outcome $\mathbf{x} = x$. In this case, the family $\{\rho_{\{\mathbf{x}=x\}} |\ x \in \Lambda\}$ of states is called the *family of posterior states* for the prior state ρ. The family $\{\rho_{\{\mathbf{x}=x\}} |\ x \in \Lambda\}$ of posterior states is postulated to be a Borel family, i.e., the function $x \mapsto \mathrm{Tr}[a\rho_{\{\mathbf{x}=x\}}]$ is a Borel function of Λ for all $a \in \mathcal{L}(\mathcal{H})$, and that two such families are identical if they differ only on a set $\Delta \in \mathcal{B}(\Lambda)$ such that $\Pr[\mathbf{x} \in \Delta \| \rho] = 0$.

By the *measurement statistics* we mean the pair of the outcome distribution and the state reduction. Two measurement models for $(\Lambda, \mathcal{H})$ are said to be *statistically equivalent* if their measurement statistics are identical.

Consider an ensemble $\mathbf{E}$ of samples of the system $\mathbf{S}$ described by a density operator ρ; in this case we say that the system $\mathbf{S}$ is in the state ρ if $\mathbf{S}$ is considered to be chosen randomly from this ensemble. Suppose that an $\mathbf{x}$-measurement described by the measurement model $\mathbf{M}$ is carried out for every sample in the ensemble $\mathbf{E}$ in a prior state ρ. For any $\Delta \in \mathcal{B}(\Lambda)$ with $\Pr[\mathbf{x} \in \Delta \| \rho] > 0$, let $\mathbf{E}_{\{\mathbf{x}\in\Delta\}}$ be the subensemble of $\mathbf{E}$ consisting of the samples satisfying $\mathbf{x} \in \Delta$. Let $\rho_{\{\mathbf{x}\in\Delta\}}$ be the state of the ensemble $\mathbf{E}_{\{\mathbf{x}\in\Delta\}}$ at the time just after measurement. In this case we say that the system $\mathbf{S}$ is in the state $\rho_{\{\mathbf{x}\in\Delta\}}$ at the time just after measurement if $\mathbf{S}$ is considered as a random sample from $\mathbf{E}_{\{\mathbf{x}\in\Delta\}}$ or equivalently if the observer knows the occurrence of $\mathbf{x} \in \Delta$ but no more details.

Since the family $\{\rho_{\{\mathbf{x}=x\}} |\ x \in \Lambda\}$ of posterior states is a Borel family, there is a sequence of $\tau c(\mathcal{H})$-valued simple Borel functions F_n on Λ such that $\lim_n \|F_n(x) - \rho_{\{\mathbf{x}=x\}}\|_{\tau c} = 0$ for all $x \in \Lambda$, where $\tau c(\mathcal{H})$ is the Banach space of trace class operators on $\mathcal{H}$ with trace norm $\|\cdot\|_{\tau c}$, so that the family $\{\rho_{\{\mathbf{x}=x\}} |\ x \in \Lambda\}$ is Bochner integrable with respect to every probability measure on $\mathcal{B}(\Lambda)$ [HP57]. The state $\rho_{\{\mathbf{x}\in\Delta\}}$ is naturally considered to be the mixture of all $\rho_{\{\mathbf{x}=x\}}$ for $x \in \Delta$ with relative frequency proportional to the outcome distribution $\Pr[\mathbf{x} \in dx \| \rho]$ and hence we have

$$\rho_{\{\mathbf{x}\in\Delta\}} = \frac{1}{\Pr[\mathbf{x} \in \Delta \| \rho]} \int_{\Delta} \rho_{\{\mathbf{x}=x\}} \Pr[\mathbf{x} \in dx \| \rho] \tag{11}$$

where the integral is Bochner integral, provided that $\Pr[\mathbf{x} \in \Delta\|\rho] > 0$. When $\Pr[\mathbf{x} \in \Delta\|\rho] = 0$, we assume for mathematical convenience that $\rho_{\{\mathbf{x}\in\Delta\}}$ is an arbitrarily chosen density operator. Note that if $\Pr[\mathbf{x} \in \{x\}\|\rho] > 0$ then $\rho_{\{\mathbf{x}=x\}} = \rho_{\{\mathbf{x}\in\{x\}\}}$ from (11). The state transformation $\rho \mapsto \rho_{\{\mathbf{x}\in\Delta\}}$ where $\Delta \in \mathcal{B}(\Lambda)$ is called the *integral state reduction* of the measurement model $\mathbf{M}$.

