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量子確率微分方程式の体系
—物理と数学の狭間—

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1 Introduction

Studies of the Langevin equation for quantum systems were started by Senitzky [1], Lax [2] and Haken [3]. They investigated the Langevin equation for a quantum mechanical damped harmonic oscillator. In the quantum Langevin equation, variables in both relevant and irrelevant systems are stochastic operators. Putting the condition that the equal-time canonical commutation relation should hold for all time even for stochastic operators, they derived commutation relations among random force operators and their correlations.

In their studies, Senitzky, Lax and Haken did not construct a representation space explicitly. In quantum theory, observable operators do not have physical meaning until a representation space is specified. As was pointed out by Kubo [4], the quantum Langevin equation is an operator equation defined on a total representation space, i.e., a space of relevant system and of random forces. Any representation space of random force operators had not been constructed by physicists.

Mathematicians such as Hudson, Parthasarathy and their co-workers [5]-[10] constructed explicitly a representation space of random force operators. With the representation space, they realized a stochastic Schrödinger equation by analogy with the usual quantum mechanics. A time-evolution generator satisfying the stochastic Schrödinger equation was determined on the requirement of its unitarity, which is one of the necessary conditions for construction of a canonical operator formalism. It seems that, for mathematicians, a construction of the stochastic Liouville equation was out of their considerations.

The stochastic Liouville equation was introduced first by Kubo [11, 12] in order to investigate classical stochastic systems. In classical systems, the stochastic Liouville equation is an equation of motion for a probability distribution function in phase space under the influence of random forces. There had been a few attempts to extend the stochastic Liouville equation to quantum systems. Gardiner et al. [13, 14] and Dekker [15] derived a quantum stochastic Liouville equation by obtaining, within the trace formalism, an adjoint operator of a time-evolution generator for quantum Langevin equation. Furthermore, Gardiner et al. [16] rederived their stochastic Liouville equation on the basis of the stochastic Schrödinger equation introduced by Hudson et al. by making use of the fact that a density operator is a functional of wave functions. Within the density operator formalism, it is impossible to extract an explicit form of the time-evolution generator satisfying the stochastic Liouville equation, since the Liouville equation has entanglements between relevant operators and a density operator due to commutators and anticommutators among them. These difficulties prevent one from constructing a canonical operator formalism based on the stochastic Liouville equation.

On the other hand, within the framework of Non-Equilibrium Thermo Field Dynamics (NETFD) [17]-[19], a unified canonical operator formalism of quantum stochastic differential equations was constructed [20]-[30] on the basis of the stochastic Liouville equation. The quantum stochastic differential equations include the quantum Langevin equation and the quantum stochastic Liouville equation together with the corresponding quantum master equation. Within NETFD, introducing two kinds of operators, *with* tilde and *without* tilde, the entanglements between relevant operators and a density operator in the stochastic Liouville equation can be disentangled. Therefore, one can extract the explicit form of the time-evolution generator satisfying the stochastic Liouville equation, which enables us to construct a unified canonical operator formalism.

In this paper, we will construct quantum Wiener processes by means of boson annihilation and creation operators with their representation space extending mathematicians' procedure and implanting it into NETFD. The thermal degree of freedom in the quantum Wiener processes will be introduced by a *Bogoliubov transformation* in the thermal space which is a representation space within NETFD. Then, starting from a stochastic Shrödinger equation, we will show how one can obtain the time-evolution generator satisfying a stochastic Liouville equation with the help of the fact that a density operator is a functional of wave functions together with the principle of correspondence between quantities in the thermal space and in the Hilbert space. We will also show how one can construct a unified canonical operator formalism of quantum stochastic differential equations on the basis of the time-evolution generator.

2 Quantum Wiener Processes

We will construct quantum Wiener processes at zero temperature according to Hudson and Parthasarathy [5, 9, 10].

2.1 Fock Space

We introduce boson operators $b(t)$ and $b^\dagger(t)$ with $t \in [0,\infty)$ satisfying the canonical commutation relations

$$[b(t),\ b^\dagger(t)] = \delta(t-s), \quad [b(t),\ b(s)] = 0, \tag{1}$$

and define the vacuums $|0\rangle\!\rangle$ and $\langle\!\langle 0|$ by

$$b(t)|0\rangle\!\rangle = 0, \quad \langle\!\langle 0|b^\dagger(t) = 0. \tag{2}$$

We introduce ket- and bra-vectors defined by

$$|t_1,\cdots,t_n\rangle\!\rangle = \frac{1}{\sqrt{n!}}b^\dagger(t_1)\cdots b^\dagger(i_n)|0\rangle\!\rangle, \quad \langle\!\langle t_1,\cdots t_n| = \langle\!\langle 0|\frac{1}{\sqrt{n!}}b(t_1)\cdots b(t_n), \tag{3}$$

which satisfy the orthonormalization condition

$$\langle\!\langle t_1,\cdots,t_n|s_1,\cdots,s_m\rangle\!\rangle = \delta_{nm}\frac{1}{n!}\sum_P \delta(t_1-s_1)\cdots\delta(t_n-s_n), \tag{4}$$

and the completeness relation

$$\sum_{n=0}^{\infty}\left(\prod_l^n \int_0^\infty dt_l\right)|t_1,\cdots,t_n\rangle\!\rangle\langle\!\langle t_1,\cdots,t_n| = I. \tag{5}$$

Here, $\sum_P$ indicates the summation over all possible permutations of $t_1,\cdots,t_n$ with $s_1,\cdots.s_n$ fixed. Therefore, the set of ket-vectors $\{|t_1,\cdots,t_n\rangle\!\rangle\}$ and that of bra-vectors $\{\langle\!\langle t_1,\cdots,t_n|\}$ form complete orthonormal systems. The vector space Γ^0 built on the complete orthonormal basic vectors $|t_1,\cdots,t_n\rangle\!\rangle$ and $\langle\!\langle t_1,\cdots,t_n|$ is called the *Fock space**.

*Since annihilation and creation operators $b(t)$, $b^\dagger(t)$ satisfy bosonic canonical commutation relations (1), the vector space Γ^0 is also called the *boson Fock space* or the *symmetric Fock space* [10].

2.2 Quantum Wiener Processes

Let us define the operators B_t and $B_t^\dagger$ on the Fock space Γ^0 by

$$B_t = \int_0^t ds\ b(s), \quad B_t^\dagger = \int_0^t ds\ b^\dagger(s). \tag{6}$$

Taking average of B_t, $B_t^\dagger$ and the product $B_t^\dagger B_s$, $B_t B_s^\dagger$ with respect to the vacuums $|0\rangle\rangle$ and $\langle\langle 0|$, we find that

$$\langle\langle 0|B_t|0\rangle\rangle = \langle\langle 0|B_t^\dagger|0\rangle\rangle = 0, \tag{7}$$

$$\langle\langle 0|B_t^\dagger B_s|0\rangle\rangle = 0, \quad \langle\langle 0|B_t B_s^\dagger|0\rangle\rangle = \min(t,s), \tag{8}$$

where we used (2) and (1). Since the correlations (7), (8) indicates that the operators B_t and $B_t^\dagger$ on the Fock space Γ^0 can be interpreted as the Wiener process for a quantum system, we call the operators the quantum Wiener processes. The processes B_t and $B_t^\dagger$ are also called annihilation and creation processes, respectively [9, 10].

2.3 Product Rules

Let us introduce the *exponential vector* $|e(f)\rangle\rangle$, $\langle\langle e(f)| \in \Gamma^0$ by

$$|e(f)\rangle\rangle = \exp\left[\int_0^\infty dt\ f(t)b^\dagger(t)\right]|0\rangle\rangle, \quad \langle\langle e(f)| = \langle\langle 0|\exp\left[\int_0^\infty dt\ f^*(t)b(t)\right], \tag{9}$$

where f is an element of the set L^2 of square integrable functions satisfying $\int_0^\infty dt|f(t)|^2 < \infty$. Since the sets $\{|e(f)\rangle\rangle\ |f \in L^2\}$ and $\{\langle\langle e(f)|\ |f \in L^2\}$ of all exponential vectors are *linearly independent* and *total* in the Fock space Γ^0 [10], any operator on the Fock space is characterized by the action on the exponential vectors [5]. The annihilation and creation operators $b(t)$ and $b^\dagger(t)$ are characterized by the relations

$$b(t)|e(f)\rangle\rangle = f(t)|e(f)\rangle\rangle, \quad \langle\langle e(f)|b^\dagger(t) = \langle\langle e(f)|f^*(t), \tag{10}$$

respectively.

With the help of the properties (10), the quantum Wiener processes B_t and $B_t^\dagger$ defined by (6) are characterized by the following relations:

$$\langle\langle e(f)|dB_t|e(f')\rangle\rangle = f'(t)dt\langle\langle e(f)|e(f')\rangle\rangle, \tag{11}$$

$$\langle\langle e(f)|dB_t^\dagger|e(f')\rangle\rangle = f^*(t)dt\langle\langle e(f)|e(f')\rangle\rangle. \tag{12}$$

The products of the increments $dB_t = B_{t+dt} - B_t$, $dB_t^\dagger = B_{t+dt}^\dagger - B_t^\dagger$ and dt are characterized by the following relations[†]:

$$\langle\!\langle e(f)|dB_t dB_t|e(f')\rangle\!\rangle = O(dt^2), \tag{13}$$

$$\langle\!\langle e(f)|dB_t dB_t^\dagger|e(f')\rangle\!\rangle = dt\langle\!\langle e(f)|e(f')\rangle\!\rangle + O(dt^2), \quad \text{etc..} \tag{14}$$

Taking into account of the terms of $O(dt)$ in L^2-space and neglecting the terms of $o(dt)$, we have, from the matrix elements (11), (12) and (13), (14), the following product rules [5]:

$$\begin{array}{c|ccc} & dB_t & dB_t^\dagger & dt \\ \hline dB_t & 0 & dt & 0 \\ dB_t^\dagger & 0 & 0 & 0 \\ dt & 0 & 0 & 0 \end{array} \tag{15}$$

3 Quantum Wiener Processes at Finite Temperatures

We will construct quantum Wiener processes at finite temperatures considering boson operators on the thermal space, which is the representation space within NETFD. This is the reconstruction of quantum Wiener processes at finite temperatures introduced by Hudson and Lindsay [7, 8], within the framework of NETFD.

