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The Remainder Term in the Dirichlet Divisor Problem

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1. Introduction

Let $d(n)$ denote the divisor function. We shall use $c, c', c_1, c_2, \cdots$ etc to denote certain constants which need not be the same at each occurrence. In this talk we shall give a survey on some recent results concerning the well-known remainder term

$$\Delta(x) := \sum_{n\leq x} d(n) - x(\log x + 2\gamma - 1), \qquad x \geq 2,$$

which occurs frequently in analytic number theory. This can also be interpreted as a lattice point problem since $\sum_{n\leq x} d(n)$ counts the number of lattice points in the first quadrant bounded by the hyperbola $uv = x$.

The first result on $\Delta(x)$ was obtained more than one and a half century ago by Dirichlet, who proved by an elementary argument that $\Delta(x) \ll \sqrt{x}$. This upper bound was successively improved upon by many authors and the best result to date is : $\Delta(x) \ll x^{\frac{7}{22}+\varepsilon}$ *for any* $\varepsilon > 0$, due to Iwaniec and Mozzochi [9]. It has been widely conjectured that $\Delta(x) \ll_\varepsilon x^{1/4+\varepsilon}$ is true for any $\varepsilon > 0$.

2. Values of $\Delta(x)$

Figure 1 below shows the graphs of $y = \Delta(x)$ for four different ranges of x. An immediate observation was that $\Delta(x)$ is highly oscillatory, it takes large values in both the positive and negative sides and yet it is slightly skewed towards the positive. Indeed, Voronoi [18] has proved in 1904 that

$$\int_2^X \Delta(x)dx = \frac{1}{4}X + \mathcal{O}(X^{3/4}), \tag{1}$$

that is, $\Delta(x)$ *has* $1/4$ *as mean value.* Concerning the large values of $\Delta(x)$, Hardy [3] showed earlier this century that

$$\Delta(x) = \begin{cases} \Omega_+\left((x\log x)^{\frac{1}{4}}\log\log x\right), \\ \Omega_-(x^{\frac{1}{4}}) \end{cases}$$

The best results in this direction to date are

$$\Delta(x) = \Omega_- \left\{ x^{\frac{1}{4}} \exp(c(\log\log x)^{\frac{1}{4}} (\log\log\log x)^{-\frac{3}{4}}) \right\}$$

and

$$\Delta(x) = \Omega_+ \left\{ (x \log x)^{\frac{1}{4}} (\log\log x)^{\frac{1}{4}(3+\log 4)} \exp(-c(\log\log\log x)^{\frac{1}{2}}) \right\}$$

for some constant $c > 0$, due to Corrádi - Kátai [1] and Selberg - Hafner [2] respectively. These Ω-results, however, do not localize the occurrence of the extreme values of $\Delta(x)$. There is an earlier result of Tong [12] which says that :

There exist positive constants c *and* c' *such that, for any* $X \geq 1$ *and for any* $t \in [-cX^{1/4}, cX^{1/4}]$ *the equation* $\Delta(x) = t$ *always has a solution* x *in the interval* $[X, X + c'\sqrt{X}]$. *In particular,* $\Delta(x)$ *changes signs in* $[X, X + c'\sqrt{X}]$ *for every* $X \geq 1$, *that is, the gap between the zeros of* $\Delta(x)$ *is* $\mathcal{O}(\sqrt{x})$.

Basing upon some numerical evidence, Ivić and te Riele [8] conjectured that $\Delta(x)$ *changes signs in every interval* $[X, X + c_\varepsilon X^{1/4+\varepsilon}]$ *for any* $\varepsilon > 0, X \geq X_0(\varepsilon)$ *and* c_ε *is a constant dependent on* ε. This conjecture, however, was shown to be too strong by the following result.