5. Integral state reduction

Suppose that an $\mathbf{x}$-measurement described by the measurement model $\mathbf{M}$ is followed immediately by a $\mathbf{y}$-measurement described by a measurement model $\mathbf{M}'$ for $(\Lambda', \mathcal{H})$ so that the time just after the $\mathbf{x}$-measurement is the time of the $\mathbf{y}$-measurement. Let ρ be the prior state of the $\mathbf{x}$-measurement. Then if the outcome of the $\mathbf{x}$-measurement is $\mathbf{x} = x$ then the state of $\mathbf{S}$ at the time of $\mathbf{y}$-measurement is $\rho_{\{\mathbf{x}=x\}}$. Thus the conditional probability distribution $\Pr[\mathbf{y} \in \Gamma|\mathbf{x} = x\|\rho]$ of $\mathbf{y}$ given $\mathbf{x} = x$ in the prior state ρ of the $\mathbf{x}$-measurement is given by the outcome distribution of the $\mathbf{y}$-measurement in the posterior state $\rho_{\{\mathbf{x}=x\}}$, i.e.,

$$\Pr[\mathbf{y} \in \Gamma|\mathbf{x} = x\|\rho] = \Pr[\mathbf{y} \in \Gamma\|\rho_{\{\mathbf{x}=x\}}]. \tag{12}$$

By the definition of the conditional probability distribution in probability theory, the joint probability distribution $\Pr[\mathbf{x} \in \Delta, \mathbf{y} \in \Gamma\|\rho]$ of the outcome variables $\mathbf{x}$ and $\mathbf{y}$ in the prior state ρ of $\mathbf{x}$-measurement satisfies the relation

$$\Pr[\mathbf{x} \in \Delta, \mathbf{y} \in \Gamma\|\rho] = \int_{\Delta} \Pr[\mathbf{y} \in \Gamma|\mathbf{x} = x\|\rho] \Pr[\mathbf{x} \in dx\|\rho]. \tag{13}$$

From the integrability of the posterior states, we have

$$\begin{aligned}
&\Pr[\mathbf{x} \in \Delta\|\rho] \Pr[\mathbf{y} \in \Gamma\|\rho_{\{\mathbf{x}\in\Delta\}}] \\
&\quad = \operatorname{Tr}\{F^{\mathbf{y}}(\Gamma) \int_{\Delta} \rho_{\{\mathbf{x}=x\}} \Pr[\mathbf{x} \in dx\|\rho] && \text{by Eq. (10), (11)} \\
&\quad = \int_{\Delta} \operatorname{Tr}[F^{\mathbf{y}}(\Gamma)\rho_{\{\mathbf{x}=x\}}] \Pr[\mathbf{x} \in dx\|\rho]\} \\
&\quad = \int_{\Delta} \Pr[\mathbf{y} \in \Gamma\|\rho_{\{\mathbf{x}=x\}}] \Pr[\mathbf{x} \in dx\|\rho] && \text{by Eq. (10).}
\end{aligned}$$

From Eq. (12), Eq. (13) we obtain

$$\Pr[\mathbf{x} \in \Delta, \mathbf{y} \in \Gamma\|\rho] = \Pr[\mathbf{x} \in \Delta\|\rho] \Pr[\mathbf{y} \in \Gamma\|\rho_{\{\mathbf{x}\in\Delta\}}]. \tag{14}$$

Recall that, if $\Pr[\mathbf{x} \in \Delta \| \rho] > 0$, the conditional probability distribution $\Pr[\mathbf{y} \in \Gamma | \mathbf{x} \in \Delta \| \rho]$ of $\mathbf{y}$ given $\mathbf{x} \in \Delta$ is defined in probability theory by

$$\Pr[\mathbf{y} \in \Gamma | \mathbf{x} \in \Delta \| \rho] = \frac{\Pr[\mathbf{x} \in \Delta, \mathbf{y} \in \Gamma \| \rho]}{\Pr[\mathbf{x} \in \Delta \| \rho]}. \tag{15}$$

Therefore we have the following statistical interpretation of the state $\rho_{\{\mathbf{x}\in\Delta\}}$ for $\Delta \in \mathcal{B}(\Lambda)$ with $\Pr[\mathbf{x} \in \Delta \| \rho] > 0$:

$$\Pr[\mathbf{y} \in \Gamma \| \rho_{\{\mathbf{x}\in\Delta\}}] = \Pr[\mathbf{y} \in \Gamma | \mathbf{x} \in \Delta \| \rho]. \tag{16}$$

In order to determine the integral state reduction of the measurement model $\mathbf{M}$, suppose that the $\mathbf{y}$-measurement is a measurement of an arbitrary observable A of $\mathbf{S}$. Recall that the outcome $\mathbf{x}$ of the $\mathbf{x}$-measurement at time t is obtained as the outcome of a measurement of an observable $f(M_1, \ldots, M_n)$ of $\mathbf{P}$ at time $t + \Delta t$. On the other hand, the outcome of the $\mathbf{y}$-measurement is obtained as the outcome of a measurement of an observable A of $\mathbf{S}$ at time $t + \Delta t$. Thus the joint probability distribution of $\mathbf{x}$ and $\mathbf{y}$ is obtained by the Born statistical formula for the joint probability distribution of the observables in two different systems, i.e.,

$$\Pr[\mathbf{x} \in \Delta, \mathbf{y} \in \Gamma \| \rho] = \mathrm{Tr}\{[E^A(\Gamma) \otimes E^{f(M_1,\ldots,M_2)}(\Delta)]U(\rho \otimes \sigma)U^\dagger\} \tag{17}$$

where $\Delta \in \mathcal{B}(\Lambda)$, $\Gamma \in \mathcal{B}(\mathbf{R})$ for any prior state ρ. Thus, if $\Pr[\mathbf{x} \in \Delta \| \rho] > 0$, by (16) and (17), we have

$$\mathrm{Tr}[E^A(\Gamma)\rho_{\{\mathbf{x}\in\Delta\}}] = \frac{\mathrm{Tr}[E^A(\Gamma)\mathrm{Tr}_{\mathcal{K}}\{[1 \otimes E^{f(M_1,\ldots,M_2)}(\Delta)]U(\rho \otimes \sigma)U^\dagger\}}{\Pr[\mathbf{x} \in \Delta \| \rho]}. \tag{18}$$