3.1 Representation Space

We introduce the *tilde* operators $(\tilde{b}(t), \tilde{b}^\dagger(t))$ on the *tilde conjugate space* $\tilde{\Gamma}^0$ associated with $(b(t), b^\dagger(t))$ on Γ^0. Here, the *tilde conjugation* $\sim$ is defined by

$$(A_1 A_2)^\sim = \tilde{A}_1 \tilde{A}_2, \tag{16}$$

$$(c_1 A_1 + c_2 A_2)^\sim = c_1^* \tilde{A}_1 + c_2^* \tilde{A}_2, \tag{17}$$

$$(\tilde{A})^\sim = A, \tag{18}$$

$$(A^\dagger)^\sim = \tilde{A}^\dagger, \tag{19}$$

[†] $O(x)$ indicates that

$$\lim_{x \to 0} \frac{O(x)}{x} = \alpha \neq 0,$$

while $o(x)$ indicates that

$$\lim_{x \to 0} \frac{o(x)}{x} = 0.$$

where A_1, A_2 and A are arbitrary operators on Γ^0 and c_1 and c_2 are c-numbers. Note that the tilde conjugate space $\tilde{\Gamma}^0$ is the Fock space built on the basic vectors made by cyclic operations of $\tilde{b}^\dagger(t)$ on the vacuum $|\tilde{0}\rangle\rangle$ and $\tilde{b}(t)$ on the vacuum $\langle\langle\tilde{0}|$, where the vacuums $|\tilde{0}\rangle\rangle$ and $\langle\langle\tilde{0}|$ are defined by

$$\tilde{b}(t)|\tilde{0}\rangle\rangle = 0, \quad \langle\langle\tilde{0}|\tilde{b}^\dagger(t) = 0. \tag{20}$$

Note that $|\tilde{0}\rangle\rangle$ and $\langle\langle\tilde{0}|$ are the tilde conjugate of $|0\rangle\rangle$ and $\langle\langle 0|$.

We consider the tensor product space

$$\Gamma = \Gamma^0 \otimes \tilde{\Gamma}^0. \tag{21}$$

The vacuum states $|0)$ and $(0|$ of Γ is defined by

$$|0) = |0\rangle\rangle \otimes |\tilde{0}\rangle\rangle, \quad (0| = \langle\langle 0| \otimes \langle\langle\tilde{0}|, \tag{22}$$

which are invariant under the tilde conjugation, i.e., $\{|0)\}^\sim = |0)$, $\{(0|\}^\sim = (0|$. Note that Γ is the Fock space built on the basic vectors made by cyclic operations of $(b^\dagger(t), \tilde{b}^\dagger(t))$ on the vacuum $|0)$ and $(b(t), \tilde{b}(t))$ on the vacuum $(0|$. In the following, we will use the notational conventions such as

$$b(t) \otimes \tilde{I} \Rightarrow b(t), \quad b^\dagger(t) \otimes \tilde{I} \Rightarrow b^\dagger(t), \tag{23}$$

$$I \otimes \tilde{b}(t) \Rightarrow \tilde{b}(t), \quad I \otimes \tilde{b}^\dagger(t) \Rightarrow \tilde{b}^\dagger(t), \tag{24}$$

where I and $\tilde{I}$ stand for identity operators on Γ^0 and $\tilde{\Gamma}^0$, respectively. The annihilation and creation operators $b(t)$, $b^\dagger(t)$, $\tilde{b}(t)$, $\tilde{b}^\dagger(t)$ on Γ satisfy the canonical commutation relations

$$[b(t),\ b^\dagger(s)] = [\tilde{b}(t),\ \tilde{b}^\dagger(s)] = \delta(t-s), \tag{25}$$

$$[b(t),\ b(s)] = [\tilde{b}(t),\ \tilde{b}(s)] = [b(t),\ \tilde{b}(s)] = [b(t),\ \tilde{b}^\dagger(s)] = 0. \tag{26}$$

Note that the vacuums $|0)$ and $(0|$ satisfy

$$b(t)|0) = \tilde{b}(t)|0) = 0, \quad (0|b^\dagger(t) = (0|\tilde{b}^\dagger(t) = 0. \tag{27}$$

The thermal degree of freedom can be introduced by *Bogoliubov transformation* in Γ. First, we require that the expectation value of $b^\dagger(t)b(s)$ should be

$$\langle b^\dagger(t)b(s)\rangle = \bar{n}\delta(t-s), \tag{28}$$

with a real positive number $\bar{n}$, where $\langle\cdots\rangle$ indicates the expectation with respect to some states $|\rangle$ and $\langle|$. We find that in order to insure the equation (28), it is sufficient to impose the following conditions on the states $|\rangle$ and $\langle|$:

$$b(t)|\rangle = \frac{\bar{n}}{1+\bar{n}}\tilde{b}^{\dagger}(t)|\rangle, \quad \langle|b^{\dagger}(t) = \langle|\tilde{b}(t). \tag{29}$$

In fact, using the conditions (29), we have

$$\langle b^{\dagger}(t)b(s)\rangle = \frac{\bar{n}}{1+\bar{n}}\left\{\langle|b^{\dagger}(t)b(s)|\rangle + \delta(t-s)\right\}, \tag{30}$$

which leads to (28). We call the states $|\rangle$ and $\langle|$ the *thermal ket-vacuum* and the *thermal bra-vacuum*, respectively, and the conditions (29) the *thermal state conditions* for the thermal ket- and bra-vacuums.

We introduce annihilation operators $c(t)$, $\tilde{c}(t)$ and creation operators $c^{\mathring{\dagger}}(t)$, $\tilde{c}^{\mathring{\dagger}}(t)$ for the thermal ket-vacuum $|\rangle$ satisfying

$$c(t)|\rangle = \tilde{c}(t)|\rangle = 0, \quad \langle|c^{\mathring{\dagger}}(t) = \langle|\tilde{c}^{\mathring{\dagger}}(t) = 0, \tag{31}$$

and the canonical commutation relations

$$[c(t),\ c^{\mathring{\dagger}}(s)] = [\tilde{c}(t),\ \tilde{c}^{\mathring{\dagger}}(s)] = \delta(t-s), \tag{32}$$

$$[c(t),\ c(s)] = [\tilde{c}(t),\ \tilde{c}(s)] = [c(t),\ \tilde{c}(s)] = [c(t),\ \tilde{c}^{\mathring{\dagger}}(s)] = 0, \tag{33}$$

$$[c^{\mathring{\dagger}}(t),\ c^{\mathring{\dagger}}(s)] = [\tilde{c}^{\mathring{\dagger}}(t),\ \tilde{c}^{\mathring{\dagger}}(s)] = [c^{\mathring{\dagger}}(t),\ \tilde{c}(s)] = [c^{\mathring{\dagger}}(t),\ \tilde{c}^{\mathring{\dagger}}(s)] = 0. \tag{34}$$

Recalling the thermal state conditions (29), we see that such operators $c(t)$, $c^{\mathring{\dagger}}(t)$ and their tilde conjugates are related to $b(t)$, $b^{\dagger}(t)$ and their tilde conjugates through the transformation

$$\begin{pmatrix} c(t) \\ \tilde{c}^{\mathring{\dagger}}(t) \end{pmatrix} = \begin{pmatrix} 1+\bar{n} & -\bar{n} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b(t) \\ \tilde{b}^{\dagger}(t) \end{pmatrix}. \tag{35}$$

The transformation (35) is called the Bogoliubov transformation[‡] [31]. The Bogoliubov transformation is the canonical one such that the canonical commutation relations do not change under this transformation.

Let the boson Fock space built on the basic ket- and bra-vectors made by cyclic operations of $(c^{\mathring{\dagger}}(t), \tilde{c}^{\mathring{\dagger}}(t))$ on the thermal ket-vacuum $|\rangle$ and of $(c(t), \tilde{c}(t))$ on the thermal bra-vacuum $\langle|$, be denoted by Γ^{β}.

‡In the expression of the Bogoliubov transformation, there is a freedom of normalization constant. The expression (35) is the only one that is linear with respect to $\bar{n}$.

