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TRANSFORMATIONS APPROXIMATING A GROUP GENERATED BY THE LÉVY LAPLACIAN

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1. Introduction

Since T. Hida [6] applied the Lévy Laplacian, which was introduced by P. Lévy [25], to his theory of generalized white noise functionals, this Laplacian has been studied within the framework of white noise calculus ([8,10,11,17,23,27,30,31], etc.). On the other hand, L. Accardi et al. [1] obtained a nice relation between the Laplacian and the Yang-Mills equation. It seems an interest to consider a relation to their results [1,2].

By H.-H. Kuo [16], an infinite dimensional Fourier-Mehler transform acting on the space $(\mathcal{S})^*$ of generalized white noise functionals was introduced and he showed a relation between the transform and the Lévy Laplacian (see [19]). There are several Laplacian operators acting on $(\mathcal{S})^*$.

In this paper we discuss integral expressions of those Laplacians and groups generated by the Laplacians. In addition, we show a transform acting on $(\mathcal{S})^*$ approximating a group generated by the Lévy Laplacian.

The paper is organized as follows. In Section 2 we assemble some basic notations of white noise calculus. In Section 3 we explain the definitions of Laplacian operators acting on Hida distributions, and give a limiting integral expression of the Lévy Laplacian with an integral expression of the Gross Laplacian. In Section 4 we define groups generated by the Laplacian operators acting on the Hida distributions and show that Kuo's Fourier-Mehler transform is given by a composition of groups generated by the number operator and the Gross Laplacian. In addition, we give a result that the group generated by the Lévy Laplacian is approximated by groups generated by the Gross Laplacian. Finally, in the last section we introduce a transform approximating a group generated by the Lévy Laplacian. This transform includes the adjoint operator of Kuo's Fourier-Mehler transform.

2. Preliminaries

In this section, we explain some basic notations of white noise analysis following [10,15,27,29]. We begin with a Gel'fand triple $\mathcal{S} \subset L^2(\mathbf{R}) \subset \mathcal{S}^*$, where $\mathcal{S} \equiv \mathcal{S}(\mathbf{R})$ is the Schwartz space consisting of rapidly decreasing C^∞-functions on $\mathbf{R}$ and $\mathcal{S}^* \equiv \mathcal{S}^*(\mathbf{R})$ is its dual space. An operator $A = -(d/du)^2 + u^2 + 1$ is a densely defined self-adjoint operator on $L^2(\mathbf{R})$. There exists an orthonormal basis $\{e_\nu; \nu \geq 0\}$ for

$L^2(\mathbf{R})$ such that $Ae_\nu = 2(\nu+1)e_\nu$. We define the norm $|\cdot|_p$ by $|f|_p = |A^p f|_0$ for $f \in \mathcal{S}$ and $p \in \mathbf{Z}$, where $|\cdot|_0$ is the $L^2(\mathbf{R})$-norm, and let $\mathcal{S}_p$ be the completion of $\mathcal{S}$ with respect to the norm $|\cdot|_p$. Then the dual space $\mathcal{S}_p'$ of $\mathcal{S}_p$ is the same as $\mathcal{S}_{-p}$ (see [13]).

The Bochner-Minlos theorem admits the existence of a probability measure μ on $\mathcal{S}^*$ such that

$$C(\xi) \equiv \int_{\mathcal{S}^*} \exp\{i\langle x, \xi\rangle\}\, d\mu(x) = \exp\{-\frac{1}{2}|\xi|_0^2\},\ \xi \in \mathcal{S}.$$

The space $(L^2) = L^2(\mathcal{S}^*, \mu)$ of complex-valued square-integrable functionals defined on $\mathcal{S}^*$ admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where H_n is the space of multiple Wiener integrals of order $n \in \mathbf{N}$ and $H_0 = \mathbf{C}$. This decomposition theorem says that each $\varphi \in (L^2)$ is uniquely represented as

$$\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n),\ f_n \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n},$$

where $\mathbf{I}_n(f_n) \in H_n$ and $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$ denotes the n-th symmetric tensor product of the complexification of $L^2(\mathbf{R})$.