Since A is arbitrary, $\rho_{\{\mathbf{x}\in\Delta\}}$ is uniquely determined by the above relation, and hence by (8) we have

$$\rho_{\{\mathbf{x}\in\Delta\}} = \frac{\mathrm{Tr}_{\mathcal{K}}\{[1 \otimes E^{f(M_1,\ldots,M_2)}(\Delta)]U(\rho \otimes \sigma)U^\dagger\}}{\mathrm{Tr}\{[1 \otimes E^{f(M_1,\ldots,M_2)}(\Delta)]U(\rho \otimes \sigma)U^\dagger\}} \tag{19}$$

Therefore, we have determined the integral state reduction $\rho \mapsto \rho_{\{\mathbf{x}\in\Delta\}}$ of the measurement model $\mathbf{M}$.

6. Operational measures

In this section, we shall introduce a useful mathematical notion which is to represent the statistics of a measurement model in a single mathematical object.

A linear map T on $\tau c(\mathcal{H})$ is said to be *completely positive (CP)* if

$$\sum_{j,k=1}^{n} \langle \xi_j | T(|\eta_j\rangle\langle\eta_k|) | \xi_k \rangle \geq 0$$

for all finite sequences $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ in $\mathcal{H}$. We shall denote the space of CP maps on $\tau c(\mathcal{H})$ by $CP[\tau c(\mathcal{H})]$. Every CP map is positive and bounded. For a bounded linear map T on $\tau c(\mathcal{H})$, the *dual* of T is a bounded linear map T^* on $\mathcal{L}(\mathcal{H})$ such that $\mathrm{Tr}[aT(\rho)] = \mathrm{Tr}[T^*(a)\rho]$ for all $a \in \mathcal{L}(\mathcal{H})$ and $\rho \in \tau c(\mathcal{H})$. The dual of a CP map T on $\tau c(\mathcal{H})$ is a CP map on $\mathcal{L}(\mathcal{H})$ in the sense that

$$\sum_{j,k=1}^{n} \langle \xi_j | T^*(a_j^\dagger a_k) | \xi_k \rangle \geq 0 \tag{20}$$

for all finite sequences $a_1, \ldots, a_n$ in $\mathcal{L}(\mathcal{H})$ and $\xi_1, \ldots, \xi_n$ in $\mathcal{H}$; for a general definition of CP maps on C*-algebras or their duals we refer to [Tak79, p. 200].

A map $\mathbf{X} : \mathcal{B}(\Lambda) \to CP[\tau c(\mathcal{H})]$ is called an *operational measure* for $(\Lambda, \mathcal{H})$ if it satisfies the following two conditions:

(1) For any disjoint sequence $\Delta_1, \Delta_2, \ldots$ in $\mathcal{B}(\Lambda)$,

$$\mathbf{X}(\bigcup_{i=1}^{\infty} \Delta_i) = \sum_{i=1}^{\infty} \mathbf{X}(\Delta_i),$$

where the sum is convergent in the strong operator topology of $CP[\tau c(\mathcal{H})]$.

(2) For any $\rho \in \tau c(\mathcal{H})$,

$$\mathrm{Tr}[\mathbf{X}(\Lambda)\rho] = \mathrm{Tr}\rho.$$

A map $\mathbf{X} : \mathcal{B}(\Lambda) \to CP[\tau c(\mathcal{H})]$ satisfying only condition (1) is called a *CP map valued measure* for $(\Lambda, \mathcal{H})$. A CP map valued measure is said to be normalized if (2) holds, so that the operational measures are the normalized CP map valued measures. General theory of operational measures are developed in [Oza84, Oza85b, Oza85a, Oza86, Oza93], where they are also called CP instruments.

For any operational measure $\mathbf{X}$, the relation

$$F(\Delta) = \mathbf{X}(\Delta)^* 1 \tag{21}$$

where $\Delta \in \mathcal{B}(\mathbf{R})$ determines a POM F, called the *POM of* $\mathbf{X}$. Conversely, any POM F has at least one operational measure $\mathbf{X}$ such that F is the POM of $\mathbf{X}$ [Oza84, Proposition 4.1].

Let $\mathbf{X}$ be an operational measure for $(\Lambda, \mathcal{H})$. A Borel family $\{\rho_x|\ x \in \mathbf{R}\}$ of density operators on $\mathcal{H}$ is called a *family of posterior states* for $(\mathbf{X}, \rho)$ if it satisfies the relation

$$\mathbf{X}(\Delta)\rho = \int_\Delta \rho_x \operatorname{Tr}[\mathbf{X}(dx)\rho] \tag{22}$$

for all $\Delta \in \mathcal{B}(\Lambda)$. For the existence of a family of posterior states, the following theorem is known [Oza85b].