Heath-Brown and Tsang [5] : *There exist positive constants* c, c_1, c_2 *such that, for any sufficiently large* X, *there are more than* $c_1\sqrt{X}\log^5 X$ *disjoint subintervals of length* $c_2\sqrt{X}\log^{-5} X$ *in* $[X, 2X]$, *throughout each of which either* $\Delta(x) > cX^{\frac{1}{4}}$ *or* $\Delta(x) < -cX^{\frac{1}{4}}$ *holds. In particular* $\Delta(x)$ *does not change signs in each of these subintervals.*

The graph of $y = \Delta(x)$ oscillates rigorously above and below the x-axis. Apparently there is no simple way to describe the values of $\Delta(x)$. However Heath-Brown [4] has shown that $\Delta(x)$ possesses a distribution function in the following sense.

There is a smooth function $f(x)$ *such that, for any interval* I, *we have*

$$X^{-1} \text{ meas } \{x \in [2, X] : x^{-1/4}\Delta(x) \in I\} \to \int_I f(\alpha) d\alpha$$

as $X \to \infty$.

3. Mean square of $\Delta(x)$

When considered in the mean, the remainder term $\Delta(x)$ exhibits much better regularity. Voronoi's formula (1) shows that $\Delta(x)$ has an asymptotic mean value of $1/4$ over intervals of length $\gg X^{3/4}$. For the mean square, we have the following formula of Tong [13]:

$$\int_2^X \Delta(x)^2 dx = cX^{3/2} + F(X),$$

where $c = (6\pi^2)^{-1}\sum_{n=1}^{\infty} d(n)^2 n^{-3/2} = 0.6542869\cdots$ *and* $F(X) \ll X\log^5 X$. Thus, $\Delta(x)^2$ has asymptotic mean value of $\frac{3}{2}c\sqrt{x}$ over intervals of length $\gg \sqrt{X}\log^5 X$. After more than thirty years have elapsed, this was then sharpened slightly by Preissmann [11] to $F(X) \ll X\log^4 X$, by using a variant of Hilbert's inequality. (Motohashi and others observed that the same can be obtained via an estimate for the sum $\sum_{m\le x} d(m)d(m+h)$.)

There is not much information on the true order of $F(X)$. Ivić [6, Theorem 3.8] observed that

$$F(X) \ll U(X) \Rightarrow \Delta(x) \ll (U(x)\log x)^{1/3}. \tag{2}$$

Consequently, in view of the Ω-results above, Ivić- Ouellet [7] showed that

$$F(x) = \Omega(x^{\frac{3}{4}}(\log x)^{-\frac{1}{4}}(\log\log x)^{\frac{3}{4}(3+\log 4)}e^{-c(\log\log\log x)^{1/2}}),$$

and it was even conjectured that $F(x) \ll_\varepsilon x^{3/4+\varepsilon}$ *is true for any* $\varepsilon > 0$. This conjecture is very strong, since by Ivić's argument in (2), it implies the long standing conjecture that $\Delta(x) \ll_\varepsilon x^{1/4+\varepsilon}$. Ivić's conjecture is indeed too optimistic. Recently Tsang [15] deduced from the lower estimate :

$$\int_X^{2X} (F(x+\sqrt{X}) - F(x))^2 dx \gg X^3 \tag{3}$$

that $F(x) = \Omega(x)$. Later, this was further sharpened by Lau - Tsang [17] to $F(x) = \Omega_-(x\log^2 x)$, which is an immediate consequence of the asymptotic formula

$$\int_2^X F(x)dx = -(8\pi^2)^{-1}X^2\log^2 X + cX^2\log X + \mathcal{O}(X^2).$$

This asymptotic formula can be reformulated as

$$\int_2^X (F(x) + (4\pi^2)^{-1}x\log^2 x - \kappa x\log x)dx \ll X^2$$

for a suitable constant κ. This leads us to the following

Conjecture :

$$F(x) = -(4\pi^2)^{-1} x \log^2 x + \kappa x \log x + \mathcal{O}(x), \tag{4}$$

that is,

$$\int_2^X \Delta(x)^2 dx = cX^{3/2} - (4\pi^2)^{-1} X \log^2 X + \kappa X \log X + \mathcal{O}(X). \tag{5}$$

This conjecture, if true, would imply that $\Delta(x)^2$ has asymptotic mean value over intervals of length $\gg \sqrt{X}$. The Ω-result for $F(x)$ explains adequately the level of difficulty one faces in improving the upper bound for $F(x)$. The gap now left behind between the upper and lower bounds for $F(x)$, though small, seems very difficult to close.