For each $p \in \mathbf{Z}$, we define the norm $\|\varphi\|_p$ of $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n)$ by

$$\|\varphi\|_p^2 = \left(\sum_{n=0}^{\infty} n! |f_n|_{p,n}^2 \right)^{1/2},\ f_n \in \mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n}$$

where $|\cdot|_{p,n}$ is the norm of $\mathcal{S}_{\mathbf{C},p}^{\otimes n}$ and $\mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n}$ is the n-th symmetric tensor product of the complexification of $\mathcal{S}_p$. The norm $\|\cdot\|_0$ is nothing but the (L^2)-norm. We put

$$(\mathcal{S}_p) = \{\varphi \in (L^2); \|\varphi\|_p < \infty\}$$

for $p \in \mathbf{Z}, p \geq 0$. Let $(\mathcal{S}_p)^*$ be the dual space of $(\mathcal{S}_p)$. Then $(\mathcal{S}_p)$ and $(\mathcal{S}_p)^*$ are Hilbert spaces with the norm $\|\cdot\|_p$ and the dual norm of $\|\cdot\|_p$, respectively. We define the space $(\mathcal{S}_p)$ for $p < 0$ by the completion of (L^2) with respect to $\|\cdot\|_p$. Then $(\mathcal{S}_p), p < 0$, is a Hilbert space with the norm $\|\cdot\|_p$. It is easy to see that for $p > 0$, the dual space $(\mathcal{S}_p)^*$ of $(\mathcal{S}_p)$ is given by $(\mathcal{S}_{-p})$. Moreover, we see that for any $p \in \mathbf{R}$,

$$(\mathcal{S}_p) = \bigoplus H_n^{(p)},$$

where $H_n^{(p)}$ is the completion of $\{\mathbf{I}_n(f); f \in \mathcal{S}_{\mathbf{C}}^{\hat{\otimes} n}\}$ with respect to $\|\cdot\|_p$.

Denote the projective limit space of the $(\mathcal{S}_p), p \in \mathbf{Z}, p \geq 0$, and the inductive limit space of the $(\mathcal{S}_p)^*, p \in \mathbf{Z}, p \geq 0$, by $(\mathcal{S})$ and $(\mathcal{S})^*$, respectively. Then $(\mathcal{S})$ is a nuclear space and $(\mathcal{S})^*$ is nothing but the dual space of $(\mathcal{S})$. The space $(\mathcal{S})^*$ is called the space of *Hida distributions* or *generalized white noise functionals.*

Since $\exp < \cdot, \xi > \in (\mathcal{S})$, the *S-transform* is defined on $(\mathcal{S})^*$ by

$$S[\Phi](\xi) = C(\xi) \ll \Phi, \exp < \cdot, \xi > \gg, \xi \in \mathcal{S},$$

where $\ll \cdot, \cdot \gg$ is the canonical pairing of $(\mathcal{S})^*$ and $(\mathcal{S})$. In [10], we can see the following fundamental properties:

i) if $S[\Phi](\xi) = S[\Psi](\xi)$ for all $\xi \in \mathcal{S}$, then $\Phi = \Psi$.
ii) if $\Phi = \sum_{n=0}^{\infty} \Phi_n \in (\mathcal{S})^*$, then there exist an integer p and distributions $f_n \in \mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n}$, $n = 0, 1, 2, \dots$, such that $\sum_{n=0}^{\infty} n! |f_n|_{p,n}^2 < \infty$ and

$$S[\Phi](\xi) = \sum_{n=0}^{\infty} \langle \xi^{\otimes n}, f_n \rangle$$

for all $\xi \in \mathcal{S}$.

We denote the above Hida distribution Φ_n in ii) by the same notation $\mathbf{I}_n(f_n)$ for $f_n \in \mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n}$.