Theorem 6.1. (Existence of posterior states) *A family of posterior states for* $(\mathbf{X}, \rho)$ *always exists for any density operator* ρ *on* $\mathcal{H}$ *and any operational measure* $\mathbf{X}$ *for* $(\Lambda, \mathcal{H})$ *uniquely up to almost everywhere with respect to* $\operatorname{Tr}[\mathbf{X}(\cdot)\rho]$ *in the following sense: if* $\{\rho'|\ x \in \Lambda\}$ *is another family of posterior states for* $(\mathbf{X}, \rho)$*, then* $\rho'_x = \rho_x$ *almost everywhere with respect to* $\operatorname{Tr}[\mathbf{X}(\cdot)\rho]$.

We call any Borel family of density operators satisfying (22) as (a *version* of) the family of posterior states for $(\mathbf{X}, \rho)$. Let $\{\rho_x|\ x \in \Lambda\}$ be a version of the family of posterior states for $(\mathbf{X}, \rho)$. Then for any $a \in \mathcal{L}(\mathcal{H})$, the function $\Delta \mapsto \operatorname{Tr}[a\mathbf{X}(\Delta)\rho]$ is a finite signed measure on $\mathcal{B}(\Lambda)$ such that the Radon-Nikodym derivative $\operatorname{Tr}[a\,\mathbf{X}(dx)\rho]/\operatorname{Tr}[\mathbf{X}(dx)\rho]$ of $\operatorname{Tr}[a\mathbf{X}(\cdot)\rho]$ with respect to the probability measure $\operatorname{Tr}[\mathbf{X}(\cdot)\rho]$ is given by the function $x \mapsto \operatorname{Tr}[a\rho_x]$. As suggested by this fact, we shall also write

$$\frac{\mathbf{X}(dx)\rho}{\operatorname{Tr}[\mathbf{X}(dx)\rho]} = \rho_x \tag{23}$$

for almost every $x \in \Lambda$ with respect to $\operatorname{Tr}[\mathbf{X}(\cdot)\rho]$.

7. Measuring processes

In order to discuss measurement statistics in the most general framework, a mathematical notion of measuring process is introduced in [Oza84]. A *measuring process* for $(\Lambda, \mathcal{H})$ is a 4-tuple $\mathcal{X} = [\mathcal{K}, \sigma, U, E]$ consisting of a Hilbert space $\mathcal{K}$, a density operator σ, a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$, and a spectral measure E for $(\Lambda, \mathcal{K})$. According to the physical interpretation of the measuring process $\mathcal{X}$, the Hilbert space $\mathcal{K}$ describes the probe system, the density operator σ describes the probe preparation, the unitary operator U describes the time evolution of the object-probe composite system during the measurement, and the spectral measure E describes the probe observable with the data processing. The measurement model $\mathbf{M} = [\mathcal{K}, \sigma, H, \langle M_1, \ldots, M_n \rangle, f]$ gives thus naturally a measuring process

$\mathcal{X}(\mathbf{M}) = [\mathcal{K}, \sigma, U, E]$ such that

$$\begin{aligned} U &= \exp(-\frac{i}{\hbar}H) \\ E &= E^{f(M_1,\ldots,M_n)}. \end{aligned}$$

The measuring process $\mathcal{X}(\mathbf{M})$ is called the *measuring process of* $\mathbf{M}$. The following proposition asserts that every measuring process arises in this way.

Proposition 7.1. *Any measuring process* $\mathcal{X}$ *for* $(\Lambda, \mathcal{H})$ *has at least one measurement model* $\mathbf{M}$ *for* $(\Lambda, \mathcal{H})$ *such that* $\mathcal{X}$ *is the measuring process of* $\mathbf{M}$.

Proof. Let $\mathcal{X} = [\mathcal{K}, \sigma, U, E]$ be a measuring process for $(\Lambda, \mathcal{H})$. Since Λ is a standard Borel space, there is a Borel isomorphism g of Λ onto a Borel subset Ω of the real line $\mathbf{R}$. (The subset Ω can be taken to be $\mathbf{R}$, $\mathbf{N}$, or a finite set, where $\mathbf{N}$ stands for the set of natural numbers.) Let M be a self-adjoint operator on $\mathcal{K}$ such that $E^M(\Delta) = E[g^{-1}(\Delta \cap \Omega)]$ for all $\Delta \in \mathcal{B}(\mathbf{R})$. Let f be a Borel function of $\mathbf{R}$ into Λ such that $f(x) = g^{-1}(x)$ for all $x \in \Omega$ and $f(x)$ is arbitrary for all $x \in \mathbf{R} \setminus \Omega$. Then we have $E = E^{f(M)}$. By the function calculus, it is easy to see that for the unitary operator U there is a self-adjoint operator H such that $U = \exp(-iH/\hbar)$. Thus $\mathcal{X}$ is the measuring process of the measurement model $\mathbf{M} = [\mathcal{K}, \sigma, H, M, f]$. □

Let $\mathcal{X} = [\mathcal{K}, \sigma, U, E]$ be a measuring process for $(\Lambda, \mathcal{H})$. It is easy to check that the relation

$$\mathbf{X}(\Delta)\rho = \mathrm{Tr}_{\mathcal{K}}\{[1 \otimes E(\Delta)]U(\rho \otimes \sigma)U^{\dagger}\} \tag{24}$$

where $\Delta \in \mathcal{B}(\Lambda)$ and $\rho \in \tau c(\mathcal{H})$ defines an operational measure for $(\Lambda, \mathcal{H})$, which is called the *operational measure of* $\mathcal{X}$.

