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Siegel modular forms: estimates for eigenvalues and Fourier coefficients *)

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We would like to report on recent progress on the estimation of eigenvalues and Fourier coefficients of Siegel modular forms of arbitrary genus g.

1. Siegel modular forms and Hecke operators

We start with recalling some basic terminology and facts on Siegel modular forms. As a reference the reader may consult [9].

Let

$$\mathcal{H}_g = \{Z\in\mathbb{C}^{(g,g)} \mid Z=Z', \mathrm{Im}(Z)>0\}$$

(Z' = transpose of Z) be the Siegel upper half-space of genus g. The group $Sp_g(\mathbb{R})$ of real symplectic matrices of size 2g operates on $\mathcal{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\circ Z = (AZ+B)(CZ+D)^{-1}.$$

We let $\Gamma_g = Sp_g(\mathbb{Z})$ be the Siegel modular group and for $k\in\mathbb{N}$ denote by $M_k(\Gamma_g)$ the space of Siegel modular forms of weight k w.r.t. Γ_g, i.e. the space of holomorphic functions $F:\mathcal{H}_g\to\mathbb{C}$ satisfying $F((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k F(Z)$ for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_g$ and having a Fourier expansion

$$F(Z) = \sum_{T\geq 0} a(T)e^{\pi i\cdot tr(TZ)},$$

where T runs over all positive semi-definite even integral matrices of size g. (We remark that for $g\geq 2$ the existence of the Fourier expansion of F as required above actually follows automatically, by Koecher's principle.)

Observe that if $U\in GL_g(\mathbb{Z})$, then

$$a(T[U]) = (\det(U))^k a(T)$$

(notation: if A and B are real matices of appropriate sizes, we put $A[B]:=B'AB$). In fact, $\begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix}\in\Gamma_g$ and $F(Z[U]) = (\det(U))^k F(Z)$.

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We will denote by $S_k(\Gamma_g)$ the subspace of cusp forms (require $a(T)=0$ unless $T>0$ is positive definite).

Basic examples of modular forms are theta series: if S is a positive definite even unimodular matrix of size m (one knows that such S exists if and only if $8|m$), then

$$\theta_S^{(g)}(Z) = \sum_{G\in\mathbb{Z}^{(m,g)}} e^{\pi i\cdot \mathrm{tr}(S[G]\cdot Z)} \qquad (Z\in\mathfrak{H}_g)$$

is in $M_{m/2}(\Gamma_g)$. Note that

$$\theta_S^{(g)}(Z) = \sum_{T\geq 0} r_S(T)e^{\pi i\cdot \mathrm{tr}(TZ)},$$

where $r_S(T)$ is the number of representations of T by S.

More generally, if $P:\mathbb{C}^{(m,g)}\to\mathbb{C}$ is a harmonic polynomial of degree $\nu\geq 1$, i.e. is a polynomial function in the variables $X=(x_{ij})$ $(1\leq i\leq m, 1\leq j\leq g)$ which satisfies $P(XU) = (\det(U))^{\nu}P(X)$ for all $U\in GL_g(\mathbb{Z})$ and which is annihilated by the Laplace operator $\sum_{i,j}\partial^2/\partial x_{ij}^2$, then the generalized theta series

$$\theta_{S,P}^{(g)}(Z) = \sum_{G\in\mathbb{Z}^{(m,g)}} P(S^{1/2}G)e^{\pi i\cdot \mathrm{tr}(S[G]\cdot Z)}$$

is in $S_{m/2+\nu}(\Gamma_g)$ (here $S^{1/2}>0$, $(S^{1/2})^2=S$).

For the moment let us set $\Gamma:=\Gamma_g$ and denote by $G=GSp_g^+(\mathbb{Q})$ the group of rational symplectic similitudes of size 2g with positive scalar factor. Let $L(\Gamma,G)$ be the free $\mathbb{C}$-module generated by the right cosets Γx $(x\in\Gamma\backslash G)$. Then Γ operates on $L(\Gamma,G)$ by right multiplication, and we let

$$H_g:=L(\Gamma,G)^{\Gamma}$$

be the subspace of Γ-invariants.

If $T_i = \sum_{x_i\in\Gamma\backslash G} a_{x_i}\Gamma x_i$ $(i=1,2)$ are in H_g, one sets

$$T_1\cdot T_2 := \sum_{x_1,x_2\in\Gamma\backslash G} a_{x_1}a_{x_2}\Gamma x_1x_2.$$

Then $T_1\cdot T_2$ is in H_g. This easily follows from the fact that H_g is "generated" by double cosets, i.e. is generated by the elements $\sum_i \Gamma x_i$ where $\Gamma\backslash\Gamma x\Gamma = \bigcup_i \Gamma x_i$ (finite, disjoint) and $x\in G$.