Note that the Bogoliubov transformation (35) is rewritten as

$$c(t) = U_B^{-1} b(t) U_B, \quad \tilde{c}^{\text{♀}}(t) = U_B^{-1} \tilde{b}^\dagger(t) U_B, \tag{36}$$

where

$$U_B = \exp\left[-\bar{n}\int_0^\infty dt\; b^\dagger(t)\tilde{b}^\dagger(t)\right]\exp\left[\int_0^\infty dt\; b(t)\tilde{b}(t)\right], \tag{37}$$

$$U_B^{-1} = \exp\left[-\int_0^\infty dt\; b(t)\tilde{b}(t)\right]\exp\left[\bar{n}\int_0^\infty dt\; b^\dagger(t)\tilde{b}^\dagger(t)\right]. \tag{38}$$

The equations (36) together with the properties (27) and (31) give the relations between the thermal vacuums in Γ^β and the vacuums in Γ as follows:

$$|\rangle = U_B^{-1}|0), \quad \langle| = (0|U_B. \tag{39}$$

Using the formula of the Lie algebra of SU(1,1) group [32]-[35], we can rewrite U_B^{-1} as normal ordered product

$$\begin{aligned} U_B^{-1} = \exp&\left[\frac{\bar{n}}{1+\bar{n}}\int_0^\infty dt\; b^\dagger(t)\tilde{b}^\dagger(t)\right] \\ &\times\exp\left[-\ln(1+\bar{n})\int_0^\infty dt\;\left\{b^\dagger(t)b(t) + \tilde{b}^\dagger(t)\tilde{b}(t) + \delta(0)\right\}\right] \\ &\times\exp\left[-\frac{1}{1+\bar{n}}\int_0^\infty dt\; b(t)\tilde{b}(t)\right]. \end{aligned} \tag{40}$$

Here, $\delta(0)$ is the delta function $\delta(t)$ with $t = 0$. The equations (39) and (40) together with the property (27) give

$$|\rangle = \exp\left[-\delta(0)\ln(1+\bar{n})\int_0^\infty dt\right]\exp\left[\frac{\bar{n}}{1+\bar{n}}\int_0^\infty dt\; b^\dagger(t)\tilde{b}^\dagger(t)\right]|0). \tag{41}$$

Since $\delta(0) = \infty$, the equation (41) shows that in the thermal ket-vacuum $|\rangle$, infinite number of the $(b(t), \tilde{b}(t))$-pairs are condensed and the Fock space Γ^β is *inequivalent* to the Fock space Γ in the sense that any vector in Γ^β can not be written as a superposition of vectors in Γ and vice versa.

On the other hand, the equation (39) together with the expression (37) of U_B gives

$$\begin{aligned} \langle| &= (0|\exp\left[-\bar{n}\int_0^\infty dt\; b^\dagger(t)\tilde{b}^\dagger(t)\right]\exp\left[\int_0^\infty dt\; b(t)\tilde{b}(t)\right] \\ &= (0|\exp\left[\int_0^\infty dt\; b(t)\tilde{b}(t)\right], \end{aligned} \tag{42}$$

where we used the property (27). We see that the equation (42) is consistent with the thermal state condition (29) of the bra-vacuum. In fact, using the equation (42) and the property (27), we can prove the thermal state condition (29).

3.2 Quantum Wiener Processes at Finite Temperatures

Quantum Wiener processes at finite temperatures are defined by the operators

$$B_t = \int_0^t ds\ b(s), \quad B_t^\dagger = \int_0^t ds\ b^\dagger(s), \tag{43}$$

and their tilde-conjugates represented in the Fock space Γ^β. The explicit representations of the processes B_t, $B_t^\dagger$, $\tilde{B}_t$ and $\tilde{B}_t^\dagger$ in Γ^β are given in terms of the Bogoliubov transformation (35) by

$$B_t = \int_0^t ds\ [c(s) + \bar{n}\tilde{c}^\natural(s)] = C_t + \bar{n}\tilde{C}_t^\natural, \tag{44}$$

$$B_t^\dagger = \int_0^t ds\ [\tilde{c}(s) + (1+\bar{n})c^\natural(s)] = \tilde{C}_t + (1+\bar{n})C_t^\natural, \tag{45}$$

and their tilde conjugates, where C_t, $C_t^\natural$, $\tilde{C}_t$ and $\tilde{C}_t^\natural$ are the annihilation and creation processes in Γ^β defined by

$$C_t = \int_0^t ds\ c(s), \quad C_t^\natural = \int_0^t ds\ c^\natural(s), \tag{46}$$

and their tilde-conjugates.

Any operator in the Fock space Γ^β can be characterized by the exponential vectors $|e(f,g)\rangle$, $\langle e(f,g)|$ in Γ^β with $f,\ g \in L^2$ defined by

$$|e(f,g)\rangle = \exp\left[\int_0^\infty dt \left\{f(t)c^\natural(t) + g^*(s)\tilde{c}^\natural(s)\right\}\right] |\rangle, \tag{47}$$

$$\langle e(f,g)| = \langle| \exp\left[\int_0^\infty dt \left\{f^*(t)c(t) + g(s)\tilde{c}(s)\right\}\right], \tag{48}$$

which satisfy the following relations:

$$c(t)|e(f,g)\rangle = f(t)|e(f,g)\rangle, \quad \langle e(f,g)|c^\natural(t) = \langle e(f,g)|f^*(t), \tag{49}$$

$$\tilde{c}(t)|e(f,g)\rangle = g^*(t)|e(f,g)\rangle, \quad \langle e(f,g)|\tilde{c}^\natural(t) = \langle e(f,g)|g(t). \tag{50}$$

As the case of the construction of annihilation and creation processes B_t, $B_t^\dagger$ in Fock space Γ^0, the evaluation of matrix elements of the products of the increments dC_t, $dC_t^\natural$, $d\tilde{C}_t$, $d\tilde{C}_t^\natural$ and dt between exponential vectors $|e(f,g)\rangle$ and $\langle e(f,g)|$ with the help of the properties (49), (50) gives the product rules of the increments dC_t, $dC_t^\natural$, $d\tilde{C}_t$, $d\tilde{C}_t^\natural$ and dt, which are summarized as the following table.

	dC_t	$dC_t^\natural$	$d\tilde{C}_t$	$d\tilde{C}_t^\natural$	dt
dC_t	0	dt	0	0	0
$dC_t^\natural$	0	0	0	0	0
$d\tilde{C}_t$	0	0	0	dt	0
$d\tilde{C}_t^\natural$	0	0	0	0	0
dt	0	0	0	0	0

(51)

By means of the expressions (44), (45) and their tilde conjugates and the product rules (51), we can evaluate the products of the increments of dB_t, $dB_t^\dagger$, $d\tilde{B}_t$, $d\tilde{B}_t^\dagger$ and dt and obtain the product rules summarized as follows:

$$\begin{array}{c|ccccc} & dB_t & dB_t^\dagger & d\tilde{B}_t & d\tilde{B}_t^\dagger & dt \\ \hline dB_t & 0 & (1+\bar{n})dt & \bar{n}dt & 0 & 0 \\ dB_t^\dagger & \bar{n}dt & 0 & 0 & (1+\bar{n})dt & 0 \\ d\tilde{B}_t & \bar{n}dt & 0 & 0 & (1+\bar{n})dt & 0 \\ d\tilde{B}_t^\dagger & 0 & (1+\bar{n})dt & \bar{n}dt & 0 & 0 \\ dt & 0 & 0 & 0 & 0 & 0 \end{array} \tag{52}$$

Using the equations (44), (45) and their tilde conjugates, the commutation relation (32) and the properties (31) of the thermal vacuums, we obtain the correlations of the increments dB_t, $dB_t^\dagger$, $d\tilde{B}_t$ and $d\tilde{B}_t^\dagger$ with respect to the thermal vacuums $|\rangle$ and $\langle|$ as follows:

$$\langle dB_t\rangle = \langle dB_t^\dagger\rangle = \langle d\tilde{B}_t\rangle = \langle d\tilde{B}_t^\dagger\rangle = 0, \tag{53}$$

$$\left\langle dB_t^\dagger dB_s\right\rangle = \left\langle d\tilde{B}_t^\dagger d\tilde{B}_s\right\rangle = \left\langle dB_t d\tilde{B}_s\right\rangle = \left\langle d\tilde{B}_t dB_s\right\rangle = \bar{n}\delta(t-s)dtds, \tag{54}$$

$$\left\langle dB_t dB_s^\dagger\right\rangle = \left\langle d\tilde{B}_t d\tilde{B}_s^\dagger\right\rangle = \left\langle dB_t^\dagger d\tilde{B}_s^\dagger\right\rangle = \left\langle d\tilde{B}_t^\dagger dB_s^\dagger\right\rangle = (1+\bar{n})\delta(t-s)dtds, \tag{55}$$

$$(\text{others}) = 0, \tag{56}$$

where $\langle\cdots\rangle \equiv \langle|\cdots|\rangle$. From the correlations (53)–(56), we see that putting $\bar{n}$ to the Planck distribution given by

$$\bar{n} = \frac{1}{e^{\beta\omega}-1}, \tag{57}$$

with some positive number ω and the inverse of the temperature, $\beta = 1/T$, the quantum Wiener processes B_t, $B_t^\dagger$ are essentially equivalent to those introduced in the problem of quantum optics [3].

4 Quantum Stochastic Calculus

On the basis of the quantum Wiener processes at finite temperatures, we will investigate the quantum stochastic calculus.

4.1 Adapted Processes

The Fock space Γ^β is decomposed as

$$\Gamma^\beta = \Gamma^\beta_{t]} \otimes \Gamma^\beta_{(t}, \tag{58}$$

in which for $f, g \in L^2$,

$$|e(f,g)\rangle = |e(f_{t]}, g_{t]})\rangle \otimes |e(f_{(t}, g_{(t)})\rangle, \quad \langle e(f,g)| = \langle e(f_{t]}, g_{t]})| \otimes \langle e(f_{(t}, g_{(t)})|, \tag{59}$$

where we set

$$f_{t]} = f\chi_{t]}, \quad f_{(t} = f\chi_{(t}, \tag{60}$$

and assumed that

$$|\rangle = |_{t]}\rangle \otimes |_{(t}\rangle, \quad \langle| = \langle_{t]}| \otimes \langle_{(t}|. \tag{61}$$

Here, $\chi_{t]}$ and $\chi_{(t}$ are defined by

$$\chi_{t]}(s) = \theta(t-s), \quad \chi_{(t}(s) = \theta(s-t), \quad \text{for } t, s > 0. \tag{62}$$

Note that $\Gamma_{t]}^{\beta}$ is the boson Fock space built on the vacuums $|_{t]}\rangle$ and $\langle_{t]}|$, while $\Gamma_{(t]}^{\beta}$ is the boson Fock space built on the vacuums $|_{(t}\rangle$ and $\langle_{(t}|$. The quantum Wiener processes B_t, $B_t^{\dagger}$, $\tilde{B}_t$, $\tilde{B}_t^{\dagger}$ are operators on the space $\Gamma_{t]}^{\beta}$.