Concerning the formula (4), one may naturally ask whether the $\mathcal{O}(x)$ term contains some other main terms. Clearly

$$\begin{aligned}\int_X^{2X} (F(x+\sqrt{X}) - F(x))dx &= \int_{2X}^{2X+\sqrt{X}} F(x)dx - \int_X^{X+\sqrt{X}} F(x)dx \\ &\ll X^{3/2} \log^4 X,\end{aligned} \tag{6}$$

by applying Preissmann's upper bound for $F(x)$. We see that $F(x+\sqrt{X}) - F(x)$ must change signs, for otherwise, by (6)

$$\int_X^{2X} (F(x+\sqrt{X})-F(x))^2 dx \ll X \log^4 X \int_X^{2X} |F(x+\sqrt{X})-F(x)| dx \ll X^{5/2} \log^8 X$$

which contradicts (3). Hence the term $\mathcal{O}(x)$ in (4) is oscillatory and cannot be $o(x)$.

In support of our conjecture (4), we prove recently that :

Tsang [16]. *For some positive constant* c, *we have*

$$\int_2^X |F(x) + (4\pi^2)^{-1} x \log^2 x - \kappa x \log x|^r dx \ll (cr)^{4r} X^{r+1}$$

for any $r \geq 1$. *Consequently, if* $H(x)$ *is any increasing function satisfying* $2 \leq H(x) \leq \log^4 x$, *we have*

$$|F(x) + (4\pi^2)^{-1} x \log x - \kappa x \log x| \leq x H(x)$$

for all but $\mathcal{O}(X e^{-cH(X)^{1/4}})$ *values of* x *in* $[2, X]$.

So at least our conjecture (4) is true for almost all x. Even though the constant κ is effectively computable, it is difficult to obtain an accurate numerical value for it. Very roughly, we estimated $\kappa \doteqdot 0.32$ and Figure 2 below shows the graphs of $y = \int_2^X \Delta(x)^2 dx - cX^{3/2} + (4\pi^2)^{-1} X \log^2 X - \kappa X \log X$ (with $\kappa = 0.32$) for four different ranges of X.

In an earlier paper [14], we have established the following higher power moments of $\Delta(x)$:

$$\int_2^X \Delta(x)^3 dx = c_3 X^{7/4} + \mathcal{O}(X^{7/4-\delta}),$$
$$\int_2^X \Delta(x)^4 dx = c_4 X^2 + \mathcal{O}(X^{2-\delta}),$$

where δ is some small positive constant. Likewise we can consider the refinement of the $\mathcal{O}$-terms in these formulas, but the machinery available does not seem to be strong enough for this purpose.

4. Conclusion

Our investigation on the error term $\Delta(x)$ can be carried out for certain other error terms in number theory which have representation by Voronoi's type formula. These include $E(T)$ defined by

$$E(T) := \int_0^T |\zeta(\tfrac{1}{2} + it)|^2 dt - T \log \tfrac{T}{2\pi} - (2\gamma - 1)T, \quad T \geq 2$$

and

$$P(x) = \sum_{n \leq x} r(n) - \pi x,$$

where $r(n)$ denotes the number of integer pairs (x, y) such that $n = x^2 + y^2$. All our results on $\Delta(x)$ hold true for $E(T)$ and $P(x)$. The details will appear in a forthcoming paper.

After this talk was presented, Professor K. Matsumoto has kindly informed me that a conjectural formula of the shape

$$\int_0^T E(t)^2 dt \sim cT^{3/2} + c' T \log^A T$$

for a certain constant A has been proposed by him earlier [10]. Eventhough this is not as precise as our conjecture in (5), it is nonetheless in that same direction.