The space H_g together with the above multiplication is called the Hecke algebra. It is a commutative associative algebra with 1 (commutativity formally follows from the fact that $\Gamma x \Gamma = \Gamma x' \Gamma$).

The following facts are known:

i) $H_g = \bigotimes_{p \text{ prim}} H_{g,p}$

where $H_{g,p}$ is defined in the same way as H_g, however with G replaced by $G^{(p)} := G \cap GL_{2g}(\mathbb{Z}[p^{-1}])$.

ii) The local component $H_{g,p}$ is generated by the g+1 double cosets

$$T(p) = \Gamma\begin{pmatrix} 1_g & 0 \\ 0 & p1_g \end{pmatrix}\Gamma$$

and

$$T_{i,j}(p^2) = \Gamma\left(\begin{array}{cc|cc} 1_i & 0 & & \\ 0 & p1_j & & 0 \\ \hline & & p^2 1_i & 0 \\ & 0 & 0 & p1_j \end{array}\right)\Gamma \qquad (0\leq i<g, i+j=g)$$

where 1_g ... denotes the unit matrix of size g ... (the elements $T(p)$, $T_{i,j}(p^2)$ are algebraically independent). Moreover, one has

$$H_{g,p} \cong \mathbb{C}[X_0^{\pm 1}, \ldots, X_g^{\pm 1}]^W,$$

where W is the Weyl group generated by the permutations of the X_i $(i=1,\ldots,g)$ and the maps $X_0 \mapsto X_0 X_j$, $X_j \mapsto X_j^{-1}$, $X_i \mapsto X_i$ $(1\leq i\leq g, i\neq j)$, for $j\in\{1,\ldots,g\}$. In particular one has

$$\mathrm{Hom}_{\mathbb{C}}(H_{g,p},\mathbb{C}) = (\mathbb{C}^*)^{g+1}/W$$

(with the obvious operation of W on $(\mathbb{C}^*)^{g+1}$).

iii) The Hecke algebra operates on $M_k(\Gamma_g)$ resp. $S_k(\Gamma_g)$ by

$$F|_k\left(\sum a_x \Gamma x\right) = \sum a_x F|_k x$$

where

$$(F|_k x)(Z) = r_x^{gk-g(g+1)/2} \cdot \det(CZ+D)^{-k} F((AZ+B)(CZ+D)^{-1})$$

$(x=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in G$, r_x = scalar factor of x).

The space $S_k(\Gamma_g)$ has a basis of common eigenfunctions of all $T\in H_g$. If F is such an eigenfunction and $F|T = \lambda(T)F$ $(T\in H_g)$, then $T\mapsto\lambda(T)$ is a homomorphism $H_{g,p}\to\mathbb{C}$ for every p, hence by ii) is determined by an element $(\alpha_{0,p},\alpha_{1,p},\ldots,\alpha_{g,p}) \in (\mathbb{C}^*)^{g+1}/W$ ("Satake p-parameters").

In what follows, we are mainly interested in $\lambda(T(p))$, and we will write λ_p for the latter.

2. Problems

The following questions are natural to ask:

A) If F is a Hecke eigenfunction, how does λ_p (and more generally $\lambda(T_{i,j}(p^2))$) grow for $p\to\infty$? What bounds hold for the Satake parameters?

B) Given an arbitary $F\in S_k(\Gamma_g)$, how do the Fourier coefficients a(T) grow? More precisely, one wants bounds of the form $a(T) \ll (\det(T))^c$ (or more generally bounds involving on the right the "successive minima" of T) with c>0 "small" and only depending on k and g.

Of course, when g=1 the theory of eigenvalues and Fourier coefficients coincides, and A) and B) have the same well-known answer given by Deligne's theorem (previously the Ramanujan-Petersson conjecture) which states that $a(n) \ll_\varepsilon n^{k/2-1/2+\varepsilon})$ $(\varepsilon>0)$.

If $g\geq 2$, then in general eigenvalues and Fourier coefficients no longer coincide, and it is not clear how to deduce estimates for all the a(T) from those for the eigenvalues. However, since the Hecke operators act on Fourier coefficients, one may ask:

C) Let $g\geq 2$. What bounds for the Fourier coefficients will imply what bounds for the eigenvalues?

For example, if g=2 then by Andrianov [1] one has (very roughly speaking) $a(pT)$ "=" $a(T)\lambda_p$ $(p\to\infty$, T fixed) hence any estimate $a(T) \ll (\det(T))^c$ will imply an estimate $\lambda_p \ll p^{gc}$. Is anything similar true for arbitrary $g\geq 2$?