Let us consider a tensor product space $\mathcal{H}_S \otimes \Gamma^{\beta}$ where $\mathcal{H}_S$ is a certain vector space. For the sake of notational convenience, we identify the quantum Wiener processes B_t, $B_t^{\dagger}$, $\tilde{B}_t$, $\tilde{B}_t^{\dagger}$ in $\Gamma_{t]}^{\beta}$ with their ampliations to $\mathcal{H}_S \otimes \Gamma^{\beta}$, i.e.

$$I_S \otimes B_t \otimes I_{(t} \Rightarrow B_t, \quad I_S \otimes B_t^{\dagger} \otimes I_{(t} \Rightarrow B_t^{\dagger}, \tag{63}$$

$$I_S \otimes \tilde{B}_t \otimes I_{(t} \Rightarrow \tilde{B}_t, \quad I_S \otimes \tilde{B}_t^{\dagger} \otimes I_{(t} \Rightarrow \tilde{B}_t^{\dagger}, \tag{64}$$

where I_S and $I_{(t}$ are the identity operators on $\mathcal{H}_S$ and $\Gamma_{(t}^{\beta}$, respectively.

An adapted process F_t is defined by

$$F_t = F_t^0 \otimes I_{(t}, \tag{65}$$

where F_t^0 is an operator on $\mathcal{H}_S \otimes \Gamma_{t]}^{\beta}$. The quantum Wiener processes B_t, $B_t^{\dagger}$, $\tilde{B}_t$, $\tilde{B}_t^{\dagger}$ are adapted. For the adapted process F_t, we have

$$[F_t,\ dB_t] = [F_t,\ dB_t^{\dagger}] = [F_t,\ d\tilde{B}_t] = [F_t,\ d\tilde{B}_t^{\dagger}] = 0. \tag{66}$$

4.2 Quantum Stochastic Calculus

Let X_t denote an arbitrary adapted process in $\mathcal{H}_S \otimes \Gamma^{\beta}$ and Q_t denote quantum Wiener processes B_t, $B_t^{\dagger}$, $\tilde{B}_t$, $\tilde{B}_t^{\dagger}$.

Quantum stochastic integrals of the Ito type are defined by

$$\int_0^T X_t dQ_t \equiv \lim \sum_{i=0}^{I-1} X_{t_i} \Delta Q_{t_i} = \lim \sum_{i=0}^{I-1} X_{t_i} (Q_{t_{i+1}} - Q_{t_i}), \tag{67}$$

and

$$\int_0^T dQ_t X_t \equiv \lim \sum_{i=0}^{I-1} \Delta Q_{t_i} X_{t_i} = \lim \sum_{i=0}^{I-1} (Q_{t_{i+1}} - Q_{t_i}) X_{t_i}, \tag{68}$$

while those of the Stratonovich type are defined by

$$\int_0^T X_t \circ dQ_t \equiv \lim \sum_{i=0}^{I-1} \frac{X_{t_{i+1}} + X_{t_i}}{2} \Delta Q_{t_i} = \lim \sum_{i=0}^{I-1} \frac{X_{t_{i+1}} + X_{t_i}}{2} (Q_{t_{i+1}} - Q_{t_i}), \tag{69}$$

and

$$\int_0^T dQ_t \circ X_t \equiv \lim \sum_{i=0}^{I-1} \Delta Q_{t_i} \frac{X_{t_{i+1}} + X_{t_i}}{2} = \lim \sum_{i=0}^{I-1} (Q_{t_{i+1}} - Q_{t_i}) \frac{X_{t_{i+1}} + X_{t_i}}{2}. \tag{70}$$

Here, the notation lim indicates taking the limit

$$\Delta t \longrightarrow +0, \quad I \longrightarrow +\infty, \tag{71}$$

keeping $T = I\Delta t$ fixed. Note that quantum stochastic integrals both of the Ito and of the Stratonovich types are adapted processes.

We introduce the differential notations of the stochastic integrals (67), (68) and (69), (70) as

$$X_t dQ_t \equiv X_t (Q_{t+dt} - Q_t), \tag{72}$$

$$dQ_t X_t \equiv (Q_{t+dt} - Q_t) X_t, \tag{73}$$

and

$$X_t \circ dQ_t \equiv \frac{X_{t+dt} + X_t}{2} (Q_{t+dt} - Q_t), \tag{74}$$

$$dQ_t \circ X_t \equiv (Q_{t+dt} - Q_t) \frac{X_{t+dt} + X_t}{2}. \tag{75}$$

We call (72) and (73) the products of the Ito type, whereas we refer to (74) and (75) as the products of the Stratonovich type.

From the property (66), we have in the stochastic calculus of the Ito type

$$[X_t,\ dQ_t] = 0, \tag{76}$$

which leads to

$$\int_0^T X_t dQ_t = \int_0^T dQ_t X_t. \tag{77}$$

In addition, using the property (61) of the thermal vacuums and the property of quantum stochastic processes

$$\langle|Q_t|\rangle = 0, \tag{78}$$

we have

$$\langle|dQ_t X_t|\rangle = \langle|X_t dQ_t|\rangle = \langle_{t]}|X_t|_{t]}\rangle\langle_{(t}|dQ_t|_{(t}\rangle = 0, \tag{79}$$

which indicates that there is no correlation between X_t and dQ_t.

It should be pointed out that while the increment dQ_t commutes with X_t, it does not commute with X_{t+dt}. Therefore, in the stochastic calculus of the Stratonovich type, the commutation relation of X_t and dQ_t of the Stratonovich type defined by

$$[X_t \overset{\circ}{,} dQ_t] \equiv X_t \circ dQ_t - dQ_t \circ X_t, \tag{80}$$

does not equal zero, i.e.

$$[X_t \overset{\circ}{,} dQ_t] \neq 0, \tag{81}$$

which leads to

$$\int_0^T X_t \circ dQ_t \neq \int_0^T dQ_t \circ X_t. \tag{82}$$

Moreover, in contrast with the case of the Ito type,

$$\langle|X_t \circ dQ_t|\rangle \neq 0, \quad \langle|dQ_t \circ X_t|\rangle \neq 0. \tag{83}$$

Substituting $X_{t+dt} = X_t + dX_t$ into (74) and (75), we obtain

$$X_t \circ dQ_t = X_t dQ_t + \frac{1}{2} dX_t dQ_t, \tag{84}$$

and

$$dQ_t \circ X_t = dQ_t X_t + \frac{1}{2} dQ_t dX_t, \tag{85}$$

which give the relations between the products of the Ito and the Stratonovich types.

4.3 Quantum Ito's Formula

We introduce an operator N_t constructed by using the stochastic integrals of the Ito type as

$$N_T = \int_0^T \left(F_t dB_t + G_t dB_t^\dagger + J_t d\tilde{B}_t + K_t d\tilde{B}_t^\dagger + H_t dt \right), \tag{86}$$

where F_t, G_t, H_t, J_t, K_t are adapted processes. The operator N_t is an adapted process which satisfies

$$[N_t, dQ_t] = 0. \tag{87}$$

The differential notation of N_t is given by

$$dN_t = F_t dB_t + G_t dB_t^\dagger + J_t d\tilde{B}_t + K_t d\tilde{B}_t^\dagger + H_t dt. \tag{88}$$

It should be noted that, for an arbitrary adapted process X_t, we have in general

$$[X_t,\ dN_t] \neq 0, \tag{89}$$

because dN_t includes not only the increments dB_t, $dB_t^\dagger$, $d\tilde{B}_t$, $d\tilde{B}_t^\dagger$ but also adapted processes F_t, G_t, J_t, K_t, H_t. Furthermore, for the adapted process X_t and the increment dN_t, the property as (79) does not hold because of the term $H_t dt$[§], i.e.

$$\langle|X_t dN_t|\rangle = \langle|X_t H_t|\rangle dt \neq 0, \quad \langle|dN_t X_t|\rangle = \langle|H_t X_t|\rangle dt \neq 0. \tag{90}$$

For stochastic integrals, *quantum Ito's formula* holds [5, 9, 10]:

Theorem 4.1 (Quantum Ito's Formula) *We set*

$$N_T = \int_0^T \left(F_t dB_t + G_t dB_t^\dagger + J_t d\tilde{B}_t + K_t d\tilde{B}_t^\dagger + H_t dt\right), \tag{91}$$

$$N'_T = \int_0^T \left(F'_t dB_t + G'_t dB_t^\dagger + J'_t d\tilde{B}_t + K'_t d\tilde{B}_t^\dagger + H'_t dt\right), \tag{92}$$

where F_t, G_t, H_t, J_t, K_t, F'_t, G'_t, H'_t, J'_t, K'_t are adapted processes. The differential notations of N_t and N'_t are given by

$$dN_t = F_t dB_t + G_t dB_t^\dagger + J_t d\tilde{B}_t + K_t d\tilde{B}_t^\dagger + H_t dt, \tag{93}$$

and

$$dN'_t = F'_t dB_t + G'_t dB_t^\dagger + J'_t d\tilde{B}_t + K'_t d\tilde{B}_t^\dagger + H'_t dt, \tag{94}$$

respectively. Then, the differential of the product $N_t N'_t$ can be evaluated by means of the formula

$$d(N_t N'_t) = dN_t \cdot N'_t + N_t \cdot dN'_t + dN_t dN'_t, \tag{95}$$

with the property (66) of adapted processes and the product rules (52).

