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An application of Mellin-Barnes' type integrals to the mean square of L-functions

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1 Introduction

Let q be a positive integer, s a complex variable and $L(s,\chi)$ the Dirichlet L-function attached to a Dirichlet character χ mod q. Note that $L(s,\chi)$ reduces to the Riemann zeta-function $\zeta(s)$ if $q=1$.

Let $\varphi(q)$ be Euler's function. The mean square

$$\varphi(q)^{-1}\sum_{\chi(\bmod q)}|L(s,\chi)|^{2}, \tag{1.1}$$

summed over all characters χ mod q, has been studied by various authors. Let $\mu(n)$ be Mobius' function. In the special case $s=\frac{1}{2}$, D. R. Heath-Brown [He] found the formula

$$\varphi(q)^{-1}\sum_{\chi(\bmod q)}|L(\tfrac{1}{2},\chi)|^{2}=q^{-1}\sum_{k|q}\mu(\tfrac{q}{k})T(k), \tag{1.2}$$

where k runs through all positive divisors of q and $T(k)$ has the asymptotic expansion

$$T(k)=k\Big(\log\frac{k}{8\pi}+\gamma\Big)+2\zeta^{2}(\tfrac{1}{2})+\sum_{n=0}^{2N-1}c_{n}k^{-\frac{n}{2}}+O(k^{-N})$$

for any integer $N\geq 1$, with Euler's constant γ and unspecified numerical constants c_n. If $q=p$ is a prime, (1.2) gives an asymptotic series in terms of $p^{-\frac{1}{2}}$, since $T(1)$ can be evaluated in a closed form. On the other hand, Y. Motohashi [Mo1], in a series of his study on higher power moments for $\zeta(s)$ and $L(s,\chi)$, applied a classical idea of F. V. Atkinson [At] to (1.1) and proved for any prime $q=p$

$$\begin{aligned}
&(p-1)^{-1}\sum_{\chi(\bmod p)}|L(\tfrac{1}{2}+it,\chi)|^{2}\\
&= \log\frac{p}{2\pi}+2\gamma+\mathrm{Re}\,\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+it)+2p^{-\frac{1}{2}}|\zeta(\tfrac{1}{2}+it)|^{2}\cos(\log p)\\
&\quad -p^{-1}|\zeta(\tfrac{1}{2}+it)|^{2}+O(p^{-\frac{3}{2}}),
\end{aligned}$$

where $\Gamma(s)$ is the gamma-function and the constant implied in the O-symbol depends on t. More general and precise formulae have been proved in [KM1], [Ka1] and [Ka2] by refining the argument of Atkinson and Motohashi.

and its meromorphic continuation.

Let $\sigma_a(n)$ denote the sum of the a-th powers of positive divisors of n. The error term $e_N(\sigma+it;k)$ in (1.6) is of the form

$$e_N(\sigma+it;k)=\mathrm{Re}\{k^{\sigma+it-N}R_N(\sigma+it,\sigma-it;k)\},$$

where $R_N(u,v;k)$ has the following expressions (cf. [Ka2, Lemma 2.2]):

For $\mathrm{Re}\,u<N$, $\mathrm{Re}\,v>-N+1$ and $\mathrm{Re}(u+v)<2$,

$$\begin{aligned} R_N(u,v;k) &= (-1)^N(2\pi)^{u+v-1}\frac{\Gamma(N+1-u)}{\Gamma(v)}\int_0^1\frac{(1-\tau)^{N-1}}{(N-1)!}\sum_{l=1}^{\infty}\sigma_{u+v-1}(l) \\ &\quad\times\left\{e^{\frac{\pi i}{2}(u+v-1)}J_-(\tau,l;k)+e^{-\frac{\pi i}{2}(u+v-1)}J_+(\tau,l;k)\right\}d\tau \end{aligned} \tag{1.8}$$

with

$$J_\pm(\tau,l;k)=\int_0^\infty y^{v+N-1}(1+k^{-1}\tau y)^{u-N-1}e^{\pm 2\pi i l y}dy,$$

while for $\mathrm{Re}\,u<N$, $\mathrm{Re}\,v>-N+1$ and $\mathrm{Re}(u+v)>0$,

$$\begin{aligned} R_N(u,v;k) &= (-1)^N\frac{\Gamma(v+N)}{\Gamma(v)}\int_0^1\frac{(1-\tau)^{N-1}}{(N-1)!}\sum_{l=1}^{\infty}\sigma_{1-u-v}(l) \\ &\quad\times\left\{\tilde{J}_-(\tau,l;k)+\tilde{J}_+(\tau,l;k)\right\}d\tau \end{aligned} \tag{1.9}$$

with

$$\tilde{J}_\pm(\tau,l;k)=\int_0^\infty y^{-u+N}(1+k^{-1}\tau y)^{-v-N}e^{\pm 2\pi i l y}dy.$$

It is in fact possible to obtain a more explicit estimate for $e_N(\sigma+it;k)$ by applying a saddle point lemma of Atkinson [At, Lemma 1] to $J_\pm(\tau,l;k)$ and $\tilde{J}_\pm(\tau,l;k)$.