3. Facts on A)

We **start** with stating the

Generalized Ramanujan-Petersson conjecture (Satake, 1963). Let **F** be a Hecke eigenfunction in $S_k(\Gamma_g)$. Then

$$|\alpha_{1,p}| = \ldots = |\alpha_{g,p}| = 1 \qquad \text{(for all p)}.$$

In particular one has:

i) $|\lambda_p| \leq 2^g \cdot p^{gk/2-g(g+1)/4}$;

ii) the standard zeta function

$$D_F(s) := \prod_p D_{F,p}(p^{-s})^{-1}$$

(where $D_{F,p}(X) := (1-X)\prod_{\nu=1}^{g}(1-\alpha_\nu X)(1-\alpha_\nu^{-1}X)$, $\alpha_\nu := \alpha_{\nu,p}$) converges absolutely for $\mathrm{Re}(s)>1$;

iii) the spinor zeta function

$$Z_F(s) := \prod_p Z_{F,p}(p^{-s})^{-1}$$

(where $Z_{F,p}(X) := (1-\alpha_0 X)\prod_{\nu=1}^{g}\prod_{1\leq i_1<i_2<\ldots<i_\nu\leq g}(1-\alpha_0\alpha_{i_1}\ldots\alpha_{i_\nu}X)$, $\alpha_\nu := \alpha_{\nu,p}$) converges absolutely for $\mathrm{Re}(s)>\frac{gk}{2}-\frac{g(g+1)}{4}+1$.

Note that i) and iii) follow from the equality $\alpha_{0,p}^2\alpha_{1,p}\ldots\alpha_{g,p}=p^{gk-g(g+1)/2}$ (which is easy to verify) and from the fact that $-\lambda_p$ = coefficient of X in $Z_{F,p}(X)$.

If $g\geq 3$ then the conjecture so far is not known in a single case. If g=2, it is known to be wrong if F is the Saito-Kurokawa lift of an elliptic cusp form $f\in S_{2k-2}(\Gamma_1)$ (k even), cf. [7]. In fact, in the latter case one knows that $Z_F(s)=\zeta(s-k+1)\zeta(s-k+2)\cdot L(f,s)$ where L(f,s) is the Hecke L-function of f, and one has a contradiction to iii).

However, recently Weissauer (1993) as an application of the trace formula for Hecke operators on Shimura varieties proved

Theorem [21]. Let g=2, $F\in S_k(\Gamma_2)$ and suppose that F is not a Saito-Kurokawa lift. Then the generalized Ramanujan-Petersson conjecture is true for F.

In the general case of arbitrary $g\geq 2$ the best bound known so far seems to be

Theorem (Duke-Howe-Li [5], 1992). Let $g\geq 2$ and $F\in S_k(\Gamma_g)$ be a Hecke eigenform. Then

$$|\lambda_p| \leq 2^g \cdot p^{gk/2-\kappa_g}$$

where one can take

$$\kappa_g = \begin{cases} \frac{g(g+1)}{12} & (g\geq 2) \\ \frac{g(g+1)}{8} & (g=2^\nu \text{ for some } \nu\in\mathbb{N}) \\ 1 & (g=2). \end{cases}$$

The proof uses local representation theory and also works for "generic" cusp forms. One knows that the above bound is sharp if $g=2$ and F is a Saito-Kurokawa lift.

4. Facts on B)

As indicated above, for the estimation of eigenvalues powerful tools (such as local representation theory or the trace formula) are at disposal. For the Fourier coefficients the situation is completely different, and so far no "structure theory" for them is known. Concerning estimates, the classical Hecke argument (properly modified) gives the result

$$a(T) \ll (\det(T))^{k/2},$$

and one may ask -being optimistic- what improvements one could expect.

In the case $g=1$, the classical Rankin method (using the analytic properties of the Dirichlet series $\sum_{n\geq 1} |a(n)|^2/n^s$ $(\mathrm{Re}(s)\gg 0)$) gives the Ramanujan-Petersson conjecture for the Fourier coefficients "on the average", i.e.

$$\sum_{n\leq N} |a(n)|^2 \ll_\varepsilon N^{k+\varepsilon} \qquad (\varepsilon>0; N\to\infty).$$

One may imitate the above procedure for $g\geq 2$ with the generalized Rankin-Dirichlet series

$$R_F(s) := \sum_{\{T>0\}/GL_g(\mathbb{Z})} \varepsilon(T)^{-1} |a(T)|^2/(\det(T))^s \qquad (\varepsilon(T) := \#\{U\in GL_g(\mathbb{Z}) \mid T[U]=T\},\ \mathrm{Re}(s)\gg 0).$$

Using its analytic properties (much more difficult to prove than in the case $g=1$) and the class number bound

$$\#\{T>0 \mid \det(T)=n\}/GL_g(\mathbb{Z}) \ll_\varepsilon n^{(g-1)/2+\varepsilon} \qquad (\varepsilon>0)$$

[10,19], and <u>assuming</u> in addition that the $a(T)$ are "equally distributed", one may arrive at the following

<u>Conjecture (Resnikoff-Saldaña [17], 1974)</u>. One has

$$a(T) \ll_\varepsilon (\det(T))^{k/2-(g+1)/4+\varepsilon} \qquad (\varepsilon>0).$$

(The motivation for the conjecture given in [17] seems to be different from the one given here.)