The following theorem, proved in [Oza84], asserts that every operational measure arises from a measuring process.

Theorem 7.2. (Realization Theorem) *For any operational measure* $\mathbf{X}$ *for* $(\Lambda, \mathcal{H})$, *there exists at least one measuring process for* $(\Lambda, \mathcal{H})$ *such that* $\mathbf{X}$ *is the operational measure of* $\mathcal{X}$.

Proof. Follows from Theorem 5.1 and Corollary 5.2 of [Oza84]. □

8. Measurement statistics

Let $\mathbf{M} = [\mathcal{K}, \sigma, H, \langle M_1, \ldots, M_n \rangle, f]$ be a measurement model and $\mathbf{x}$ the outcome variable of a measurement described by $\mathbf{M}$. Let $\mathbf{X}$ be the operational measure of the measuring process $\mathcal{X}(\mathbf{M})$ of $\mathbf{M}$, which will be called as the *operational measure of* $\mathbf{M}$. Then the statistics of measurement model $\mathbf{M}$ is represented by $\mathbf{X}$ as follows. By (8), the outcome distribution of $\mathbf{M}$ is given by

$$\Pr[\mathbf{x} \in \Delta \| \rho] = \mathrm{Tr}[\mathbf{X}(\Delta)\rho] \tag{25}$$

for any prior state ρ where $\Delta \in \mathcal{B}(\Lambda)$. By Eq. (19), the integral state reduction of $\mathbf{M}$ is given by

$$\rho \mapsto \rho_{\{\mathbf{x} \in \Delta\}} = \frac{\mathbf{X}(\Delta)\rho}{\mathrm{Tr}[\mathbf{X}(\Delta)\rho]} \tag{26}$$

for any prior state ρ where $\Delta \in \mathcal{B}(\Lambda)$. By Eq. (11) and Eq. (22), the state reduction of $\mathbf{M}$ is given by

$$\rho \mapsto \rho_{\{\mathbf{x} = x\}} = \frac{\mathbf{X}(dx)\rho}{\mathrm{Tr}[\mathbf{X}(dx)\rho]} \tag{27}$$

for any prior state ρ and almost every x with respect to $\Pr[\mathbf{x} \in dx \| \rho]$. We have shown that the measurement statistics of a measurement model is determined by its operational measure. The following theorem states that two measurement models are statistically equivalent if and only if they have the same operational measure and that any operational measure has at least one associated measurement model.

Theorem 8.1. *The correspondence from measurement models* $\mathbf{M}$ *to their operational measures* $\mathbf{X}$ *gives a one-to-one correspondence between the statistical equivalence classes of measurement models for* $(\Lambda, \mathcal{H})$ *and the operational measures for* $(\Lambda, \mathcal{H})$.

Proof. By Eq. (25) and Eq. (27), two measurement models are statistically equivalent if they have the same operational measure. Conversely, if two measurement models are statistically equivalent, then by Eq. (11) they have the same integral state reduction so that by Eq. (26) they have the same operational measure. It follows that the correspondence $\mathbf{M} \mapsto \mathbf{X}$ gives an injective mapping from the statistical equivalence classes of measurement models to the operational measures. To show this mapping is surjective, let $\mathbf{X}$ be an operational measure for $(\Lambda, \mathcal{H})$. By the Realization Theorem, there is a measuring process $\mathcal{X}$ associated with $\mathbf{X}$. By Proposition 7.1, there is a measurement model $\mathbf{M}$ such that $\mathcal{X} = \mathcal{X}(\mathbf{M})$. Then the operational measure $\mathbf{X}$ is the operational measure of $\mathbf{M}$. It follows that for any statistical equivalence class of measurement models there is an operational measure associated with it. □

9. Statistics of pure measurements

Let $\mathcal{H}$ be a Hilbert space and Λ a standard Borel space. In what follows, we shall consider an operational measure $\mathbf{X}$ for $(\Lambda, \mathcal{H})$ and a measurement with the outcome variable $\mathbf{x}$ the measurement statistics of which is described by $\mathbf{X}$. In the context where the reference to $\mathbf{x}$ is obvious, we shall write $P(\Delta|\rho) = \Pr[\mathbf{x} \in \Delta \| \rho]$ and $\rho_x = \rho_{\{\mathbf{x}=x\}}$ for the measurement statistics determined by $\mathbf{X}$.