Theorem 2 ([Ka2, Theorem 1 with $h=0$]) *For any integer $N\geq 1$, the inequality*

$$e_N(\sigma+it;k)=O\{k^{\sigma-N}(|t|+1)^{2N+\frac{1}{2}-\sigma}\} \tag{1.10}$$

holds in the region

$$\{\sigma+it;\ -N+1<\sigma<N,\ t\text{: real}\},$$

where the O-constant depends only on σ and N.

Remark. It is reasonable that such a bound as in (1.10) follows, since

$$\frac{(-1)^n}{n!}k^{\sigma+it-n}\frac{\Gamma(\sigma-it+n)}{\Gamma(\sigma-it)}\zeta(\sigma+it-n)\zeta(\sigma-it+n)\ll k^{\sigma-n}(|t|+1)^{2n+\frac{1}{2}-\sigma} \tag{1.11}$$

for $-n+1<\sigma<n$ ($n\geq 1$), see (1.5). Note that (1.11) is the best-possible, since $\zeta(\sigma+it)=\Omega(1)$ for $\sigma>1$ as $t\to\pm\infty$.

The main aim of this paper is to provide alternative simple proofs of Theorems 1 and 2. It should be remarked that the introduction of a Mellin transform (2.3) below is a key to the considerable simplification. In Sections 2 and 3, we shall prove Theorems 1 and 2, respectively. In the final section, the inner connections between different expressions for $R_N(u,v;k)$ (see (1.8), (1.9), (2.9) and (4.1)) will be examined.

Theorem 1 ([KM1, Theorem 1], [Ka1, Theorem 3], [Ka2, Theorem 3 with $h = 0$]) *Let*

$$E = \{1, 2, 3, \ldots\} \cup \{\tfrac{n}{2} + it;\ n\text{: integer} \le 2,\ t\text{: real}\}.$$

Then for any integer $N \ge 1$, in the region

$$\{\sigma + it;\ -N + 1 < \sigma < N + 1,\ t\text{: real}\} \tag{1.3}$$

except the points of E, the formula

$$\begin{aligned} &\varphi(q)^{-1} \sum_{\chi (\bmod q)} |L(\sigma + it, \chi)|^2 \\ &= \zeta(2\sigma) \prod_{p|q} (1 - p^{2\sigma}) + 2q^{-2\sigma} \varphi(q) \Gamma(2\sigma - 1) \zeta(2\sigma - 1) \operatorname{Re} \left\{ \frac{\Gamma(1 - \sigma - it)}{\Gamma(\sigma - it)} \right\} \\ &\quad + 2q^{-2\sigma} \sum_{k|q} \mu(\tfrac{q}{k}) T(\sigma + it; k) \end{aligned} \tag{1.4}$$

holds, where p runs through all prime divisors of q and $T(\sigma + it; k)$ has the asymptotic expansion

$$\begin{aligned} T(\sigma + it; k) &= \sum_{n=0}^{N-1} \frac{(-1)^n k^{-n}}{n!} \operatorname{Re} \left\{ k^{\sigma + it} \frac{\Gamma(\sigma - it + n)}{\Gamma(\sigma - it)} \zeta(\sigma + it - n) \zeta(\sigma - it + n) \right\} \\ &\quad + e_N(\sigma + it; k). \end{aligned} \tag{1.5}$$

Here $e_N(\sigma + it; k)$ is the error term satisfying

$$e_N(\sigma + it; k) = O(k^{\sigma - N}) \tag{1.6}$$

in the region (1.3), with the O-constant depends only on σ, N and t. In particular, if $q = p$ is a prime, the asymptotic series

$$\begin{aligned} &(p - 1)^{-1} \sum_{\chi (\bmod p)} |L(\sigma + it, \chi)|^2 \\ &= \zeta(2\sigma) + 2p^{1 - 2\sigma} \Gamma(2\sigma - 1) \zeta(2\sigma - 1) \operatorname{Re} \left\{ \frac{\Gamma(1 - \sigma - it)}{\Gamma(\sigma - it)} \right\} \\ &\quad - p^{-2\sigma} |\zeta(\sigma + it)|^2 + 2p^{-2\sigma} T(\sigma + it; p) \end{aligned} \tag{1.7}$$

holds.