If $g \geq 2$, the conjecture is not known in a single case. One knows that it is wrong if $g=2$ and F is a Saito-Kurokawa lift [3,§2]. Also for higher g with $8|g$ Freitag constructed counter examples of weight $1+\frac{g}{2}$ by looking at certain generalized theta series with the sperical harmonic $G \mapsto \det(G)$ $(G \in \mathbb{Z}^{(g,g)})$, cf. [3,§2].

Let us now briefly discuss what -in fact- has been proved positively.

Theorem (Böcherer-Raghavan [3], Fomenko [8], 1988). One has

$$a(T) \ll_{\varepsilon} (\det(T))^{k/2-\alpha_g+\varepsilon} \qquad (\varepsilon > 0)$$

where

$$\alpha_g^{-1} := 2g+2+4\left[\frac{g}{2}\right]+\frac{2}{g+1}.$$

The proof uses the analytic properties of the Rankin-Dirichlet series $R_F(s)$ defined above and the classical theorem of Landau in the slightly improved version given by Sato-Shintani [18].

Theorem (Kitaoka [11], 1984). Suppose $g=2$. Then

$$a(T) \ll_{\varepsilon} (\det(T))^{k/2-1/4+\varepsilon} \qquad (\varepsilon > 0).$$

The proof uses Poincaré series of exponential type on Γ_2 and estimates for generalized matrix-argument Kloosterman sums which occur in their Fourier expansion.

Theorem (Böcherer-Kohnen [4], 1993). Let $g \geq 2$ and $k > g+1$. Then

$$a(T) \ll_{\varepsilon} (\det(T))^{k/2-1/2g-(1-1/g)\beta_g+\varepsilon} \qquad (\varepsilon > 0)$$

where

$$\beta_g^{-1} := 4(g-1)+4\left[\frac{g-1}{2}\right]+\frac{2}{g+2}.$$

For the proof one expands $F(Z)$ in terms of a Fourier-Jacobi series and then combines in a proper way estimates (uniform w.r.t. the index) for the Fourier coefficients of Jacobi forms and estimates for scalar products of Fourier-Jacobi coefficients. For the case $g=2$ cf. [13].

If $g=2$, using the above estimate one can get a slight improvement upon the error term in an estimate given by Duke [6] for the average of numbers of representations of T by a positive definite even unimodular matrix S.

It should be noted that (since $\alpha_g, \beta_g \to 0$ for $g \to \infty$) the above estimates for $g \to \infty$ still are of the same order of magnitude as Hecke's bound. However, one can show

Theorem ([14], 1994). Suppose that $4 | g$. Then there is $N_0 = N_0(g) \in \mathbb{N}$ with the following property: for every $N \in \mathbb{N}$ there is $k \in \{N, N+1, \dots N+N_0-1\}$ and $F \in S_k(\Gamma_g)$, $F \neq 0$ such that

$$a(T) \ll_\varepsilon (\det(T))^{k/2-1/2+\varepsilon} \qquad (\varepsilon > 0).$$

The functions F are constructed explicitly using generalized theta series of positive definite even unimodular quadratic forms S of rank 2g. To estimate their Fourier coefficients one uses Siegel's main theorem on quadratic forms and its "primitive" variant [2,12,20].

5. Facts on C)

One has the following result

Theorem (Duke-Howe-Li [5], 1992). Let F be a Hecke eigenfunction with Fourier coefficients a(T) and eigenvalues λ_p. Then

$$\lambda_p a(T) - a(pT) \ll (\det(T))^{k/2} p^{gk/2-1}.$$

Choosing T with $a(T) \neq 0$ one obtains

Corollary. Suppose that the bound $a(T) \ll (\det(T))^{k/2-\alpha}$ holds for all $T>0$, where $\alpha>0$ is fixed. Then

$$\lambda_p \ll p^{gk/2-\min\{1,\alpha g\}}.$$

The theorem is proved by looking at the action of T(p) on Fourier coefficients and using the Hecke bound for them.

By studying the action of T(p) on the Fourier coefficients in much more detail as in [16] (in particular making use of the cancellation of certain exponential sums) one can deduce under the assumption that $a(T) \ll (\det(T))^{k/2-\alpha}$ a much better estimate (depending on α) for the λ_p than given in the Corollary. For details we refer to [15]. It should be noted, however, that the bounds implied for the eigenvalues in this way (using the results quoted in sect. 4) are much weaker than the bounds discussed in sect. 3.

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