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Simultaneous Approximations and Dynamical Systems

(On the simultaneous approximation of (α,α^2) satisfying $\alpha^3+k\alpha-1=0$)

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For each k $\in \boldsymbol{N}$, let us consider the following matrix A_k and its characteristic polynomial $\Phi_k(x)$ of A_k in this talk:

$$A_k=\begin{pmatrix} k & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$\Phi_k(x)=x^3-kx^2-1.$$

The positive solution λ of $\Phi_k(x)=0$ has the following properties:

(1) λ is a complex Pisot number, that is,

$$\lambda>1>|\lambda'|=|\lambda''|$$

where λ', λ'' are algebraic conjugates of λ and moreover the number λ satisfies

$$k+1>\lambda>k,$$

(2) put $\alpha=\frac{1}{\lambda}$, then the column eigen vector of A_k is given by

$$A_k\begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix}=\lambda\begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix},$$

$<1,\alpha,\alpha^2>$ is a basis of the cubic field Q(α) and discriminant of α is given by $d_k=-4k^3-27$,

(3) the pair of rational numbers $\left(\frac{p_n}{q_n},\frac{r_n}{q_n}\right)$ given by

$$A_k^n=\begin{pmatrix} q_n & q_{n-2} & q_{n-1} \\ p_n & p_{n-2} & p_{n-1} \\ r_n & r_{n-2} & r_{n-1} \end{pmatrix}$$

give a simultaneous approximation of (α,α^2). Moreover, there exists $C_k>0$ such that the inequality

$$max\left(\left|\alpha-\frac{p_n}{q_n}\right|,\left|\alpha^2-\frac{r_n}{q_n}\right|\right)>\frac{C_k-\epsilon}{q_n{}^{\frac{3}{2}}}\quad for\ any\ \epsilon>0$$

holds for any n. (We know that the point (α, α^2) is a purely periodic point with period 1 by *Modified Jacobi-Perron algorithm* [2].)

The aim of this talk is to claim the following theorem.

Theorem 1 *For each $k \in \boldsymbol{N}$, put the sets of points J_k and L_k:*

$$J_k := \left\{\left(\sqrt{q_n}(q_n\alpha - p_n), \sqrt{q_n}(q_n\alpha^2 - r_n)\right) \mid n = 1, 2, \cdots\right\},$$

$$L_k := \left\{\left(\sqrt{q}(q\alpha - p), \sqrt{q}(q\alpha^2 - r)\right) \mid (q, p, r) \in \boldsymbol{Z}^3, q > 0\right\}.$$

Then there exists a domain D_k exactly, which is the interior of an ellipse (the explicit form of the ellipse is found in [1]), such that

(1) the limit set of J_k is equal to the ellipse ∂D_k,

(2) the limit set of $L_k \backslash J_k$ is included in the complement of $\overline{D_k}$,

(3) the volume of D_k is equal to $\frac{2\pi}{\sqrt{|d_k|}}$, where d_k is the discriminant of α given by $d_k = -4k^3 - 27$.

As a collorary, we have

Corollary 1 *For each $k \in \boldsymbol{N}$, let C_k be the infimum of $C > 0$ such that*

$$max\sqrt{q}\ (\ |\ q\alpha - p\ |,\ |\ q\alpha^2 - r\ |\ ) < C$$

has infinitely sulutions (q,p,r). Then C_k is given explicitly by

$$C_k{}^2 = \frac{l(k^2\lambda^2 + 3\lambda + k)}{(k^4 + k^3 + 5k^2 + 4k + 3)\lambda^2 + (k^2 + k + 6)\lambda + (k^3 + k^2 + 5k + 3)},$$

where $l = \frac{\lambda^3}{\lambda^3 + 2}$.

We note that the constant C_k satisfies the relation:

$$C_k{}^2 < \frac{1}{\sqrt{|d_k|}}.$$

The proof of the theorem is obtained by using the substitution like as the Diophantine approximation algorithm. In fact, for each k $\in \boldsymbol{N}$ let us introduce the substitution σ_k on $W^* = \cup_{n=1}^{\infty}\{1, 2, 3\}^n$:

$$\sigma_k : \begin{array}{ccc} 1 & \rightarrow & \overbrace{1 \cdots 1}^{k}2 \\ 2 & \rightarrow & 3 \\ 3 & \rightarrow & 1 \end{array}.$$

Then the abelianization of σ_k is given by A_k, that is, the following commutative relation holds:

$$\begin{array}{ccc} W^* & \xrightarrow{\sigma_k} & W^* \\ \downarrow f & & \downarrow f \\ \boldsymbol{Z}^3 & \xrightarrow{A_k} & \boldsymbol{Z}^3 \end{array}$$

where $f : W^* \to W^*$ be the canonical homomorphism.
Let us denote the fixed point of σ_k by

$$w_k = (w(1), w(2), \cdots, w(l), \cdots)$$

and consider the lattice points:

$$S_k = \left\{ \sum_{i=1}^{l} f_{w(i)} \mid l = 1, 2, \cdots \right\}.$$

Then we can see that the set of the lattice points S_k is enough to consider the appoximation points of (α, α^2), that is, we have the following key lemma (See figure).

Lemma 1 *Let π be the projection to y-z plane along $(1, \alpha, \alpha^2)$. Then the domain X_k with fractal boundary given by*

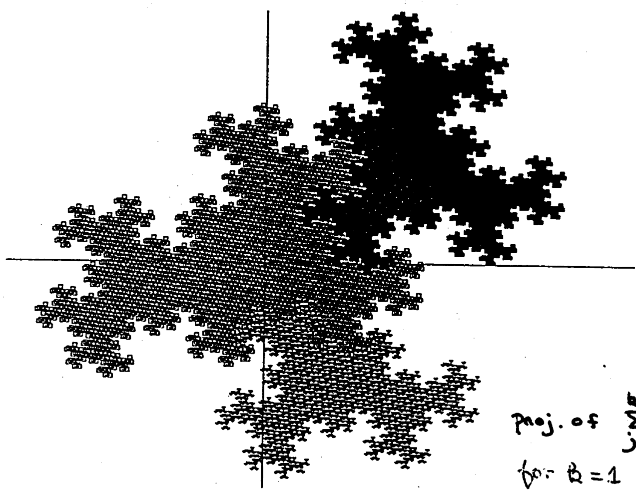
$$X_k = \overline{\pi S_k}$$

satisfies the following property:

(1) $0 \in$ interior of X_k

(2) For any lattice point (q,p,r) the projection point $(p\alpha - q, p\alpha^2 - r)$ $(= \pi(p, q, r))$ is the outside of X_k if $(q,p,r) \notin S_k$.

We see also that the points $\sqrt{q_n}(q_n\alpha - p_n, q_n\alpha^2 - r_n)$, $n = 1, 2, 3, \cdots$ are the nearest points in $\left(\sqrt{q}(q\alpha - p), \sqrt{q}(q\alpha^2 - p)\right)$, $(q, p, r) \in S_k$ from the origin point. Therefore we have the theorem.



proj. of $\sum_{j=1}^{k} e_{s_j}$, $k = 1, 2, \cdots, l_n$ $(n = 23)$
for $k = 1$

References

[1] J. Fujii, H. Higashino and Sh. Ito, On the simultaneous approximation of (α, α^2) satisfying $\alpha^3 + k\alpha - 1 = 0$ (preprint).

[2] Sh. Ito and M. Ohtsuki, Modified Jacobi-Pirron Algorithm and Generating Markov Partitions for Special Hyperbolic Toral Automorphisms, Tokyo J. Math., **16** (1993), 441-472.