Remark 1. Asymptotic formulae as in (1.4) for the exceptional points $s \in E$ can be deduced as limiting cases of Theorem 1. Important cases $\operatorname{Re} s = \frac{1}{2}$ and $s = 1$ are treated in [KM1, Theorem1] and [KM2, Theorems 1 and 4], respectively.

Remark 2. In this paper, the region (1.3) in which (1.4) remains valid will be slightly improved upon our eariler results [KM1, Theorem 1], [Ka1, Theorem 3] and [Ka2, Theorem 3].

Remark 3. Similar asymptotic results for (1.1) have been independently obtained by W. Zhang [Zh2]–[Zh7] and V. V. Rane [Ra]. Their proofs are based on the use of the Hurwitz zeta-function $\zeta(s, \alpha)$ defined by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \qquad (\operatorname{Re} s > 1,\ \alpha > 0),$$

2 Proof of Theorem 1

Let

$$Q(u,v;q)=\varphi(q)^{-1}\sum_{\chi(\mathrm{mod}\,q)}L(u,\chi)L(v,\bar{\chi}).$$

We suppose first that $\mathrm{Re}\,u>1$ and $\mathrm{Re}\,v>1$. Then by the orthogonality and the periodicity of characters

$$Q(u,v;q)=\sum_{\substack{h,k=1\\ h\equiv k(\mathrm{mod}\,q)\\ (h,q)=(k,q)=1}}^{\infty}h^{-u}k^{-v}=\sum_{\substack{a=1\\ (a,q)=1}}^{q}\sum_{m,n=0}^{\infty}(qm+a)^{-u}(qn+a)^{-v}.$$

Classifying the last inner double sum according to the conditions $m=n$, $m<n$ and $m>n$, we get

$$Q(u,v;q)=L(u+v,\chi_0)+f(u,v;q)+f(v,u;q), \tag{2.1}$$

where χ_0 is the principal character mod q and

$$f(u,v;q)=\sum_{\substack{a=1\\ (a,q)=1}}^{q}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(qm+a)^{-u}(q(m+n)+a)^{-v}. \tag{2.2}$$

Atkinson succeeded in obtaining the analytic continuation of $f(u,v;1)$ (namely in the case of $\zeta(s)$), which led him to the eventual application on taking $u=\frac{1}{2}+it$ and $v=\frac{1}{2}-it$. Several ways are known to prove the analytic continuation of $f(u,v;q)$. T. Meurman [Me] generalizes Atkinson's original proof to treat $f(u,v;q)$ by Poisson's summation formula, while Motohashi [Mo1] makes use of certain loop-integral expressions for $f(u,v;q)$. In this paper we apply

$$(qm+a)^{-u}(q(m+n)+a)^{-v}=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)}(qm+a)^{-s-u-v}(qn)^{s}ds, \tag{2.3}$$

where c is a constant fixed with $-\mathrm{Re}\,v<c<-1$ and (c) denotes the vertical straight line from $c-i\infty$ to $c+i\infty$. This can be obtained by taking $-z=qn/(qm+a)$ in

$$\Gamma(\alpha)(1-z)^{-\alpha}=\frac{1}{2\pi i}\int_{(b)}\Gamma(\alpha+s)\Gamma(-s)(-z)^{s}ds\quad(|\arg(-z)|<\pi,\ -\mathrm{Re}\,\alpha<b<0),$$

which is a special case of Mellin-Barnes' integral expression for Gauss' hypergeometric function $F(\alpha,\beta;\gamma;z)$ (cf. [WW, p.289, 14.51 Corollary]). Integrals of the type (2.3) were firstly introduced by Motohashi [Mo2] to investigate the fourth power mean of $\zeta(s)$. Recently, A. Ivić [Iv2, Chapter 2] applied Motohashi's argument to treat the mean square of $\zeta(s)$.