3. Laplacian operators acting on Hida distributions

We introduce the definitions of Laplacian operators following [10] (see also [20]). Let F be a *Fréchet differentiable* function defined on $\mathcal{S}$, i.e. we assume that there exists a map F' from $\mathcal{S}$ to $\mathcal{S}^*$ such that

$$F(\xi + \eta) = F(\xi) + F'(\xi)(\eta) + o(\eta), \eta \in \mathcal{S},$$

where $o(\eta)$ means that there exists $p \in \mathbf{Z}$, $p \geq 0$, depending on ξ such that $o(\eta)/|\eta|_p \to 0$ as $|\eta|_p \to 0$. If the first variation is expressed in the form

$$F'(\xi)(\eta) = \int_{\mathbf{R}} F'(\xi; u) \eta(u) \, du$$

for every $\eta \in \mathcal{S}$ by using the generalized function $F'(\xi; \cdot)$, we define the *Hida derivative* $\partial_t \Phi$ of Φ to be the generalized white noise functional whose S-transform is given by $F'(\xi; t)$. The differentiation ∂_t is continuous from $(\mathcal{S})$ into itself. Its adjoint operator ∂_t^* is continuous from $(\mathcal{S})^*$ into itself.

Let $(\mathcal{H}, \mathcal{B})$ be an abstract Wiener space. Suppose ψ is a real-valued twice $\mathcal{H}$-differentiable function on $\mathcal{B}$ such that the second $\mathcal{H}$-derivative $D^2\psi(x)$ at x is a trace class operator of $\mathcal{H}$. Then the *Gross Laplacian* Δ_G ([4,5]) is defined by

$$\Delta_G \psi(x) = \text{Trace}_{\mathcal{H}} D^2 \psi(x).$$

The Laplacian Δ_G has the expression $\Delta_G \Phi = \int_{\mathbf{R}} \partial_t^2 \Phi dt$ on $(\mathcal{S})$ (see [17]). The Gross Laplacian is a continuous linear operator from $(\mathcal{S})$ into itself.

For any $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})^*$, the *number operator* N is defined by

$$N\Phi = \sum_{n=0}^{\infty} n\mathbf{I}_n(f_n).$$

The number operator is a continuous linear operator from $(\mathcal{S})^*$ into itself. The operator N has the expression $N\Phi = \int_{\mathbf{R}} \partial_t^* \partial_t \Phi dt$ on $(\mathcal{S})$ (see [17]).

A Hida distribution Φ is called an *L-functional* if for each $\xi \in \mathcal{S}$, there exist $(S[\Phi])'(\xi;\cdot) \in L^1_{loc}(\mathbf{R}) \cap \mathcal{S}^*, (S[\Phi])''_s(\xi;\cdot) \in L^1_{loc}(\mathbf{R}) \cap \mathcal{S}^*$ and $(S[\Phi])''_r(\xi;\cdot,\cdot) \in L^1_{loc}(\mathbf{R}^2) \cap \mathcal{S}^*(\mathbf{R}^2)$ such that the first and second variations are uniquely expressed in the forms:

$$(S[\Phi])'(\xi)(\eta) = \int_{\mathbf{R}} (S[\Phi])'(\xi;u)\eta(u)\, du,$$

and

$$(S[\Phi])''(\xi)(\eta,\zeta) = \int_{\mathbf{R}} (S[\Phi])''_s(\xi;u)\eta(u)\zeta(u)\, du$$

$$+ \int_{\mathbf{R}^2} (S[\Phi])''_r(\xi;u,v)\eta(u)\zeta(v)\, dudv,$$

for each $\eta, \zeta \in \mathcal{S}$, respectively and for any finite interval T, $\int_T (S[\Phi])''_s(\cdot;u)\, du$ is in $S[(\mathcal{S})^*]$. For any L-functional $\Phi \in D_L$ and any finite interval T in $\mathbf{R}$, the *Lévy Laplacian* Δ_L^T is defined by

$$\Delta_L^T \Phi = S^{-1}\left[\frac{1}{|T|}\int_T (S[\Phi])''_s(\cdot;u)\, du\right].$$

This Laplacian has the following interesting properties.