Making use of the relations (84) and (85) between the Ito and the Stratonovich products, we have

$$N_t \circ dN'_t = N_t dN'_t + \frac{1}{2} dN_t dN'_t, \tag{96}$$

[§] When $H_t = 0$, dN_t satisfies $\langle\langle 0|X_t dN_t|0\rangle\rangle = \langle\langle 0|dN_t X_t|0\rangle\rangle = 0$, although dN_t still does not commute with X_t, i.e. $[X_t,\ dN_t] \neq 0$. When $H_t = 0$, N_t is called the *martingale* [7, 36].

and

$$dN_t \circ N_t' = dN_t N_t' + \frac{1}{2} dN_t dN_t'. \tag{97}$$

Therefore, we find that quantum Ito's formula (95) is expressed in terms of the Stratonovich products as

$$d(N_t N_t') = dN_t \circ N_t' + N_t \circ dN_t', \tag{98}$$

which is identical to the well-known formula of the ordinary differential calculus.

5 Stochastic Schrödinger Equation

In this section, we consider the stochastic Schrödinger equation investigated by Hudson and Lindsay [7].

5.1 The Ito Type

We consider a boson system which is described by the operators a, $a^\dagger$ on a Hilbert space $\mathcal{H}_S^0$ satisfying the commutation relations

$$[a,\ a^\dagger] = 1, \quad [a,\ a] = 0, \tag{99}$$

and which interacts with a reservoir at finite temperatures. Let us suppose that the effect of the reservoir on the system is taken into account by the random force operators represented by the quantum Wiener processes at finite temperatures constructed on the Fock space Γ^β. We sometimes call the boson system the relevant system and the reservoir system the irrelevant system.

The state of the system is described by the state vector $|\psi_f(t)\rangle\!\rangle$ in the space $\mathcal{H}_S^0 \otimes \Gamma^\beta$. The state vector $|\psi_f(t)\rangle\!\rangle$ is assumed to evolve in time according to the Schrödinger equation

$$d|\psi_f(t)\rangle\!\rangle = -i\mathcal{H}_{f,t} dt |\psi_f(t)\rangle\!\rangle, \tag{100}$$

with an infinitesimal time-evolution generator $\mathcal{H}_{f,t}$ including random force operators. We call the equation (100) *stochastic Schrödinger equation.*

The formal solution of (100) is written by

$$|\psi_f(t)\rangle\!\rangle = V_f(t)|\psi_f(0)\rangle\!\rangle, \tag{101}$$

where $V_f(t)$ is the stochastic time-evolution generator satisfying the equation

$$dV_f(t) = -i\mathcal{H}_{f,t} dt V_f(t), \tag{102}$$

with $V_f(0) = 1$. Note that the bra-vector $\langle\!\langle \psi_f(t)|$ is defined by

$$\langle\!\langle \psi_f(t)| = \langle\!\langle \psi_f(0)| V_f^{-1}(t), \tag{103}$$

where $V_f^{-1}(t)$ is the inverse of $V_f(t)$.

For bi-linear and phase invariant boson system with the interaction

$$i\sqrt{2\kappa}\left(a^\dagger dB_t - a\, dB_t^\dagger\right),$$

$\mathcal{H}_{f,t}dt$ has the form

$$\mathcal{H}_{f,t}dt = Zdt + i\sqrt{2\kappa}\left(a^\dagger dB_t - a\, dB_t^\dagger\right), \tag{104}$$

with $Z \in \mathcal{H}_S^0 \otimes \Gamma^\beta$ being operators having the forms

$$Z = Z_S \otimes I, \tag{105}$$

where Z_S are a operator on $\mathcal{H}_S^0$ and I is the identity operator on Γ^β. dB_t, $dB_t^\dagger$ are the increment of the quantum Wiener processes at finite temperatures and κ is a positive c-number. Note that we adopt the same notations for B_t, $B_t^\dagger$ and their tilde conjugates $\tilde{B}_t$, $\tilde{B}_t^\dagger$ as (63) and (64). In the following, we will put $\bar{n}$ to the Planck distribution function (57).

Note that since the equation (102) with the infinitesimal time-evolution generator (104) is the quantum stochastic differential equation of the Ito type, the time-evolution generator $V_f(t)$ is the quantum stochastic integral of the Ito type which is an adapted process.

We require that the time-evolution generator $V_f(t)$ should be unitary, i.e.

$$V_f^\dagger(t)V_f(t) = V_f(t)V_f^\dagger(t) = 1. \tag{106}$$

Therefore, we have the algebraic identities

$$d[V_f^\dagger(t)V_f(t)] = dV_f^\dagger(t) \cdot V_f(t) + V_f^\dagger(t) \cdot dV_f(t) + dV_f^\dagger(t)dV_f(t) = 0, \tag{107}$$

and

$$d[V_f(t)V_f^\dagger(t)] = dV_f(t) \cdot V_f^\dagger(t) + V_f(t) \cdot dV_f^\dagger(t) + dV_f(t)dV_f^\dagger(t) = 0, \tag{108}$$

where we have made use of the calculus rule of the Ito type (quantum Ito's formula). The identities (107) and (108) with the equation (102) and its hermitian conjugates give the following relation

$$i(Z^\dagger - Z) + 2\kappa\left[(\bar{n}+1)a^\dagger a + \bar{n}aa^\dagger\right] = 0, \tag{109}$$

where use has been made of the product rules (52). Thus, we obtain

$$\mathcal{H}_{f,t}dt = H_S dt - i\kappa[(\bar{n}+1)a^\dagger a + \bar{n}aa^\dagger]dt + i\sqrt{2\kappa}\left(a^\dagger dB_t - adB_t^\dagger\right), \tag{110}$$

where we put $(Z+Z^\dagger)/2 = H_S$. Note that H_S is hermitian.

Applying the state vector $|\psi_f(0)\rangle\rangle$ to the equation (102) of $V_f(t)$, we have the stochastic Schrödinger equation of the Ito type

$$d|\psi_f(t)\rangle\rangle = -i\mathcal{H}_{f,t}dt|\psi_f(t)\rangle\rangle, \tag{111}$$

with the infinitesimal time-evolution generator (110).

5.2 The Stratonovich Type

Using the relation (85) between the Ito and the Stratonovich products, we transform the stochastic differential equation (102) of the Ito type into that of the Stratonovich type as

$$\begin{aligned} dV_f(t) &= -i\mathcal{H}_{f,t}dtV_f(t) \\ &= -i\left\{\mathcal{H}_{f,t}dt \circ V_f(t) - \frac{1}{2}\mathcal{H}_{f,t}dt\, dV_f(t)\right\} \\ &\equiv -iH_{f,t}dt \circ V_f(t), \end{aligned} \tag{112}$$

where we have substituted (102) into the right hand side of the second equality. Here, we defined the infinitesimal time-evolution generator $H_{f,t}$ of the Stratonovich type by

$$H_{f,t}dt \equiv \mathcal{H}_{f,t}dt + i\frac{1}{2}\mathcal{H}_{f,t}dt\mathcal{H}_{f,t}dt. \tag{113}$$

With the help of the product rules (52), we obtain the hermitian stochastic infinitesimal time-evolution generator $H_{f,t}dt$ as

$$H_{f,t}dt = H_S dt + i\sqrt{2\kappa}\left(a^\dagger dB_t - adB_t^\dagger\right). \tag{114}$$

The hermiticy of $H_{f,t}dt$ guarantees the unitarity of $V_f(t)$.

Applying the state vector $|\psi_f(0)\rangle\rangle$ to the equation (112) of $V_f(t)$, we obtain the stochastic Schrödinger equation of the Stratonovich type

$$d|\psi_f(t)\rangle\rangle = -iH_{f,t}dt \circ |\psi_f(t)\rangle\rangle, \tag{115}$$

with the infinitesimal time-evolution generator (114).

6 Stochastic Time-Evolution in Thermal Space

On the basis of the stochastic Schrödinger equation, investigated in the previous section, we will construct a stochastic Liouville equation in thermal space and obtain the explicit form of the time-evolution generator satisfying the stochastic Liouville equation within the framework of NETFD. Using the time-evolution generator, we will construct a unified canonical operator formalism of quantum stochastic differential equations.

6.1 Thermal Vacuums

Let us define the density operator $\rho_f(t)$ corresponding to the state vector $|\psi_f(t)\rangle\!\rangle$ by

$$\rho_f(t) \equiv |\psi_f(t)\rangle\!\rangle\langle\!\langle\psi_f(t)| = V_f(t)|\psi_f(0)\rangle\!\rangle\langle\!\langle\psi_f(0)|V_f^\dagger(t) = V_f(t)\rho_f(0)V_f^\dagger(t), \tag{116}$$

where use has been made of (101) and (103) with the unitary time-evolution generator $V_f(t)$. The density operator $\rho_f(t)$ satisfies

$$\mathrm{tr}_{tot}\ \rho_f(t) = 1, \tag{117}$$

where the trace operation tr_{tot} is defined by

$$\mathrm{tr}_{tot} \equiv \mathrm{tr} \otimes \mathrm{tr}_R, \tag{118}$$

with the trace operations tr of the relevant system and tr_R of the reservoir. The expectation value of any observable A is given by $\mathrm{tr}_{tot}\ A\rho_f(t)$.