We assume for brevity that all the singularities appearing in the following argument are at most simple poles, since other cases can be treated by taking limits (see Remark

1 of Theorem 1). Substituting (2.3) into each term in the right-hand side of (2.2), we obtain

$$f(u,v;q)=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)}q^{-u-v}\sum_{\substack{a=1\\(a,q)=1}}^{q}\zeta(u+v+s,\tfrac{a}{q})\zeta(-s)ds, \tag{2.4}$$

where the interchange of the order of summation and integration can be justified, since, by virtue of the choice of δ, the variables $u+v+s$ and $-s$ are both in the region of absolute convergence. As we shall see in the following, the formula (2.4) will provide the analytic continuation of $f(u,v;q)$ by deforming suitably the path of integration. Note that (c) separates the poles at $s=-1+n$ $(n=0,1,2,\dots)$ from the poles at $s=1-u-v$, $-v-n$ $(n=0,1,2,\dots)$ of the integrand. If we replace (c) by the contour C which is suitably indented in such a manner as to separate the poles at $s=1-u-v$, $-1+n$ $(n=0,1,2,\dots)$ from the poles at $s=-v-n$ $(n=0,1,2,\dots)$, then we get by the theorem of residues

$$f(u,v;q)=\frac{\Gamma(u+v-1)\Gamma(1-u)}{\Gamma(v)}\zeta(u+v-1)q^{1-u-v}\prod_{p|q}(1-p^{-1})+g(u,v;q), \tag{2.5}$$

where

$$\begin{aligned} g(u,v;q) &= q^{-u-v}\sum_{k|q}\mu(\tfrac{q}{k})\frac{1}{2\pi i}\int_{C}\frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)}\zeta(u+v+s)\zeta(-s)k^{u+v+s}ds \\ &= q^{-u-v}\sum_{k|q}\mu(\tfrac{q}{k})S(u,v;k), \end{aligned} \tag{2.6}$$

say. Here we applied the identities

$$\sum_{\substack{a=1\\(a,q)=1}}^{q}\zeta(w,\tfrac{a}{q})=\zeta(w)\sum_{k|q}\mu(\tfrac{q}{k})k^{w}=\zeta(w)q^{w}\prod_{p|q}(1-p^{-w}).$$

Hence from (2.1), (2.5) and (2.6),

$$\begin{aligned} &Q(u,v;q) \\ &= \zeta(u+v)\prod_{p|q}(1-p^{-u-v})+q^{-u-v}\varphi(q)\zeta(u+v-1)\Gamma(u+v-1)\times \\ &\quad\times\left\{\frac{\Gamma(1-u)}{\Gamma(v)}+\frac{\Gamma(1-v)}{\Gamma(u)}\right\}+q^{-u-v}\sum_{k|q}\mu(\tfrac{q}{k})\{S(u,v;k)+S(v,u;k)\} \end{aligned} \tag{2.7}$$

holds in the region $\operatorname{Re}u>1$ and $\operatorname{Re}v>1$, where $S(v,u;k)$ is expressed in the same manner as $S(u,v;k)$.

Next we shift the path of integration to the left. We suppose at this stage that $\operatorname{Re}u<1$ and $\operatorname{Re}v>1$, where C can be taken as a straight line (c_0) with $-\operatorname{Re}v<c_0<\min(-1,1-\operatorname{Re}(u+v))$. Let N be a positive integer and c_N a constant fixed with $-\operatorname{Re}v-N<c_N<-\operatorname{Re}v-N+1$. Since the order of the integrand in (2.6) is $O(|\operatorname{Im}s|^{C}e^{-\pi|\operatorname{Im}s|})$ as $\operatorname{Im}s\to\pm\infty$ (C is a positive constant depending only on $\operatorname{Re}s$, Re u and Re v), we can shift the path from (c_0) to (c_N). Collecting the residues at the poles $s = -v - n$ $(n = 0, 1, \ldots, N-1)$, we obtain

$$S(u,v;k) = \sum_{n=0}^{N-1} \frac{(-1)^n}{n!}\frac{\Gamma(v+n)}{\Gamma(v)}\zeta(u-n)\zeta(v+n)k^{u-n} + r_N(u,v;k), \tag{2.8}$$

where

$$r_N(u,v;k) = \frac{1}{2\pi i}\int_{(c_N)} \frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)}\zeta(u+v+s)\zeta(-s)k^{u+v+s}ds. \tag{2.9}$$

Here the condition on u and v can be relaxed as

$$\operatorname{Re} u < N+1 \qquad \text{and} \qquad \operatorname{Re} v > -N+1. \tag{2.10}$$

Under (2.10) we can choose c_N satisfying the condition

$$-\operatorname{Re} v - N < c_N < \min(-1, -\operatorname{Re} v - N + 1, 1 - \operatorname{Re}(u+v)),$$

by which (c_N) separates the poles at $s = -v - n$ $(n = N, N+1, N+2, \ldots)$ from the poles at $s = 1 - u - v$, $-1 + n$ $(n = 0, 1, 2, \ldots)$, $-v - n$ $(n = 0, 1, \ldots, N-1)$.