1) $\Delta_L^T = 0$ on (L^2) (see [7,26]).
2) Δ_L^T is a derivation under the Wick product (see [23]).

A Hida distribution Φ is called to be *normal* if its S-transform $S[\Phi]$ is given by a finite linear combination of

$$\int_{T^k} f(u_1,\dots,u_k)\xi(u_1)^{p_1}\cdots\xi(u_k)^{p_k}\, du_1\cdots du_k, \tag{3.1}$$

where T is a finite interval in $\mathbf{R}$, $f \in L^1(T^k)$ and $p_1,\dots,p_k \in \mathbf{N}\cup\{0\}, k \in \mathbf{N}$. For any $p \geq 1$, the normal functional with the S-transform given as in (3.1) is in $D_L^T \cap (\mathcal{S}_{-p})$, because the kernel

$$\int_{T^k} f(u_1,\dots,u_k)\delta_{u_1}^{\otimes p_1} \otimes \cdots \otimes \delta_{u_k}^{\otimes p_k}\, du_1\cdots du_k$$

is in $\mathcal{S}_{-1}^{\otimes(p_1+\cdots+p_k)}$. This functional plays the role of the polynomial in the infinite dimensional analysis. Let $\mathcal{N}_T$ denote the set of all normal functionals in D_L^T. For $p \geq 1$ and $\Phi \in D_L^T$, we define a $(-p)$-norm $|||\cdot|||_{-p}$ by

$$|||\Phi|||_{-p}^2 = \sum_{k=0}^{\infty} \|(\Delta_L^T)^k\Phi\|_{-p}^2 \ (\in [0,\infty])$$

and denote the completion of $\mathcal{N}_T$ with respect to the norm $|||\cdot|||_{-p}$ by $\mathbf{D}_{-p}$. Then $\mathbf{D}_{-p}$ is the Hilbert space with the norm $|||\cdot|||_{-p}$ and Δ_L^T is a bounded linear operator from $\mathbf{D}_{-p}$ into itself satisfying $|||\Delta_L^T\Phi|||_{-p} \le |||\Phi|||_{-p}$ for $\Phi \in \mathbf{D}_{-p}$. We put $\mathbf{D} = \bigcup_{p=1}^{\infty} \mathbf{D}_{-p}$ with the inductive limit topology. Then the Laplacian Δ_L^T is a continuous linear operator on $\mathbf{D}$.

Let D_L^T denote the set of all L-functionals Φ satisfying $S[\Phi](\eta) = 0$ for η with $\mathrm{supp}(\eta) \subset T^c$. In [22], Kuo obtained the following result.

Theorem 3.1. *Suppose $\{j_\epsilon; \epsilon > 0\}$ is a family of continuous linear operators from $\mathcal{S}^*$ into $\mathcal{S}$ satisfying the following conditions:*
(a) $j_\epsilon^* \to I$ strongly on $L^2(\mathbf{R})$ as $\epsilon \to 0$.
(b) $\lim_{\epsilon\to 0} |j_\epsilon|_{HS}^{-2} |j_\epsilon^* j_\epsilon|_{HS} = 0$.
(c) There exists a uniformly bounded orthonormal basis $\{e_k; k \ge 0\}$ for $L^2(T)$ such that as $\epsilon \to 0$,

$$|j_\epsilon|_{HS}^{-2} \sum_{k=0}^{\infty} (j_\epsilon e_k)(t)^2 \longrightarrow \frac{1}{|T|} \quad \text{in } L^2(T).$$

Then for any Φ in D_L^T,

$$S[\Delta_L^T \Phi](\xi) = \lim_{\epsilon\to 0} |j_\epsilon|_{HS}^{-2} S[\Delta_G S^{-1}(S[\Phi] \circ j_\epsilon)](\xi).$$

If $\varphi \in (\mathcal{S})$, the functional $S[\varphi]''(\xi)(\eta,\zeta)$, $\eta,\zeta \in \mathcal{S}$ has an extension $S[\varphi]''(\xi)(x,y)$, $x, y \in \mathcal{S}^*$, such that $S[\varphi]''(\xi)(x,x)$ is in $(\mathcal{S})$.