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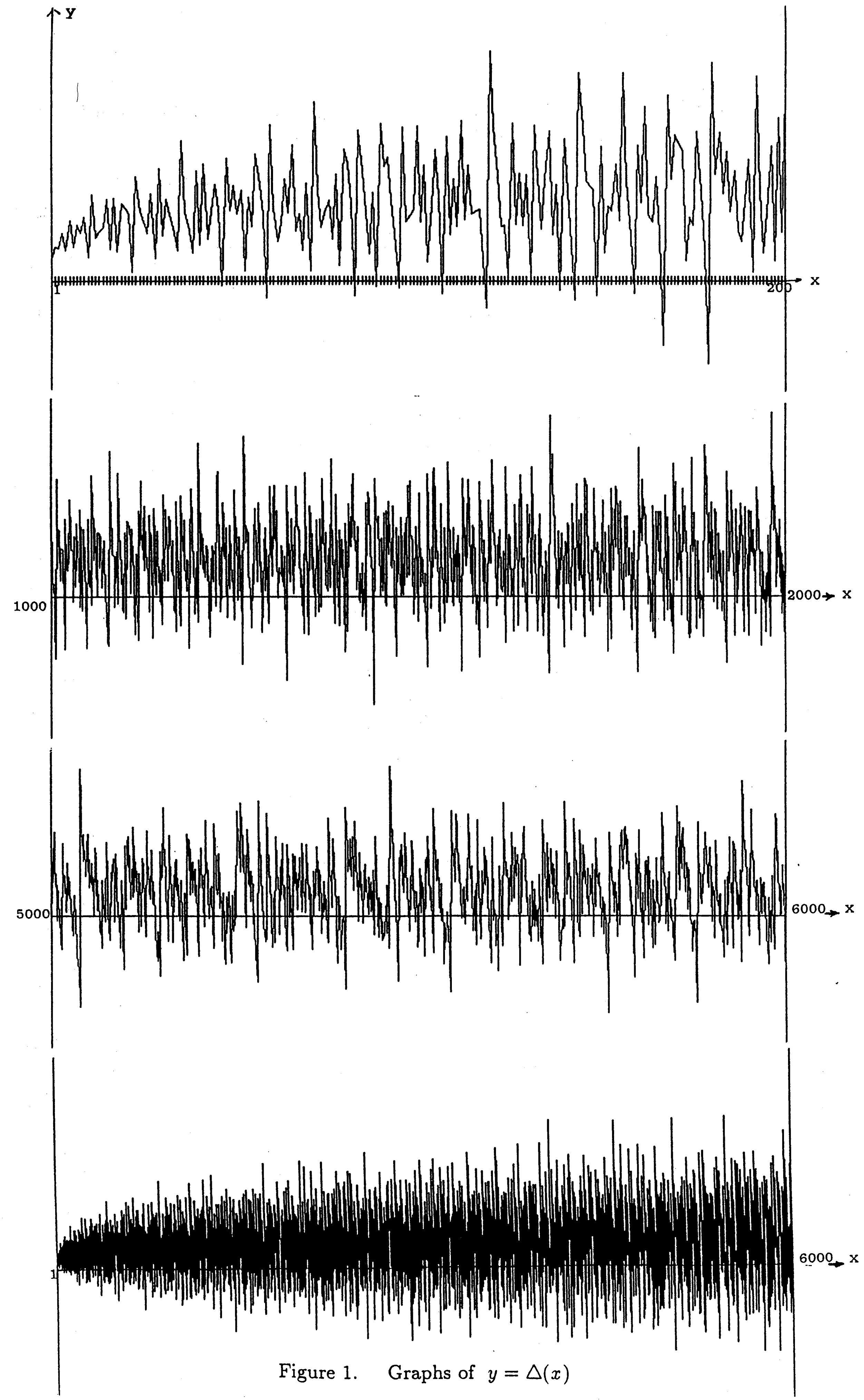