Now we proceed to prove Theorem 1. Taking $u = \sigma + it$ and $v = \sigma - it$ in (2.7), (2.8) and (2.9), we obtain (1.4) and (1.5), by noticing (2.10) and putting

$$T(\sigma+it;k) = \operatorname{Re}\{S(\sigma+it, \sigma-it;k)\} \quad \text{and} \quad e_N(\sigma+it;k) = \operatorname{Re}\{r_N(\sigma+it,\sigma-it;k)\}.$$

The error estimate (1.6) follows from

$$\begin{aligned} r_N(u,v;k) &= \frac{(-1)^N}{N!}\frac{\Gamma(v+N)}{\Gamma(v)}\zeta(u-N)\zeta(v+N)k^{u-N} \\ &\quad + \frac{1}{2\pi i}\int_{(c_{N+1})}\frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)}\zeta(u+v+s)\zeta(-s)k^{u+v+s}ds \\ &\ll k^{\operatorname{Re} u - N} + k^{\operatorname{Re}(u+v)+c_{N+1}} \ll k^{\operatorname{Re} u - N}, \end{aligned}$$

by $-\operatorname{Re} v - N - 1 < c_{N+1} < -\operatorname{Re} v - N$. Furthermore (1.7) can be deduced from (2.7) by noting

$$\begin{aligned} S(u,v;1) + S(v,u;1) &= \zeta(u)\zeta(v) - \zeta(u+v) \\ &\quad - \zeta(u+v-1)\Gamma(u+v-1)\left\{\frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)}\right\}, \end{aligned}$$

which is the special case $q = 1$ of (2.7). The proof of Theorem 1 is now complete.

3 Proof of Theorem 2

Throughout this section, let $-N+1 < \sigma < N$ and δ a constant fixed with $0 < \delta < \frac{1}{2}\min(N-\sigma, N-1+\sigma, 1)$. We write $s = -\sigma - N + \xi + i\tau$ in (2.9). For the proof of Theorem 2, we need

Lemma *For any real τ, t and ξ with $|\xi| \le \delta$, we have*

$$\Gamma(\sigma+N-\xi-i\tau) \ll (|\tau|+1)^{\sigma+N-\xi-\frac{1}{2}}e^{-\frac{\pi}{2}|\tau|}, \tag{3.1}$$

$$\Gamma(-N+\xi+i(\tau-t)) \ll \begin{cases} |\tau-t|^{-N+\xi-\frac{1}{2}}e^{-\frac{\pi}{2}|\tau-t|} & \text{for } |\tau-t|\ge 1, \\ |\xi+i(\tau-t)|^{-1} & \text{for } |\tau-t|\le 1, \end{cases} \tag{3.2}$$

$$\Gamma(\sigma-it)^{-1} \ll (|t|+1)^{\frac{1}{2}-\sigma}e^{\frac{\pi}{2}|t|}, \tag{3.3}$$

$$\zeta(\sigma-N+\xi+i\tau) \ll (|\tau|+1)^{\frac{1}{2}-\sigma+N-\xi}, \tag{3.4}$$

$$\zeta(\sigma+N-\xi-i\tau) \ll 1. \tag{3.5}$$

Here and in what follows the implied constants depend at most on σ and N.

Proof. (3.1)–(3.3) follow from Stirling's formula (cf. [Iv1, p.492, (A.34)]) and the trivial bounds for $\Gamma(w)$ near the real axis. By virtue of the choice of δ, (3.5) is an immediate consequence of the inequality $\zeta(w) \ll 1$ for $\operatorname{Re} w > 1$, while (3.4) can be proved by applying the functional equation of $\zeta(w)$. □

For the proof of Theorem 2 we may restrict ourselves to the case $t \ge 2$, since the case $t \le -2$ follows from this case by the reflection principle, and the case $|t| \le 2$ is a simple consequence of Theorem 1.

Let $\sigma_N = \sigma + N$ and L the infinite broken line joining the points $-\sigma_N - i\infty$, $-\sigma_N + i(t-\delta)$, $-\sigma_N+\delta+i(t-\delta)$, $-\sigma_N+\delta+i(t+\delta)$, $-\sigma_N+i(t+\delta)$ and $-\sigma_N+i\infty$. Taking $u = \sigma - it$ and $v = \sigma + it$ in (2.9), and then replacing the path (c_N) by L, we have

$$r_N(\sigma+it,\sigma-it;k) = \frac{1}{2\pi i}\int_L \frac{\Gamma(-s)\Gamma(\sigma-it+s)}{\Gamma(\sigma-it)}\zeta(2\sigma+s)\zeta(-s)k^{2\sigma+s}ds. \tag{3.6}$$