The chaos expansions of $\Delta_G\varphi$ and $S[\varphi]''(\xi)(x,x)$ for $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n)$ in $(\mathcal{S})$ are given by

$$\Delta_G \varphi = \sum_{n=0}^{\infty} \mathbf{I}_n \left((n+2)(n+1) \int_{\mathbf{R}} f_{n+2}(\cdot, t, t) dt \right)$$

and

$$S[\varphi]''(\xi)(x,x) = \sum_{n=0}^{\infty} n(n-1) \int_{\mathbf{R}^n} f_n(\mathbf{u}) \xi(u_1) \cdots \xi(u_{n-2}) x(u_{n-1}) x(u_n)\, d\mathbf{u},$$

respectively. Hence the expectation of $S[\varphi]''(\xi)(\cdot,\cdot)$ is given by

$$\int_{\mathcal{S}^*} S[\varphi]''(\xi)(x,x)\, d\mu(x) = \sum_{n=0}^{\infty} n(n-1) \int_{\mathbf{R}^{n-1}} f_n(\mathbf{v}, t, t) \xi^{\otimes(n-2)}(\mathbf{v})\, d\mathbf{v} dt.$$

Thus we come to get Lemma 3.2.

Lemma 3.2. *For any $\varphi \in (\mathcal{S})$, we have*

$$S[\Delta_G \varphi](\xi) = \int_{\mathcal{S}^*} S[\varphi]''(\xi)(x,x)\, d\mu(x).$$

We introduce an operator J_ϵ on $(\mathcal{S})^*$ into $(\mathcal{S})$ by

$$S[J_\epsilon \Phi](\xi) = S[\Phi](j_\epsilon(\xi)), \quad \Phi \in (\mathcal{S})^*.$$

Using the operator J_ϵ, we can obtain the following result.

Theorem 3.3. *Let T be a finite interval in* $\mathbf{R}$ *and* Φ *an* L*-functional in* D_L^T. *Then we have*

$$S[\Delta_L^T\Phi](\xi) = \lim_{\epsilon\to 0}(\theta_\epsilon)^2 \int_{\mathcal{S}^*} S[J_\epsilon\Phi]''(\xi)(x,x)\ d\mu(x),$$

where $\theta_\epsilon = |j_\epsilon|_{HS}^{-1}$.

4. Groups generated by infinite dimensional Laplacians

We now introduce an operator $e^{z\Delta_G}$, $z \in \mathbf{C}$ by

$$e^{z\Delta_G}\Phi = \sum_{n=0}^{\infty} \frac{(z\Delta_G)^n}{n!}\Phi$$

for $\Phi \in (\mathcal{S})$. This operator satisfies the following properties.

Theorem 4.1 [32]. *The* $e^{z\Delta_G}$ *is a continuous linear operator from* $(\mathcal{S})$ *into itself given by*

$$e^{z\Delta_G}\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(\ell_n(\Phi;z)),\ \ell_n(\Phi;z) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} z^m Tr^{\otimes m} * f_{n+2m} \qquad (4.1)$$

for $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})$.

Theorem 4.2 [32]. *For any* $\Phi \in (\mathcal{S})$, *we have*

$$S[e^{\frac{z}{2}\Delta_G}\Phi](\xi) = \int_{\mathcal{S}^*} S[\Phi](\xi + \sqrt{z}\,x)\ d\mu(x),$$

where the integral is defined independent of choices of the branch of $\sqrt{z}$ *since* μ *is symmetric.*