With the help of the principle of correspondence (see appendix A), the density operator $\rho_f(t)$ defined by (116) is expressed as a thermal ket-vacuum, i.e.

$$|0_f(t)\rangle \equiv |\rho_f(t)\rangle = \hat{V}_f(t)|0_f(0)\rangle, \tag{119}$$

where we have defined the stochastic time-evolution generator by

$$\hat{V}_f(t) = V_f(t)\tilde{V}_f(t). \tag{120}$$

Note that, since $V_f(0) = 1$, we have $\hat{V}_f(0) = 1$. The vector space to which the thermal vacuum $|0_f(t)\rangle$ belongs is assumed to be $\mathcal{H}_S \otimes \Gamma^\beta$ where $\mathcal{H}_S$ is the space of relevant system and Γ^β is the Fock space of the system of random force operators constructed in section 3. The operator $\hat{V}_f(t)$ defined by (120) is on the space $\mathcal{H}_S \otimes \Gamma^\beta$ and turns out to be unitary from the relation

$$\hat{V}_f^\dagger(t) = V_f^\dagger(t)\tilde{V}_f^\dagger(t) = V_f^{-1}(t)\tilde{V}_f^{-1}(t) = \hat{V}_f^{-1}(t), \tag{121}$$

where use has been made of the unitarity of $V_f(t)$. Note that the space of states $\mathcal{H}_S$ of relevant system is expressed as $\mathcal{H}_S = \mathcal{H}_S^0 \otimes \tilde{\mathcal{H}}_S^0$ with the usual Hilbert space $\mathcal{H}_S^0$ for wave function and its tilde conjugate space $\tilde{\mathcal{H}}_S^0$.

The equation (117) requires that

$$\langle 1_{tot}|0_f(t)\rangle = 1, \tag{122}$$

where the thermal bra-vacuum $\langle 1_{tot}|$ is defined by

$$\langle 1_{tot}| \equiv \langle|\langle 1|, \tag{123}$$

with the thermal bra-vacuum $\langle 1|$ in the space $\mathcal{H}_S$ of the relevant system and the thermal bra-vacuum $\langle|$ in the space Γ^β of the irrelevant system. The expectation value $\mathrm{tr}_{tot}\, A\rho_f(t)$ is expressed as the expectation with respect to the thermal ket-vacuum $|0_f(t)\rangle$ and the thermal bra-vacuum $\langle 1_{tot}|$, i.e.

$$\langle 1_{tot}|A|0_f(t)\rangle = \mathrm{tr}_{tot}\, A\rho_f(t). \tag{124}$$

Note that for any relevant system operator A, we have

$$\langle 1|A^\dagger = \langle 1|\tilde{A}, \tag{125}$$

which is the basic property of thermal space [17]-[19]. Furthermore, for the random force operators dB_t, $dB_t^\dagger$, we have

$$\langle|dB_t^\dagger = \langle|d\tilde{B}_t, \tag{126}$$

which follows from (29).

The equation (122) together with (119) yields

$$\langle 1_{tot}|\hat{V}_f(t)|0_f(0)\rangle = 1. \tag{127}$$

Since the equation (127) should hold for any time t and for any initial thermal vacuum $|0_f(0)\rangle$, we have

$$\langle 1_{tot}|\hat{V}_f(t) = \langle 1_{tot}|\hat{V}_f(0) = \langle 1_{tot}|, \tag{128}$$

where we used the fact that $\hat{V}_f(0) = 1$.

6.2 Stochastic Liouville Equation

6.2.1 The Ito Type

Using the calculus rule of the Ito type, we have from (120)

$$d\hat{V}_f(t) = dV_f(t) \cdot \tilde{V}_f(t) + V_f(t) \cdot d\tilde{V}_f(t) + dV_f(t)d\tilde{V}_f(t). \tag{129}$$

Substituting (102) and its tilde conjugate:

$$d\tilde{V}_f(t) = i\tilde{\mathcal{H}}_{f,t}dt\tilde{V}_f(t), \tag{130}$$

into (129), we have

$$d\hat{V}_f(t) = -i\hat{\mathcal{H}}_{f,t}dt\hat{V}_f(t), \tag{131}$$

where

$$\hat{\mathcal{H}}_{f,t}dt \equiv \mathcal{H}_{f,t}dt - \tilde{\mathcal{H}}_{f,t}dt + i\mathcal{H}_{f,t}dt\tilde{\mathcal{H}}_{f,t}dt. \tag{132}$$

With the help of (110) and the product rules (52), $\mathcal{H}_{f,t}dt\tilde{\mathcal{H}}_{f,t}dt$ is calculated as

$$\mathcal{H}_{f,t}dt\tilde{\mathcal{H}}_{f,t}dt = 2\kappa[(\bar{n}+1)a\tilde{a} + \bar{n}a^\dagger\tilde{a}^\dagger]dt. \tag{133}$$

Putting (110) and (133) into (132), we obtain

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}_S dt + i\left(\hat{\Pi}_R + \hat{\Pi}_D\right)dt + d\hat{M}_t, \tag{134}$$

where

$$\hat{H}_S = H_S - \tilde{H}_S, \tag{135}$$

$$\hat{\Pi}_R = -\kappa\left[(a^\dagger - \tilde{a})(\mu a + \nu\tilde{a}^\dagger) + \text{t.c.}\right], \tag{136}$$

$$\hat{\Pi}_D = 2\kappa(\bar{n}+\nu)(a^\dagger - \tilde{a})(\tilde{a}^\dagger - a), \tag{137}$$

and

$$d\hat{M}_t = i\left\{\left[(a^\dagger - \tilde{a})dW_t + \text{t.c.}\right] - \left[(\mu a + \nu\tilde{a}^\dagger)dW_t^{\natural} + \text{t.c.}\right]\right\}, \tag{138}$$

with real numbers μ, ν satisfying $\mu + \nu = 1$. The operators dW_t and $dW_t^{\natural}$ are defined by

$$dW_t \equiv \sqrt{2\kappa}\left(\mu dB_t + \nu d\tilde{B}_t^\dagger\right), \quad dW_t^{\natural} \equiv \sqrt{2\kappa}\left(dB_t^\dagger - d\tilde{B}_t\right). \tag{139}$$

Making use of the relations (125) and (126), we see that (134) satisfies

$$\langle 1_{tot}|\hat{\mathcal{H}}_{f,t}dt = \langle|\langle 1|\hat{\mathcal{H}}_{f,t}dt = 0, \tag{140}$$

which is consistent with the relation (128). Note that

$$\langle 1|\hat{\mathcal{H}}_{f,t}dt \neq 0, \tag{141}$$

which indicates that the conservation of probability does not hold within only the space of states of relevant system, i.e.

$$\langle 1|0_f(t)\rangle \neq 1. \tag{142}$$

Similarly, from the definition

$$\hat{V}_f^\dagger(t) = V_f^\dagger(t)\tilde{V}_f^\dagger(t), \tag{143}$$

we obtain

$$d\hat{V}_f^\dagger(t) = i\hat{V}_f^\dagger(t)\hat{\mathcal{H}}_{f,t}^{(-1)}dt, \tag{144}$$

with

$$\hat{\mathcal{H}}_{f,t}^{(-1)}dt = \hat{H}_S dt - i\left(\hat{\Pi}_R + \hat{\Pi}_D\right)dt + d\hat{M}_t. \tag{145}$$

We see that the equations (131) with (134) and (144) with (145) satisfy

$$d\hat{V}_f^\dagger(t)\cdot\hat{V}_f(t) + \hat{V}_f^\dagger(t)\cdot d\hat{V}_f(t) + d\hat{V}_f^\dagger(t)d\hat{V}_f(t) = 0, \tag{146}$$

and

$$d\hat{V}_f(t)\cdot\hat{V}_f^\dagger(t) + \hat{V}_f(t)\cdot d\hat{V}_f^\dagger(t) + d\hat{V}_f(t)d\hat{V}_f^\dagger(t) = 0, \tag{147}$$

which are consistent of the unitarity of $\hat{V}_f(t)$.

Since $\hat{V}_f(t)$ and $\hat{V}_f^\dagger(t)$ are subject to the stochastic differential equations (131) with (134) and (144) with (145) of the Ito type, respectively, they are quantum stochastic processes consisting of quantum stochastic integrals of the Ito type. Therefore, $\hat{V}_f(t)$ and $\hat{V}_f^\dagger(t)$ are adapted processes.

Applying the state vector the thermal vacuum $|0_f(0)\rangle$ to the equation (131) of $\hat{V}_f(t)$, we obtain the quantum stochastic Liouville equation of the Ito type

$$d|0_f(t)\rangle = -i\hat{\mathcal{H}}_{f,t}dt|0_f(t)\rangle, \tag{148}$$

with the infinitesimal time-evolution generator (134).

6.2.2 The Stratonovich Type

Using the calculus rule of the Stratonovich type, we have from (120)

$$d\hat{V}_f(t) = dV_f(t) \circ \tilde{V}_f(t) + V_f(t) \circ d\tilde{V}_f(t). \tag{149}$$

Substituting (112) and its tilde conjugate

$$d\tilde{V}_f(t) = i\tilde{H}_{f,t}dt \circ \tilde{V}_f(t), \tag{150}$$

into (149), we obtain

$$d\hat{V}_f(t) = -i\hat{H}_{f,t}dt \circ \hat{V}_f(t), \tag{151}$$

where

$$\hat{H}_{f,t}dt \equiv H_{f,t}dt - \tilde{H}_{f,t}dt. \tag{152}$$

Putting (114) into (152), we get

$$\hat{H}_{f,t}dt = \hat{H}_S dt + d\hat{M}_t, \tag{153}$$

which is apparently hermitian.

Taking the hermitian conjugate of the equation (151), we have the equation of $\hat{V}_f^\dagger(t)$ as

$$d\hat{V}_f^\dagger(t) = i\hat{V}_f^\dagger(t) \circ \hat{H}_{f,t}dt, \tag{154}$$

where use has been made of the hermiticy of $\hat{H}_{f,t}dt$.

The equations (151) and (154) with (153) satisfy the following equations

$$d\hat{V}_f^\dagger(t) \circ \hat{V}_f(t) + \hat{V}_f^\dagger(t) \circ d\hat{V}_f(t) = 0, \tag{155}$$

$$d\hat{V}_f(t) \circ \hat{V}_f^\dagger(t) + \hat{V}_f(t) \circ d\hat{V}_f^\dagger(t) = 0, \tag{156}$$

which show the unitarity of $\hat{V}_f(t)$.