We shall estimate the right-hand integral in (3.6) by dividing

$$r_N(\sigma+it,\sigma-it;k) = \frac{1}{2\pi i}\left\{\sum_{\substack{\mu=1\\ \mu\ne 5}}^{7} I_\mu + \sum_{\nu=1}^{3} I_{5,\nu}\right\},$$

where

$$I_1 = \int_{-\sigma_N-i\infty}^{-\sigma_N-i}, \quad I_2 = \int_{-\sigma_N-i}^{-\sigma_N+i}, \quad I_3 = \int_{-\sigma_N+i}^{-\sigma_N+i(t-1)}, \quad I_4 = \int_{-\sigma_N+i(t-1)}^{-\sigma_N+i(t-\delta)},$$

$$I_{5,1} = \int_{-\sigma_N+i(t-\delta)}^{-\sigma_N+\delta+i(t-\delta)}, \quad I_{5,2} = \int_{-\sigma_N+\delta+i(t-\delta)}^{-\sigma_N+\delta+i(t+\delta)}, \quad I_{5,3} = \int_{-\sigma_N+\delta+i(t+\delta)}^{-\sigma_N+i(t+\delta)},$$

$$I_6 = \int_{-\sigma_N+i(t+\delta)}^{-\sigma_N+i(t+1)}, \quad I_7 = \int_{-\sigma_N+i(t+1)}^{-\sigma_N+i\infty}.$$

The treatment of $I_{5,\nu}$ ($\nu = 1,2,3$) is more delicate than that of other I_μ's. By Lemma and the assumption $t \ge 2$, we get

$$I_1 \ll k^{\sigma-N}t^{\frac{1}{2}-\sigma}\int_{-\infty}^{-1}(-\tau)^{2N}(t-\tau)^{-N-\frac{1}{2}}e^{\pi\tau}d\tau \ll k^{\sigma-N}t^{-\sigma-N}, \tag{3.7}$$

$$I_2 \ll k^{\sigma-N}t^{\frac{1}{2}-\sigma}\int_{-1}^{1}(t-\tau)^{-N-\frac{1}{2}}d\tau \ll k^{\sigma-N}t^{-\sigma-N}. \tag{3.8}$$

Moreover

$$I_3 \ll k^{\sigma-N}t^{\frac{1}{2}-\sigma}\int_1^{t-1}\tau^{2N}(t-\tau)^{-N-\frac{1}{2}}d\tau \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}, \tag{3.9}$$

$$I_4 \ll k^{\sigma-N}t^{\frac{1}{2}-\sigma}\int_{t-1}^{t-\delta}\tau^{2N}(t-\tau)^{-1}d\tau \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}\log\delta^{-1}, \tag{3.10}$$

where the last upper bounds in (3.9) and (3.10) are obtained by integrating by parts. Similarly to I_3 and I_4,

$$I_6 \ll k^{\sigma-N}t^{\frac{1}{2}-\sigma}\int_{t+\delta}^{t+1}\tau^{2N}(t-\tau)^{-1}d\tau \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}\log\delta^{-1}, \tag{3.11}$$

$$I_7 \ll k^{\sigma-N}t^{\frac{1}{2}-\sigma}\int_{t+1}^{\infty}\tau^{2N}(\tau-t)^{-N-\frac{1}{2}}e^{-\pi(\tau-t)}d\tau \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}. \tag{3.12}$$

For $I_{5,\nu}$ $(\nu = 1,2,3)$, we procced as follows. By $0<\delta<\frac{1}{2}$,

$$\begin{aligned} I_{5,1} &\ll k^{\sigma-N}t^{\frac{1}{2}-\sigma}\int_0^{\delta}(t-\delta)^{2N-2\xi}e^{\frac{\pi}{2}\delta}|\xi-i\delta|^{-1}k^{\xi}d\xi \\ &\ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}\delta^{-1}\int_0^{\delta}(t^{-2}k)^{\xi}d\xi \le k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}\max(1,(kt^{-2})^{\delta}), \quad (3.13) \\ I_{5,2} &\ll k^{\sigma-N+\delta}t^{\frac{1}{2}-\sigma}\int_{t-\delta}^{t+\delta}\tau^{2N-2\delta}e^{\frac{\pi}{2}(t-\tau)}|\delta+i(t-\tau)|^{-1}d\tau \\ &\ll k^{\sigma-N+\delta}t^{\frac{1}{2}-\sigma}\delta^{-1}\int_{t-\delta}^{t+\delta}\tau^{2N-2\delta}d\tau \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}(kt^{-2})^{\delta}, \quad (3.14) \\ I_{5,3} &\ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}\max(1,(kt^{-2})^{\delta}), \quad (3.15) \end{aligned}$$

where the treatment of $I_{5,3}$ is similar to (3.13). Combining (3.7)–(3.15), we obtain

$$r_N(\sigma+it,\sigma-it;k) \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}\{\log\delta^{-1}+\max(1,(kt^{-2})^{\delta})\}. \tag{3.16}$$