An infinite dimensional Fourier-Mehler transform $\mathbf{F}_\theta$, $\theta \in \mathbf{R}$, on $(\mathcal{S})^*$ was defined by H.-H. Kuo [19] as follows. The transform $\mathbf{F}_\theta\Phi$, $\theta \in \mathbf{R}$ of $\Phi \in (\mathcal{S})^*$ is defined by the unique Hida distribution with the S-transform

$$S[\mathbf{F}_\theta\Phi](\xi) = S[\Phi](e^{i\theta}\xi)\exp\left[\frac{i}{2}e^{i\theta}\sin\theta|\xi|_0^2\right],\ \xi \in \mathcal{S}.$$

Moreover, the adjoint operator $\mathbf{F}_\theta^*$ of $\mathbf{F}_\theta$ is given by

$$\mathbf{F}_\theta^*\Phi = \textstyle\sum_{n=0}^{\infty} \mathbf{I}_n(h_n(\Phi;\theta)) \text{ for } \Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S}),$$

where

$$h_n(\Phi;\theta) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!}(\frac{i}{2}\sin\theta)^m e^{i(m+n)\theta} Tr^{\otimes m} * f_{n+2m};$$

$$Tr = \int_{\mathbf{R}} \delta_t \otimes \delta_t dt.$$

This operator $\mathbf{F}_\theta^*$ is a continuous linear operator on $(\mathcal{S})$. (For details, see [19] and also [9].) The operator $e^{i\theta N}$ is called the *Fourier-Wiener transform*, which is given by

$$e^{i\theta N}\Phi = \sum_{n=0}^{\infty} e^{in\theta}\Phi_n$$

for $\Phi = \sum_{n=0}^{\infty} \Phi_n \in (\mathcal{S})$ (see [9]). The families $\{e^{i\theta\Delta_G}; \theta \in \mathbf{R}\}$, $\{e^{i\theta N}; \theta \in \mathbf{R}\}$ and $\{\mathbf{F}_\theta^*; \theta \in \mathbf{R}\}$ are groups generated by $i\Delta_G$, iN and $iN + \frac{i}{2}\Delta_G$, respectively (see [9]). Take $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})$. From (4.1), we see that

$$e^{\frac{i}{2}(e^{i\theta}\sin\theta)\Delta_G}\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(\ell_n(\Phi; \frac{i}{2}e^{i\theta}\sin\theta)).$$

Hence,

$$e^{i\theta N}(e^{\frac{i}{2}(e^{i\theta}\sin\theta)\Delta_G}\Phi) = \sum_{n=0}^{\infty} \mathbf{I}_n(e^{in\theta}\ell_n(\Phi; \frac{i}{2}e^{i\theta}\sin\theta)).$$

Since $e^{in\theta}\ell_n(\Phi; \frac{i}{2}e^{i\theta}\sin\theta) = h_n(\Phi;\theta)$, we obtain the following relation.

Theorem 4.3 [31].

$$\mathbf{F}_\theta^* = e^{i\theta N} \circ e^{\frac{i}{2}(e^{i\theta}\sin\theta)\Delta_G}.$$

Remark: Details of Lie algebras containing Δ_G and N are discussed in [28].

A (C_0)-group $\{G_t, t \in \mathbf{R}\}$ is given by

$$G_t = \lim_{\epsilon\to 0} \sum_{k=0}^{n} \frac{t^k}{k!}(\Delta_L^T)^k,$$

as an operator on $\mathbf{D}$. The group G_t has naturally an analytic extension G_z, $z \in \mathbf{C}$. It is easily checked that for any $\Phi \in \mathbf{D}$ and $t \in \mathbf{R}$ there exists $p \geq 1$ such that $|||G_z\Phi|||_{-p} \leq e^{|z|}|||\Phi|||_{-p}$.