With the help of the properties (125) and (126), we see that the expression (153) satisfies

$$\langle 1_{tot}|\hat{H}_{f,t}dt = \langle|\langle 1|\hat{H}_{f,t}dt = 0, \tag{157}$$

which is consistent with (128).

The time-evolution equation (131) of the Ito type with (134) is connected to the equation (151) of the Stratonovich type with (153) by the relation (85) between the Ito and the Stratonovich products. In the same way, the equation (144) of the Ito type with

(145) is connected to the equation (154) of the Stratonovich type with (153) through the relation (84).

Applying the state vector the thermal vacuum $|0_f(0)\rangle$ to the equation (151) of $\hat{V}_f(t)$, we have the quantum stochastic Liouville equation of the Stratonovich type

$$d|0_f(t)\rangle = -i\hat{H}_{f,t}dt \circ |0_f(t)\rangle, \tag{158}$$

with the infinitesimal time-evolution generator (153).

6.3 Fokker-Planck Equation

Applying the random force bra-vacuum $\langle|$ to the stochastic Liouville equation (148) of the Ito type with the infinitesimal time-evolution generator (134), we have

$$\begin{aligned} d\langle|0_f(t)\rangle &= -i\langle|\hat{\mathcal{H}}_{f,t}dt|0_f(t)\rangle \\ &= -i\left\{\hat{H}dt\langle|0_f(t)\rangle + \langle|d\hat{M}_t|0_f(t)\rangle\right\}. \end{aligned} \tag{159}$$

Under the assumption

$$|0_f(0)\rangle = |0_S\rangle|\rangle, \tag{160}$$

with the thermal vacuum $|0_S\rangle$ of relevant system at $t=0$, $\langle|d\hat{M}_t|0_f(t)\rangle$ can be evaluated as

$$\langle|d\hat{M}_t|0_f(t)\rangle = 0, \tag{161}$$

where we used the definition (119) of the thermal vacuum $|0_f(t)\rangle$ and the property (79) of the products of the Ito type. Therefore, putting $|0(t)\rangle = \langle|0_f(t)\rangle$, we finally obtain the Fokker-Planck equation for bi-linear and phase invariant system as

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{162}$$

where the infinitesimal time-evolution generator $\hat{H}$ is given by

$$\hat{H} = \hat{H}_S + i\hat{\Pi}, \tag{163}$$

with

$$\begin{aligned} \hat{\Pi} &= \hat{\Pi}_R + \hat{\Pi}_D \\ &= -\kappa\left[(1+2\bar{n})\left(a^\dagger a + \tilde{a}^\dagger\tilde{a}\right) - 2(1+\bar{n})\,a\tilde{a} - 2\bar{n}a^\dagger\tilde{a}^\dagger\right] - 2\kappa\bar{n}, \end{aligned} \tag{164}$$

which is identical to that obtained within the framework of NETFD [17]-[19]. Note that we can also derive the Fokker-Planck equation (162) by applying the random force bra-vacuum $\langle|$ to the stochastic Liouville equation (158) of the Stratonovich type with the infinitesimal time-evolution generator (153) [17, 20].

Recalling the equation (119) and taking (160) into account, we find that

$$|0(t)\rangle = \langle|\hat{V}_f(t)|\rangle|0_S\rangle. \tag{165}$$

On the other hand, the time-evolution generator $\hat{V}(t)$ of the thermal ket-vacuum $|0(t)\rangle$ is defined by

$$|0(t)\rangle = \hat{V}(t)|0(0)\rangle. \tag{166}$$

Provided that $|0(0)\rangle = |0_S\rangle$, the equations (165) and (166) yield

$$\hat{V}(t) = \langle|\hat{V}_f(t)|\rangle. \tag{167}$$

6.4 Quantum Langevin Equation

Since $\hat{V}_f(t)$ is unitary, we have

$$\hat{V}_f^\dagger(t)\hat{V}_f(t) = \hat{V}_f(t)\hat{V}_f^\dagger(t) = 1. \tag{168}$$

From the equation (168), we see with the help of the property (128) that

$$\langle 1_{tot}|\hat{V}_f^\dagger(t) = \langle 1_{tot}|. \tag{169}$$

The expectation value of any observable A with respect to the state $|0_f(t)\rangle$ is given by

$$\begin{aligned}\langle 1_{tot}|A|0_f(t)\rangle &= \langle 1_{tot}|A\hat{V}_f(t)|0_f(0)\rangle \\ &= \langle 1_{tot}|\hat{V}_f^\dagger(t)A\hat{V}_f(t)|0_f(0)\rangle,\end{aligned} \tag{170}$$

where we have used the equation (119) and the property (169). If we define the operator in the Heisenberg representation

$$A(t) = \hat{V}_f^\dagger(t)A\hat{V}_f(t), \tag{171}$$

we consider (170) to be the expectation value of $A(t)$ with respect to the initial state $|0_f(0)\rangle$. Note that, as $\hat{V}_f(t)$ and $\hat{V}_f^\dagger(t)$ are adapted processes, the operator $A(t)$ defined by (171) is also an adapted process. Therefore, the following commutation relation holds:

$$[A(t),\ dB_t] = [A(t),\ dB_t^\dagger] = [A(t),\ d\tilde{B}_t] = [A(t),\ d\tilde{B}_t^\dagger] = 0, \tag{172}$$

for quantum Wiener processes B_t, $B_t^\dagger$ and their tilde conjugates $\tilde{B}_t$, $\tilde{B}_t^\dagger$, which comes from (76).

Any operators in the Heisenberg representation defined by (171) keep the equal-time commutation relations, such as

$$[a(t),\ a^\dagger(t)] = 1, \quad [\tilde{a}(t),\ \tilde{a}^\dagger(t)] = 1. \tag{173}$$

Note that, using the properties (125) and (169), we have for $A(t)$ defined by (171)

$$\langle 1_{tot}|A^\dagger(t) = \langle 1_{tot}|\tilde{A}(t). \tag{174}$$

Using the calculus rule of the Ito type, we have the algebraic identity for the operator $A(t)$ defined by (171)

$$dA(t) = d\hat{V}_f^\dagger(t)A\hat{V}_f(t) + \hat{V}_f^\dagger(t)Ad\hat{V}_f(t) + d\hat{V}_f^\dagger(t)Ad\hat{V}_f(t). \tag{175}$$

Substituting the equations (131) with (134) and (144) with (145) into (175), we obtain the quantum Langevin equation of the Ito type.

On the other hand, making use of the calculus rule of the Stratonovich type, we have the algebraic identity for the operator $A(t)$ defined by (171)

$$dA(t) = d\hat{V}_f^\dagger(t) \circ A\hat{V}_f(t) + \hat{V}_f^\dagger(t)A \circ d\hat{V}_f(t). \tag{176}$$

With the help of the equation (151) and its hermitian conjugate (154) together with the identity (176), we obtain the quantum Langevin equation of the Stratonovich type. We see that the quantum Langevin equation of the Stratonovich type can be expressed as the Heisenberg equation for $A(t)$:

$$dA(t) = i\left[\hat{H}_f(t)dt \,\text{\textdegree}\!\!,\; A(t)\right], \tag{177}$$

where we defined

$$\hat{H}_f(t)dt = \hat{V}_f^\dagger(t) \circ \hat{H}_{f,t}dt \circ \hat{V}_f(t). \tag{178}$$

The symbol $[\cdot \,\text{\textdegree}\!\!,\; \cdot]$ is the commutator defined by (80).

6.5 Equation of Motion of Expectation Value

Let us assume that the initial vacuum $|0_f(0)\rangle$ can be expressed by the product of the vacuums of the relevant and the irrelevant systems as

$$|0_f(0)\rangle \equiv |0_f\rangle = |0_S\rangle|\rangle, \tag{179}$$

where $|0_S\rangle \in \mathcal{H}_S$ is the thermal ket-vacuum of the relevant system at time $t = 0$.

Applying the bra-vacuum $\langle 1_{tot}|$ to the quantum Langevin equation of the Ito type, we have

$$\begin{aligned} d\langle 1_{tot}|A(t) = & \, i\langle 1_{tot}|\,[H_S(t),\ A(t)]\,dt \\ & +\kappa\left(\langle 1_{tot}|a^\dagger(t)\,[A(t),\ a(t)] + \langle 1_{tot}|\left[a^\dagger(t),\ A(t)\right]a(t)\right)dt \\ & +2\kappa\bar{n}\langle 1_{tot}|\left[a^\dagger(t),\ [A(t),\ a(t)]\right]dt \\ & +\langle 1_{tot}|\left[A(t),\ a^\dagger(t)\right]dF_t + \langle 1_{tot}|\,[a(t),\ A(t)]\,dF_t^\dagger, \end{aligned} \tag{180}$$

where we used the properties (126) and (174).

Putting the ket-vacuum $|0_f(0)\rangle \equiv |0_f\rangle$ to (180), we obtain the equation of motion of the expectation value of an arbitrary operator A of the relevant system as

$$\begin{aligned} \frac{d}{dt}\langle 1_{tot}|A(t)|0_f\rangle = & \, i\langle 1_{tot}|\,[H_S(t),\ A(t)]\,|0_f\rangle \\ & +\kappa\left(\langle 1_{tot}|a^\dagger(t)\,[A(t),\ a(t)]\,|0_f\rangle + \langle 1_{tot}|\left[a^\dagger(t),\ A(t)\right]a(t)|0_f\rangle\right) \\ & +2\kappa\bar{n}\langle 1_{tot}|\left[a^\dagger(t),\ [A(t),\ a(t)]\right]|0_f\rangle. \end{aligned} \tag{181}$$

Here, we used the property (79) of the Ito products.