In most examples from real physical experiments, the state reduction reduces a pure prior state ρ to a pure posterior state ρ_x for all possible outcomes x. Thus the characterization of this kind of statistics has a particular importance in applications. For this purpose, we say that an operational measure $\mathbf{X}$ for $(\Lambda, \mathcal{H})$ is *pure* if for any pure state ρ the family $\{\rho_x | \; x \in \Lambda\}$ of posterior states for $(\mathbf{X}, \rho)$ satisfies the condition that ρ_x is a pure state for almost all $x \in \Lambda$ with respect to $\mathrm{Tr}[\mathbf{X}(\cdot)\rho]$; such operational measures are said to be "quasicomplete" in [Oza86]. A measurement model is said to be *pure* if its operational measure is pure. For a pure measurement model, the measurement statistics is represented for prior state vectors ψ as follows:

$$\begin{aligned} \text{outcome distribution:} \quad & P(dx|\psi), \\ \text{state reduction:} \quad & \psi \mapsto \psi_x, \end{aligned}$$

where $P(dx|\psi) = P(dx| \; |\psi\rangle\langle\psi|)$ and $\{\psi_x | \; x \in \Lambda\}$ is a family of state vectors such that $|\psi_x\rangle\langle\psi_x| = |\psi\rangle\langle\psi|_x$.

10. Information theoretical characterization

The pure measurement models are know to have the following information theoretical characterization. Let ρ be the prior state of a measurement. Then the entropy of ρ, called the *prior entropy*, is

$$S(\rho) = -\mathrm{Tr}[\rho \log \rho]. \tag{28}$$

If the measuring process is given by $\mathcal{X} = [\mathcal{K}, \sigma, U, E]$, then the object-probe interaction changes the object state as follows:

$$\rho \mapsto \mathbf{X}(\Lambda)\rho = \mathrm{Tr}_{\mathcal{K}}[U(\rho \otimes \sigma)U^{\dagger}]. \tag{29}$$

This process is an irreversible open-system dynamics which increases the entropy by the amount

$$S(\mathbf{X}(\Lambda)\rho) - S(\rho) \geq 0.$$

The observer is, however, informed of the outcome $\mathbf{x} = x$ of the measurement. This information changes the state from $\mathbf{X}(\Lambda)\rho$ to the posterior state ρ_x. This process gains the *information on the system*, or equivalently decreases the entropy of the system, in average by the amount

$$S(\mathbf{X}(\Lambda)\rho) - \int_\Lambda S(\rho_x)\,\mathrm{Tr}[\mathbf{X}(dx)\rho] \geq 0. \tag{30}$$

If the outcome gives enough information about the system, we can expect that this information gain compensates the dynamical entropy increase so that the total information gain is nonnegative, i.e.,

$$I(\mathbf{X}|\rho) = S(\rho) - \int_\Lambda S(\rho_x)\,\mathrm{Tr}[\mathbf{X}(dx)\rho] \geq 0. \tag{31}$$

Relation (31) is a quantum mechanical generalization of Shannon's fundamental inequality [Khi57, p. 36]; note that original Shannon's inequality describes the classical process in which the information on the state of a system is obtained without any dynamical interaction so that the first process of entropy increase is neglected. For a von Neumann-Lüders measurement [Lud51] of a purely discrete observable A, the operational measure of which is given by

$$\mathbf{X}(\Delta)\rho = \sum_{a\in\Delta} E^A(\{a\})\rho E^A(\{a\}), \tag{32}$$

where $\Delta \in \mathcal{B}(\mathbf{R})$, inequality (31) was first conjectured by Groenewold [Gro71] and proved by Lindblad [Lin72]. The following theorem characterizes generally the measurements which satisfy this inequality [Oza86].

Theorem 10.1. (Generalized Groenewold-Lindblad Inequality) *An operational measure* $\mathbf{X}$ *is pure if and only if it satisfies* $I(\mathbf{X}|\rho) \geq 0$ *for every density operator* ρ *with* $S(\rho) < \infty$.

Theorem 10.1 clarifies the significance of pure measurement models. In order to start the structure theory of pure measurement models, we shall consider typical constructions of pure operational measures in the following subsections.

11. Von Neumann-Davies type

Let μ be a σ-finite measure on $\mathcal{B}(\Lambda)$. The space $L^2(\Lambda, \mu, \mathcal{H})$ is defined as the linear space of $\mathcal{H}$-valued Borel functions f on Λ satisfying

$$\int_{\Lambda} \|f(x)\|^2 \, \mu(dx) < \infty.$$

With identifying two functions which differ only on a μ-null set, the space $L^2(\Lambda, \mu, \mathcal{H})$ is a Hilbert space with the inner product defined by

$$\langle f|g\rangle = \int_{\Lambda} \langle f(x)|g(x)\rangle \, \mu(dx)$$

for all $f, g \in L^2(\Lambda, \mu, \mathcal{H})$. Then, by the correspondence $f(\cdot)\xi \mapsto \xi \otimes f$ for all $f \in L^2(\Lambda, \mu)$ and $\xi \in \mathcal{H}$, the Hilbert space $L^2(\Lambda, \mu, \mathcal{H})$ is isometrically isomorphic to $\mathcal{H} \otimes L^2(\Lambda, \mu)$.

Theorem 11.1. *Let W be a linear isometry from $\mathcal{H}$ to $L^2(\Lambda, \mu, \mathcal{H})$. Then the Bochner integral formula*

$$\mathbf{X}_W(\Delta)|\xi\rangle\langle\xi| = \int_{\Delta} |(W\xi)(x)\rangle\langle(W\xi)(x)| \, \mu(dx), \tag{33}$$

where $\Delta \in \mathcal{B}(\Lambda)$ and $\xi \in \mathcal{H}$, defines uniquely a pure operational measure $\mathbf{X}_W$.