An characterization of Hida distributions was obtained by J. Potthoff and L. Streit [29]. They say that for any F in $S[(\mathcal{S})^*]$ and ξ, η in $\mathcal{S}$, the function $F(\xi + \lambda\eta)$, $\lambda \in \mathbf{R}$, extends to an entire function $F(\xi + z\eta)$, $z \in \mathbf{C}$. We define an operator g_z, $z \in \mathbf{C}$, acting on a Hida distribution Φ by

$$S[g_z\Phi](\xi) = \lim_{\epsilon\to 0} S[e^{z(\theta_\epsilon)^2\Delta_G} J_\epsilon\Phi](\xi)$$

if the limit exists in $S[(\mathcal{S})^*]$. For $\Phi \in \mathcal{N}_T$ and $z \in \mathbf{C}$, we have $g_z\Phi \in \mathcal{N}_T$. For $p \geq 1$, let $\mathcal{E}_{-p}$ denote the collection of Hida distributions $\Phi = \sum_{n=0}^{\infty} \Phi_n$ in $(\mathcal{S}_{-p})$ such that $\Phi_n \in \mathcal{N}_T \cap H_n^{(-p)}$, $n = 0, 1, 2, \ldots$, and $\sum_{n=0}^{\infty} |||\Phi_n|||_{-p} < \infty$. Set $\mathcal{E} = \bigcup_p \mathcal{E}_{-p}$. It is clear that $\mathcal{E}_{-p} \subset \mathbf{D}_{-p}$ for $p \geq 1$ and $\mathcal{E} \subset \mathbf{D}$. By calculations of $g_z\Phi$ and $G_z\Phi$ for Φ whose S-transform $S\Phi$ is given as in (3.1), we get $g_z = G_z$ on $\mathcal{N}_T$ for $z \in \mathbf{C}$. The continuity of G_z implies the following result.

Theorem 4.4. *If $\Phi = \sum_{n=0}^{\infty} \Phi_n$ is in $\mathcal{E}_{-p}$ for $p \geq 1$, then $\sum_{n=0}^{\infty} g_z \Phi_n \in \mathbf{D}_{-p}$ and $G_z \Phi = \sum_{n=0}^{\infty} g_z \Phi_n$ for $z \in \mathbf{C}$. Moreover if*

$$\sum_{n=0}^{\infty} \sup_{\epsilon} \int_{\mathcal{S}^*} \left| S[J_\epsilon \Phi_n](\xi + \sqrt{2z}\theta_\epsilon x) \right| \, d\mu(x) < \infty$$

holds for any $z \in \mathbf{C}$ and $\xi \in \mathcal{S}$, then $g_z \Phi$ exists in $\mathbf{D}_{-p}$ and $g_z \Phi = G_z \Phi$.

5. A generalization

For any $\varphi \in (\mathcal{S})$, $\xi \in \mathcal{S}$ and $z_1, z_2 \in \mathbf{C}$, the functional $S[\varphi](z_1\xi + z_2\eta)$, $\eta \in \mathcal{S}$, can be extended to a functional $\widetilde{S[\varphi]}(z_1\xi + z_2 y)$, $y \in \mathcal{S}^*$, in $(\mathcal{S})$ (cf. [15]). We denote this functional by the same symbol $S[\varphi](z_1\xi + z_2 y)$. Thus we can define an operator $\mathcal{G}_{\alpha,\beta}$ from $(\mathcal{S})$ into itself by

$$S[\mathcal{G}_{\alpha,\beta}\varphi](\xi) = \int_{\mathcal{S}^*} S[\varphi](\alpha\xi + \beta x) \, d\mu(x). \tag{5.1}$$

Here we note that the right hand side of (5.1) is in $S[(\mathcal{S})]$. If $\alpha = 1$ or -1, $\mathcal{G}_{\alpha,\beta}$ is equal to Lee's transform $\mathcal{L}_{\alpha,\beta}$ ([24]) given by

$$\mathcal{L}_{\alpha,\beta}\varphi(x) = \int_{\mathcal{S}^*} \varphi(\alpha x + \beta y) \, d\mu(y), \quad \varphi \in (\mathcal{S}).$$

The transform $\mathcal{L}_{\alpha,\beta}$ is applied to the heat equation associated with the operator $(a\Delta_G + bN)^k$, $k \geq 1$, $a, b \in \mathbf{C}$ with $\mathrm{Re} b^k \leq 0$. (For details, see [3] and [14].) By the proof analogous to that of Theorem 3.2 in [32], we can obtain the following Lemma.

