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ASYMPTOTIC SERIES FOR DOUBLE ZETA AND DOUBLE GAMMA FUNCTIONS OF BARNES

Kohji Matsumoto(*) (松本耕二)

Department of Mathematics, Faculty of Education,
Iwate University, Ueda, Morioka 020, Japan (岩手大教育)

The double zeta-function with positive parameters α, w is defined by

$$(1) \qquad \zeta_2(v;\alpha,w)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha+m+nw)^{-v}$$

for any complex variable v. We can notice the resemblance of $\zeta_2(v;\alpha,w)$ to the function

$$(2) \qquad f(u,v;\alpha)=\sum_{m=0}^{\infty}(\alpha+m)^{-u}\sum_{n=1}^{\infty}(\alpha+m+n)^{-v},$$

where u, v are independent complex variables and $\alpha>0$, which has appeared in the study of M.Katsurada and the author([2][3][4]) on certain mean square of Hurwitz zeta-functions. In view of this resemblance, it seems natural to introduce

$$(3) \qquad \zeta_2(u,v;\alpha,w)=\sum_{m=0}^{\infty}(\alpha+m)^{-u}\sum_{n=1}^{\infty}(\alpha+m+nw)^{-v},$$

(*)Present address: Graduate School of Polymathematics, Nagoya University, Chikusa-ku, Nagoya 464-01, Japan

which is a generalization of both (1) and (2). In fact, we see that $\zeta_2(u,v;\alpha,1)=f(u,v;\alpha)$, and on the other hand

$$\zeta_2(v;\alpha,w)=\zeta(v,\alpha)+\zeta_2(0,v;\alpha,w),$$

where $\zeta(v,\alpha)$ is the Hurwitz zeta-function with the parameter α. The function $f(u,v;\alpha)$ has been studied in detail in the afore-mentioned works of Katsurada and the author. Therefore, studying $\zeta_2(u,v;\alpha,w)$ by the same method and then putting $u=0$, we may expect to deduce some information on the double zeta-function $\zeta_2(v;\alpha,w)$. Carrying out this procedure, we can indeed prove the following asymptotic expansions of double zeta-functions.

THEOREM 1. For any positive integer N and any complex v with $\mathrm{Re}(v)>-N+1$ and $v\notin\{2,1,0,-1,-2,-3,\ldots\}$, we have

$$\zeta_2(v;\alpha,w)=\zeta(v,\alpha)+\frac{\zeta(v-1)}{v-1}w^{1-v}$$
$$+\sum_{n=0}^{N-1}\binom{-v}{n}\zeta(-n,\alpha)\zeta(v+n)w^{-v-n}+O_{v,N}(w^{-\mathrm{Re}(v)-N}),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

The double gamma-function $\Gamma_2(\alpha,(1,w))$ of Barnes is defined by

$$\log\left(\frac{\Gamma_2(\alpha,(1,w))}{\rho_2(1,w)}\right)=\zeta_2'(0;\alpha,w),$$

where

$$-\log\rho_2(1,w)=\lim_{\alpha\to 0}(\zeta_2'(0;\alpha,w)+\log\alpha).$$

The theory of double gamma-functions has a long history, but it was extensively studied by Barnes at the beginning of the 20th century, and therefore we call double gamma-functions with his name. In his research Barnes introduced double zeta-functions, and studied their properties, too.

From Theorem 1 we can deduce the asymptotic expansion of $\zeta_2'(0;\alpha,w)$, and from which we can prove

THEOREM 2. For any $N \geqq 2$, we have

$$\log \Gamma_2(\alpha, (1,w)) = -(\alpha/2)\log w + \log \Gamma(\alpha) + (\alpha/2)\log 2\pi$$
$$+(\zeta(-1,\alpha)-\zeta(-1))w^{-1}\log w - (\zeta(-1,\alpha)-\zeta(-1))\gamma w^{-1}$$
$$+\sum_{n=2}^{N-1}\frac{(-1)^n}{n}(\zeta(-n,\alpha)-\zeta(-n))\zeta(n)w^{-n} + O_N(w^{-N}(|\log w|+1)),$$

where γ denotes Euler's constant.

Remark. It is possible to show some different type of asymptotic series for $\zeta_2(v;\alpha,w)$ and $\log \Gamma_2(\alpha, (1,w))$, in which the error estimates are given by strict inequalities.

Nowadays double zeta and double gamma functions play important roles in number theory and also in mathematical physics. Therefore we may expect that the above theorems would have various applications in these areas. We conclude this note with mentioning an application to the Hecke L-function $L_F(s,\chi)$ of a real quadratic field F, associated with a certain type of character χ. Shintani[6] expressed the special value $L_F(1,\chi)$ in terms of double gamma-functions. Combining Shintani's result with our Theorem 2, we can deduce a kind of asymptotic series for $L_F(1, \chi)$ with respect to the fundamental unit $\varepsilon > 1$. We omit the statement here, only noting that in this application it is essential that the error estimate is independent of α in Theorem 2. Arakawa[1] proved that certain special values of $L_F(s, \chi)$ with a certain Grössencharacter χ can be written in terms of double zeta-functions, so, using our Theorem 1, we can similarly show a kind of asymptotic series for these special values. The proofs of Theorems 1 and 2, and the statements of the other results mentioned in this note, are given in the author's preprint[5].

References

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