The pure operational measure $\mathbf{X}_W$ is called the operational measure for $(\Lambda, \mathcal{H})$ *of the von Neumann-Davies (ND) type determined by W*. The measurement statistics represented by $\mathbf{X}_W$ is given by

$$\begin{aligned} \text{outcome distribution:} \quad & P(dx|\psi) = \|(W\psi)(x)\|^2 \, \mu(dx), \\ \text{state reduction:} \quad & \psi \mapsto \psi_x = \|(W\psi)(x)\|^{-1}(W\psi)(x). \end{aligned}$$

It is easy to see that the dual of $\mathbf{X}_W(\Delta)$ is given by

$$\mathbf{X}_W(\Delta)^* a = W^*(a \otimes \chi_\Delta)W$$

for all $a \in \mathcal{L}(\mathcal{H})$. Thus the POM F_W of $\mathbf{X}_W$ is given by

$$F_W(\Delta) = W^*(1 \otimes \chi_\Delta)W$$

for all $\Delta \in \mathcal{B}(\Lambda)$.

For the later discussion, we say that a CP map valued measure $\mathbf{X}$ is of the *von Neumann-Davies type* if it is of the form of Eq. (33) with a bounded linear transformation $W : \mathcal{H} \to L^2(\Lambda, \mu, \mathcal{H})$.

12. Gordon-Lousell type

Another type of pure operational measure is given as follows. Let $\{\Psi_x | \, x \in \Lambda\}$ be a fixed Borel family of state vectors in $\mathcal{H}$ and F a POM for $(\Lambda, \mathcal{H})$. Then the relation

$$\mathbf{X}(\Delta)\rho = \int_\Delta |\Psi_x\rangle\langle\Psi_x| \mathrm{Tr}[\rho F(dx)] \tag{34}$$

for all $\Delta \in \mathcal{B}(\Lambda)$ and $\rho \in \tau c(\mathcal{H})$ defines a unique pure operational measure $\mathbf{X}$ [Oza85a, appendix], the measurement statistics of which is given by

$$\begin{aligned} \text{outcome distribution:} \quad & P(dx|\psi) = \langle\psi|F(dx)|\psi\rangle, \\ \text{state reduction:} \quad & \psi \mapsto \psi_x = \Psi_x. \end{aligned}$$

We shall call this type of pure operational measure as the *Gordon-Louisell type* [GL66, Oza89]. A pure operational measure $\mathbf{X}$ of the von Neumann-Davies type is also of the Gordon-Louisell type if the linear isometry W is given by a family $\{A_x | \, x \in \Lambda\}$ of rank one operators on $\mathcal{H}$ such that $(W\xi)(x) = A_x\xi$ for almost all $x \in \Lambda$.

For the later discussion, we say that a CP map valued measure $\mathbf{X}$ is of the *Gordon-Louisell type* if it is of the form of Eq. (34) with a positive operator valued measure $F : \mathcal{B}(\Lambda) \to \mathcal{L}(\mathcal{H})$.

13. Structure of pure measurements

The following Stinespring type dilation theorem is obtained in [Oza84].

Theorem 13.1. *For any operational measure $\mathbf{X}$ for $(\Lambda, \mathcal{H})$, there exist a (separable) Hilbert space $\mathcal{K}$, a spectral measure E for $(\Lambda, \mathcal{K})$, and an isometry $V : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ such that*

$$\mathbf{X}(\Delta)^* a = V^\dagger[a \otimes E(\Delta)]V \tag{35}$$

for all $a \in \mathcal{L}(\mathcal{H})$ and $\Delta \in \mathcal{B}(\Lambda)$ and that

$$\mathcal{H} \otimes \mathcal{K} = \{[a \otimes E(\Delta)]V\xi | \; a \in \mathcal{L}(\mathcal{H}), \; \Delta \in \mathcal{B}(\Lambda), \; \xi \in \mathcal{H}\}^{\perp\perp}, \tag{36}$$

where $^\perp$ stands for the operation of orthogonal complement.

The triple $\langle\mathcal{K}, E, V\rangle$ satisfying the above conditions is called a *minimal dilation* of an operational measure $\mathbf{X}$.

Let μ be a σ-finite measure on $\mathcal{B}(\Lambda)$. An operational measure $\mathbf{X}$ for $(\Lambda, \mathcal{H})$ is called *μ-continuous*, in symbols $\mathbf{X} \ll \mu$, whenever $\mu(\Delta) = 0$ implies $\mathbf{X}(\Delta) = 0$ for all $\Delta \in \mathcal{B}(\Lambda)$. A σ-finite measure μ is called a *base measure* of an operational measure $\mathbf{X}$ whenever $\mu(\Delta) = 0$ if and only if $\mathbf{X}(\Delta) = 0$ for all $\Delta \in \mathcal{B}(\Lambda)$.