Figure 1. Graphs of $y = \triangle(x)$

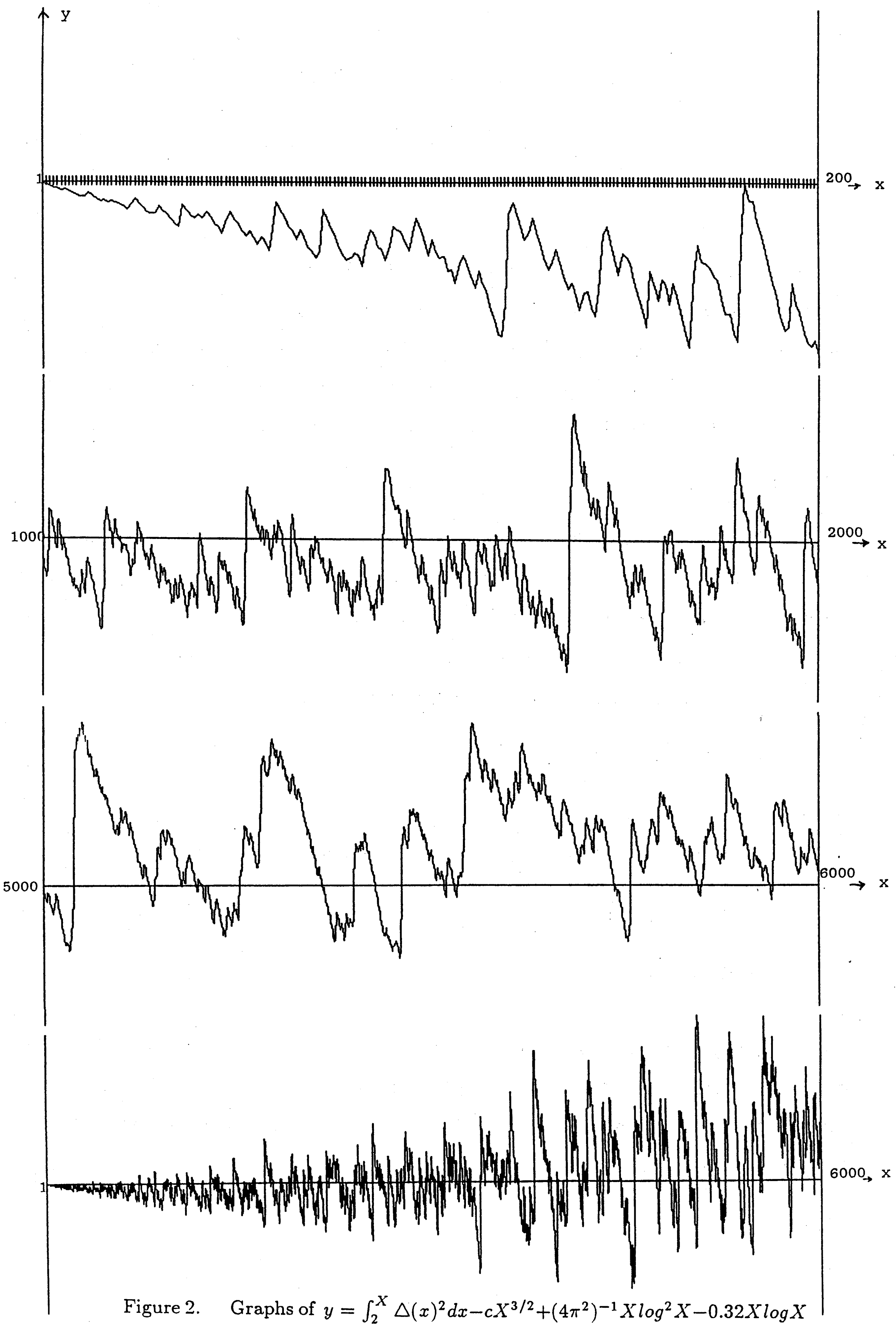

Figure 2. Graphs of $y = \int_2^X \Delta(x)^2 dx - cX^{3/2} + (4\pi^2)^{-1} X log^2 X - 0.32 X log X$