Remembering (167) and the definition (171) of $A(t)$, we find with the assumption (160) that

$$\langle 1_{tot}|A(t)|0_f\rangle = \langle|\langle 1|\hat{V}_f^\dagger(t)A\hat{V}_f(t)|0_S\rangle|\rangle = \langle 1|A|0(t)\rangle, \tag{182}$$

where we have used the property (169) and the assumption that $|0(0)\rangle = |0_S\rangle$. Taking account of the relation (182), we see that the equation (181) of expectation value is identical to the equation derived from the Fokker-Planck equation (162) with (163), (164), which shows the consistency of the framework.

7 Summary and Discussion

In this paper, we constructed the quantum Wiener processes together with their representation space by extending the work of mathematicians and by implanting it into NETFD. Then, we constructed a unified system of quantum stochastic differential equations on the basis of the stochastic Schrödinger equation which was studied by mathematicians.

The quantum Wiener processes, which we employed as random force operators, are constructed by using boson operators with time indices together with their representation space. When we adopted the Fock space Γ^0 for the representation space, in the same way as Hudson and Parthasarathy, we obtained the quantum Wiener processes at zero temperature (section 2). Whereas, we obtained the quantum Wiener processes at finite temperatures by extending the representation space to the Fock space Γ^β which is obtained by the Bogoliubov transformation in the tensor product space $\Gamma = \Gamma^0 \otimes \tilde{\Gamma}^0$. There, the thermal degree of freedom was introduced by the thermal state conditions or the Bogoliubov transformation, which is a manifestation of the unitary inequivalence between the thermal vacuums of zero and finite temperatures (section 3). This notion of the unitary inequivalence between the vacuums with different temperatures is one of the remarkable features within NETFD or TFD. The quantum Wiener processes and the quantum stochastic calculus given in this paper provide the foundation for those used in quantum optics [3] and quantum stochastic differential equations within NETFD [20]-[30].

We constructed the stochastic Schrödinger equation with the quantum Wiener processes at finite temperatures on the requirement that the time-evolution generator should be unitary (section 5). Then, we introduced the density operator corresponding to the stochastic wave function. By means of the principle of correspondence between quantities in the thermal space and in the Hilbert space, we obtained the stochastic thermal ket-vacuum corresponding to the density operator and a stochastic Liouville equation. On the basis of the time-evolution of the thermal ket-vacuum, we constructed the system of quantum stochastic differential equations within NETFD (section 6).

The time-evolution equation of the thermal ket-vacuum gave the quantum stochastic Liouville equation. On the other hand, the Heisenberg equation with the infinitesimal time-evolution generator of the quantum stochastic Liouville equation gave the quantum Langevin equation. Using the quantum stochastic calculus constructed in section 4, we constructed the quantum stochastic differential equations both of the Ito and of the Stratonovich types.

Applying the random force bra-vacuum $\langle |$ to the stochastic Liouville equation of the Ito type, we obtained the Fokker-Planck equation, which is identical to that derived in the papers [17]-[19]

Taking the expectation with respect to the thermal ket-vacuum $|0_f\rangle$ and the thermal bra-vacuum $\langle 1_{tot}|$ of the quantum Langevin equation of the Ito type, we obtained the equation of motion of expectation value of an arbitrary relevant system operator. This equation of motion is equivalent to that derived by the Fokker-Planck equation, which shows the self-consistency of the system.

Hudson and Lindsay constructed a unitary stochastic time-evolution in the vector space $\mathcal{H}_S^0 \otimes \Gamma^\beta$, where $\mathcal{H}_S^0$ for relevant system is a usual Hilbert space and Γ^β for random force operators is a Fock space in thermal space, which was briefly reviewed in section 5. The fact that the vector space for relevant system is not a thermal space prevents the system from introducing the quantum stochastic Liouville equation. In this paper, we completed the motivation of Hudson and Lindsay by adopting a thermal space for the space of states of relevant system as well as random force system and giving the quantum stochastic Liouville equation.

The stochastic time-evolution constructed in this paper is unitary. On the other hand, non-unitary stochastic time-evolution was constructed within the framework of NETFD [20]-[30]. In this way, it turned out that there exist two kinds of systems in quantum stochastic differential equations within NETFD, one of them is the system of non-unitary stochastic time-evolution and the other is that of unitary stochastic time-evolution. The two systems are equivalent in the sense that they give the same equation of motion of expectation value of any observable. The relation between the two systems will be investigated in the forthcoming paper.

A The Principle of Correspondence

The correspondence between vectors in the thermal space and operators in a Hilbert space is given by the following rule [37, 38, 39]:

$$\rho_S(t) \quad \longleftrightarrow \quad |0(t)\rangle, \tag{183}$$

$$A_1 \rho_S(t) A_2 \quad \longleftrightarrow \quad A_1 \tilde{A}_2^{\dagger} |0(t)\rangle. \tag{184}$$

Here, $\rho_S(t)$ is a density operator on the Hilbert space, whereas $|0(t)\rangle$ is a thermal ket-vacuum in the thermal space. A_1, A_2 are arbitrary operators on the Hilbert space.

It was noticed first by Crawford [40] that the introduction of two kinds of operators for each operator enables us to handle the Liouville equation as the Schrödinger equation.

References

[1] I. R. Senitzky, Phys. Rev. **119** (1960) 670.

[2] M. Lax, Phys. Rev. **145** (1966) 110.

[3] H. Haken, *Optics. Handbuch der Physik* vol. XXV/2c (1970), [*Laser Theory* (Springer, Berlin, 1984)], and the references therein.

[4] R. Kubo, J. Phys. Soc. Japan **26** Suppl. (1969) 1.

[5] R. L. Hudson and K. R. Parthasarathy, Commun. Math. Phys. **93** (1984) 301.

[6] R. L. Hudson and K. R. Parthasarathy, Acta Appl. Math. **2** (1984) 353.

[7] R. L. Hudson and J. M. Lindsay, in *Quantum Probability and Applications II*, eds. L. Accardi and W. von Waldenfels, Lecture Notes in Mathematics **1136** (Springer-Verlag, 1984) 276.

[8] R. L. Hudson and J. M. Lindsay, J. Funct. Anal. **61** (1985) 202.

[9] K. R. Parthasarathy, Rev. Math. Phys. **1** (1989) 89.

[10] K. R. Parthasarathy, Monographs in Mathematics Vol. 85, *An Introduction to Quantum Stochastic Calculus* (Birkhäuser Verlag, 1992).

[11] R. Kubo, J. Math. Phys. **4** (1963) 174.

[12] R. Kubo, M. Toda and N. Hashitsume, *Statistical Physics II* (Springer, Berlin 1985).

[13] A. S. Parkins and C. W. Gardiner, Phys. Rev. **A37** (1988) 3867.

[14] C. W. Gardiner, *Quantum Noise* Springer Series in Synergetics **56** (Springer-Verlag, 1991).

[15] H. Dekker, Physica **A175** (1991) 485.

[16] C. W. Gardiner, A. S. Parkins and P. Zoller, Phys. Rev. **A46** (1992) 4363.

[17] T. Arimitsu, in *Thermal Field Theories*, eds. H. Ezawa, T. Arimitsu and Y. Hashimoto (North-Holland, 1991) 207.

[18] T. Arimitsu, Lecture Note of the *Summer School for Younger Physicists in Condensed Matter Physics* [published in "Bussei Kenkyu" (Kyoto) **60** (1993) 491-526, written in English], and the references therein.

[19] T. Arimitsu, to be published in *Condensed Matter Physics*, a scientific journal in Ukraine.

[20] T. Arimitsu, Phys. Lett. **A153** (1991) 163.

[21] T. Saito and T. Arimitsu, Modern Phys. Lett. B **6** (1992) 1319.

[22] T. Arimitsu and T. Saito, Bussei Kenkyu **59-2** (1992) 213, in Japanese.

[23] T. Arimitsu and T. Saito, *A Unified Framework of Quantum Stochastic Differential Equations*, in Proceedings of the Conference on Field Theory and Collective Phenomena (1993) in press.

[24] T. Arimitsu and T. Saito, Vistas in Astronomy **37** (1993) 99.

[25] T. Arimitsu, M. Ban and T. Saito, Physica **A177** (1991) 329.

[26] T. Arimitsu, M. Ban and T. Saito, in *Structure: from Physics to General Systems*, eds. M. Marinaro and G. Scarpetta (World Scientific, 1991) 163.

[27] T. Saito and T. Arimitsu, Modern Phys. Lett. B **7** (1993) 623.

[28] T. Saito and T. Arimitsu, Modern Phys. Lett. B **7** (1993) 1951.

[29] T. Arimitsu and T. Saito, *Quantum Stochastic Differential Equations in Phase-Space Methods*, Mod. Phys. Lett. B (1994) submitted.

[30] T. Arimitsu and T. Saito, *General Structure of the Time-Evolution Generator for Quantum Stochastic Liouville Equation* (1995) in preparation to submit.

[31] H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North-Holland 1982).

[32] M. Ban, J. Math. Phys. **33** (1992) 3213.

[33] R. M. Wilcox, J. Math. Phys. **8** (1967) 962.

[34] W. Witschel, J. Phys. **A7** (1974) 1847.

[35] M. A. Santiago and A. N. Vaidya, J. Phys. **A9** (1976) 897.

[36] L. Accardi, Rev. Math. Phys. **2** (1990) 127.

[37] M. Schmutz, Z. Phys. **B 30** (1978) 97.

[38] T. Arimitsu and H. Umezawa, Prog. Theor. Phys. **74** (1985) 429.

[39] T. Arimitsu and H. Umezawa, Prog. Theor. Phys. **77** (1987) 32.

[40] J. A. Crawford, Nuovo Cim. **10** (1958) 698.