Next let L' be the infinite broken line joining the points $-\sigma_N - i\infty$, $-\sigma_N + i(t-\delta)$, $-\sigma_N - \delta + i(t-\delta)$, $-\sigma_N - \delta + i(t+\delta)$, $-\sigma_N + i(t+\delta)$ and $-\sigma_N + i\infty$. Then

$$\begin{aligned} &r_N(\sigma+it,\sigma-it;k) \\ &= \frac{(-1)^N}{N!}\frac{\Gamma(\sigma-it+N)}{\Gamma(\sigma-it)}\zeta(\sigma+it-N)\zeta(\sigma-it+N)k^{\sigma+it-N} \\ &\quad + \frac{1}{2\pi i}\int_{L'}\frac{\Gamma(-s)\Gamma(\sigma-it+s)}{\Gamma(\sigma-it)}\zeta(2\sigma+s)\zeta(-s)k^{2\sigma+s}ds. \quad (3.17) \end{aligned}$$

The first term in the right-hand side of (3.17) is bounded as $\ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}$ by (1.11). To estimate the second term, we divide

$$\int_{L'} = \sum_{\substack{\mu=1\\ \mu\neq5}}^{7} I_\mu + \sum_{\nu=1}^{3} I'_{5,\nu},$$

where

$$I'_{5,1} = \int_{-\sigma_N+i(t-\delta)}^{-\sigma_N-\delta+i(t-\delta)}, \qquad I'_{5,2} = \int_{-\sigma_N-\delta+i(t-\delta)}^{-\sigma_N-\delta+i(t+\delta)}, \qquad I'_{5,3} = \int_{-\sigma_N-\delta+i(t+\delta)}^{-\sigma_N+i(t+\delta)}.$$

Similarly to the previous case,

$$I'_{5,\nu} \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}\max(1,(k^{-1}t^2)^\delta) \qquad (\nu=1,3),$$
$$I'_{5,2} \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}(k^{-1}t^2)^\delta.$$

Therefore

$$r_N(\sigma+it,\sigma-it;k) \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}\{\log\delta^{-1}+\max(1,(k^{-1}t^2)^\delta)\}. \qquad (3.18)$$

Theorem 2 now follows from (3.16) if $t \geq k^{\frac{1}{2}}$, and from (3.18) if $t \leq k^{\frac{1}{2}}$, respectively.

4 Additional remarks

The first purpose of this section is to show how we can deduce (2.9) directly from (1.8) or (1.9). To do this we introduce a confluent hypergeometric function $\Psi(\alpha,\gamma;z)$ defined by

$$\Psi(\alpha,\gamma;z)=\frac{1}{\Gamma(\alpha)}\int_0^{\infty e^{i\phi}} e^{-zw}w^{\alpha-1}(1+w)^{\gamma-\alpha-1}dw$$

for $\mathrm{Re}\,\alpha>0$, $|\phi|<\pi$ and $|\phi+\arg z|<\frac{\pi}{2}$ (cf. [Er, p.256, 6.5(3)]). Rotating suitably the path of integrations for $J_\pm(\tau,l;k)$ and $\tilde{J}_\pm(\tau,l;k)$, we find

$$J_\pm(\tau,l;k)=(k\tau^{-1})^{v+N}\Gamma(v+N)\Psi(v+N,u+v;2\pi kl\tau^{-1}e^{\mp\frac{\pi i}{2}})$$

and

$$\tilde{J}_\pm(\tau,l;k)=(k\tau^{-1})^{N+1-u}\Gamma(N+1-u)\Psi(N+1-u,2-u-v;2\pi kl\tau^{-1}e^{\mp\frac{\pi i}{2}}).$$