Proposition 13.2. *Any operational measure for $(\Lambda, \mathcal{H})$ has its base measure.*

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. For any normal CP map $\Phi : \mathcal{M} \to \mathcal{N}$, denote by Φ_* the predual map $\Phi_* : \mathcal{N}_* \to \mathcal{M}_*$ defined by $\langle a, \Phi_* \sigma \rangle = \langle \Phi a, \sigma \rangle$ for all $a \in \mathcal{M}$ and $\sigma \in \mathcal{N}_*$. Denote by $L^\infty(\Lambda, \mu, \mathcal{L}(\mathcal{H}))$ the von Neumann algebra of essentially bounded weakly* μ-locally measurable $\mathcal{L}(\mathcal{H})$-valued functions on Λ. The predual of $L^\infty(\Lambda, \mu, \mathcal{L}(\mathcal{H}))$ is the space $L^1(\Lambda, \mu, \tau c(\mathcal{H}))$ of Bochner μ-integrable $\tau c(\mathcal{H})$-valued functions with duality pairing

$$\langle A, F \rangle = \int_\Lambda \mathrm{Tr}[A(x)F(x)]\, \mu(dx)$$

for all $A \in L^\infty(\Lambda, \mu, \mathcal{L}(\mathcal{H}))$ and $F \in L^1(\Lambda, \mu, \tau c(\mathcal{H}))$ [Sak71, p. 68]. The μ-continuous operational measures are characterized as follows.

Theorem 13.3. *Let μ be a σ-finite measure on $\mathcal{B}(\Lambda)$. The Bochner integral formula*

$$\mathbf{X}(\Delta)\rho = \int_\Delta (\Phi_* \rho)(x)\, \mu(dx), \tag{37}$$

where $\rho \in \tau c(\mathcal{H})$ and $\Delta \in \mathcal{B}(\Lambda)$, sets up a one-to-one correspondence between the μ-continuous operational measures $\mathbf{X}$ and the unit-preserving normal CP maps Φ from $L^\infty(\Lambda, \mu, \mathcal{L}(\mathcal{H}))$ to $\mathcal{L}(\mathcal{H})$.

The unit-preserving normal CP map Φ in Eq. (37) is called the *operational distribution* for $(\mathbf{X}, \mu)$.

The following corollary is an immediate consequence from the proof of Theorem 13.3.

Corollary 13.4. *Let μ be a σ-finite measure on $\mathcal{B}(\Lambda)$. Let $\mathbf{X}$ be a μ-continuous operational measure for $(\Lambda, \mathcal{H})$, Φ the operational distribution for $(\mathbf{X}, \mu)$, and $\langle \mathcal{K}, E, V \rangle$ the minimal dilation of $\mathbf{X}$. Then, there is a nondegenerate normal *-representation $\pi : L^\infty(\Lambda, \mu, \mathcal{L}(\mathcal{H})) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ satisfying the following conditions:*

(1) *For all $a \in \mathcal{L}(\mathcal{H})$ and $\Delta \in \mathcal{B}(\Lambda)$,*

$$\mathbf{X}(\Delta)^* a = \Phi(a \otimes \chi_\Delta) = V^\dagger \pi(a \otimes \chi_\Delta) V.$$

(2) $\pi[L^\infty(\Lambda,\mu,\mathcal{L}(\mathcal{H}))]V\mathcal{H} = \mathcal{H}\otimes\mathcal{K}$.

The following proposition shows that the relation between the family of posterior states and an operational distribution.

Proposition 13.5. *Let μ be a σ-finite measure on $\mathcal{B}(\Lambda)$. Let $\mathbf{X}$ be a μ-continuous operational measure associated with an operational distribution Φ. Let $\{\rho_x \mid x\in\Lambda\}$ be a version of the family of posterior states for $(\mathbf{X},\rho)$. Then we have*

$$(\Phi_*\rho)(x) = \frac{\mathrm{Tr}[\mathbf{X}(dx)\rho]}{\mu(dx)}\rho_x \tag{38}$$

μ-almost everywhere on Λ, and

$$\rho_x = \left(\frac{\mathrm{Tr}[\mathbf{X}(dx)\rho]}{\mu(dx)}\right)^{-1}(\Phi_*\rho)(x) \tag{39}$$

$\mathrm{Tr}[\mathbf{X}(\cdot)\rho]$-almost everywhere on Λ.

Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be CP map valued measures for $(\Lambda_1,\mathcal{H})$ and $(\Lambda_2,\mathcal{H})$, respectively. Suppose $\Lambda_1\cap\Lambda_2=\emptyset$. The *direct sum* of $\mathbf{X}_1$ and $\mathbf{X}_2$ is a CP map valued measure $\mathbf{X}$ for $(\Lambda_1\cup\Lambda_2,\mathcal{H})$ such that $\mathbf{X}(\Delta)=\mathbf{X}_1(\Delta\cap\Lambda_1)+\mathbf{X}_2(\Delta\cap\Lambda_2)$ for all $\Delta\in\mathcal{B}(\Lambda_1\cup\Lambda_2)$. The following theorem shows that pure measurements are classified essentially into two types, the ND type and the GL type.

Theorem 13.6. *Every pure operational measure is a direct sum of two CP map valued measures of the ND type and of the GL type.*

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