Lemma 5.1. *If a Hida distribution Φ is in $\mathcal{N}_T$, then*

$$\lim_{\epsilon \to 0} \int_{\mathcal{S}^*} S[J_\epsilon \Phi](\alpha_\epsilon(z)\xi + \beta_\epsilon(z)x) d\mu(x) = S[g_z \Phi](\xi)$$

holds for any $\xi \in \mathcal{S}$, where $\alpha_\epsilon(z)$ and $\beta_\epsilon(z)$ are complex-valued functions of $z \in \mathbf{C}$ depending $\epsilon > 0$ such that $\alpha_\epsilon(z) \to 1$ and $\beta_\epsilon(z)/\theta_\epsilon \to \sqrt{2it}$ as $\epsilon \to 0$.

Proof. The proof comes from Theorem 4.4 and the following formula:

$$\int_{\mathcal{S}^*} S[\varphi](\alpha\xi + \beta x) \, d\mu(x) = S[e^{N \log \alpha} \circ e^{\frac{\beta^2}{2}\Delta_G} \varphi](\xi), \quad \varphi \in (\mathcal{S}), \alpha, \beta \in \mathbf{C}.$$

□

By Lemma 5.1, we have the following result which is a generalization of Theorem 4.7 in [32].

Theorem 5.2. *Let Φ be a Hida distribution in $\mathcal{E}$ satisfying the condition*

$$\sum_{n=0}^{\infty} \sup_{\epsilon} \int_{\mathcal{S}^*} |S[J_\epsilon \Phi_n](\alpha_\epsilon(z)\xi + \beta_\epsilon(z)x)| \, d\mu(x) < \infty.$$

Then

$$\lim_{\epsilon \to 0} S[\mathcal{G}_{\alpha_\epsilon(z),\beta_\epsilon(z)} J_\epsilon \Phi](\xi) = S[G_z \Phi](\xi), \ z \in \mathbf{C}, \xi \in \mathcal{S}. \tag{5.2}$$

Proof. From the assumption and the Lebesgue convergence theorem, we can calculate as follows:

$$\lim_{\epsilon \to 0} S[\mathcal{G}_{\alpha_\epsilon(z),\beta_\epsilon(z)} J_\epsilon \Phi](\xi) = \lim_{\epsilon \to 0} \int_{\mathcal{S}^*} S[J_\epsilon \Phi](\alpha_\epsilon(z)\xi + \beta_\epsilon(z)x) d\mu(x)$$

$$= \sum_{n=0}^{\infty} \lim_{\epsilon \to 0} \int_{\mathcal{S}^*} S[J_\epsilon \Phi_n](\alpha_\epsilon(z)\xi + \beta_\epsilon(z)x) d\mu(x).$$

Consequently, by Lemma 5.1, we obtain (5.2). □

Theorem 4.3 admits an integral expression of the adjoint operator of Kuo's Fourier-Mehler transform:

$$S[\mathbf{F}_\theta^* \varphi](\xi) = \int_{\mathcal{S}^*} S[\varphi](e^{i\theta}\xi + \sqrt{ie^{i\theta}\sin\theta}\, x) \, d\mu(x), \ \varphi \in (\mathcal{S}).$$

Hence Theorem 5.2 implies the following

Corollary 5.3. *Let Φ be a Hida distribution in $\mathcal{E}$ satisfying the condition in Theorem 5.2 with*

$$\alpha_\epsilon(it) = e^{2it(\theta_\epsilon)^2} \ \textit{and} \ \beta_\epsilon(it) = \sqrt{ie^{2it(\theta_\epsilon)^2}\sin(2t(\theta_\epsilon)^2)}.$$

Then

$$\lim_{\epsilon \to 0} S[\mathbf{F}^*_{2t(\theta_\epsilon)^2} J_\epsilon \Phi](\xi) = S[G_{it}\Phi](\xi), \ t \in \mathbf{R}, \xi \in \mathcal{S}.$$

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