Furthermore, $J_\pm(\tau,l;k)$ and $\tilde{J}_\pm(\tau,l;k)$ can be expressed in terms of Mellin-Barnes' type integrals by using

$$\Psi(\alpha,\gamma;z)=\frac{1}{2\pi i}\int_{(b)}\frac{\Gamma(\alpha+s)\Gamma(-s)\Gamma(1-\gamma-s)}{\Gamma(\alpha)\Gamma(\alpha-\gamma+1)}z^s ds,$$

where $-\mathrm{Re}\,\alpha<b<\min(0,1-\mathrm{Re}\,\gamma)$ and $|\arg z|<\frac{3\pi}{2}$ (cf. [Er, p.256, 6.5(5)]). Substituting these integrals into each term in the right-hand infinite series in (1.8) and (1.9), respectively, and then applying the functional equation of $\zeta(w)$, we can see that either (1.8) or (1.9) directly yields (2.9), by noting

$$r_N(u,v;k)=k^{u-N}R_N(u,v;k). \qquad (4.1)$$

On the other hand, (1.8) and (1.9) are connected by the transformation formula

$$\Psi(\alpha,\gamma;z)=z^{1-\gamma}\Psi(\alpha-\gamma+1,2-\gamma;z)$$

(cf. [Er, p.257, 6.5(6)]), for details see [Ka1, Section 3]. In view of the consideration given above, this connection is embodied in (2.9) with the functional equation of $\zeta(w)$.

In this occasion we point out an error in the preceding article [KM3]. In Section 2, we should mention that the same result as R. Sitaramachandrarao [Si] with a slightly different log-factor was independently obtained by Zhang [Zh1, Corollary], whose main theorem is proved in a more general setting.

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References

[At] Atkinson, F. V.: The mean-value of the Riemann zeta function, Acta Math. **81** (1949), 353–376.

[Er] Erdélyi, A., et al. (the Bateman Manuscript Project): Higher Transcendental Functions Vol. I, MacGraw-Hill, New York, 1953.

[He] Heath-Brown, D. R.: An asymptotic series for the mean value of Dirichlet L-functions, Comment. Math. Helv. **56** (1981), 148–161.

[Iv1] Ivić, A.: The Riemann Zeta-function, John Wiley & Sons, New York, 1985.

[Iv2] Ivić, A.: Mean Values of the Riemann Zeta Function, Tata Institute of Fundamental Research, Bombay, 1991.

[Ka1] Katsurada, M.: Asymptotic expansions of the mean values of Dirichlet L-functions II, in "Analytic Number Theory and Related Topics", K. Nagasaka (ed.), World Scientific, Singapore, 1993, pp. 61–71.

[Ka2] Katsurada, M.: Asymptotic expansions of the mean values of Dirichlet L-functions III, Manuscr. Math. **83** (1994), 425–442.

[KM1] Katsurada, M and Matsumoto, K.: Asymptotic expansions of the mean values of Dirichlet L-functions, Math. Zeit. **208** (1991), 23–39.

[KM2] Katsurada, M and Matsumoto, K.: The mean values of Dirichlet L-functions at integer points and class numbers of cyclotomic fields, Nagoya Math. J. **134** (1994), 151–172.

[KM3] Katsurada, M. and Matsumoto, K.: Some asymptotic results on Hurwitz zeta-functions, Kôkyûroku **886**, RIMS Kyoto Univ. (1994), 114–123.

[Me] Meurman, T.: A generalization of Atkinson's formula to L-functions, Acta Arith. **47** (1986), 351–370.

[Mo1] Motohashi, Y.: A note on the mean value of the zeta and L-functions I, Proc. Japan Acad. **61** Ser. A (1985), 222–224.

[Mo2] Motohashi, Y.:An explicit formula for the fourth power mean of the Riemann zeta-function, Acta Math. **170** (1993), 181–220.

[Ra] Rane, V. V.: Dirichlet L-function and power series for Hurwitz zeta function, Proc. Indian Acad. Sci. Math. Sci. **103** (1993), 27–39.

[Si] Sitaramachandrarao, R.: A mean value theorem for Hurwitz zeta-function, preprint.

[WW] Whittaker, E. T. and Watson, G. N.: A course of Modern Analysis, Cambridge University Press, Cambridge, 1927.

[Zh1] Zhang, Wenpeng: On the Hurwitz zeta-function, Northeastern Math. J. **6** (1990), 261–267.

[Zh2] Zhang, Wenpeng: On the mean square value of the Dirichlet L-functions, Adv. in Math. (China) **19** (1990), 321–333.

[Zh3] Zhang, Wenpeng: On the mean value of L-functions, J. Math. Res. Exposition **10** (1990), 355–360.

[Zh4] Zhang, Wenpeng: On an elementary result of L-functions, Adv. Math. (China) **19** (1990), 478–487.

[Zh5] Zhang, Wenpeng: On the mean value formula of Dirichlet L-functions (II), Science in China (Ser. A) **34** (1991), 660–675.

[Zh6] Zhang, Wenpeng: On the Dirichlet L-functions, Acta Math. Sinica NS **7** (1991), 103–118.

[Zh7] Zhang, Wenpeng: On the mean square value of Dirichlet's L-functions, Compos. Math. **84** (1992), 59–69.

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