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Iteration of some birational polynomial quadratic maps of $\mathbb{P}^2$

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1 Introduction

Recently, several authors (for example, J. Hubbard and P. Papadopol [HP], J. E. Fornaess and N. Sibony [FS3], [FS4], T. Ueda [U2], [U3]) began to construct the general theory of the iteration of rational maps of $\mathbb{P}^2$ or $\mathbb{P}^n$ with $n \geq 2$. Some examples were also investigated by [FS2] and [U1]. In this note, we study further examples of rational maps of $\mathbb{P}^2$.

Let us take and fix a homogeneous coordinate system $z : w : t$ of $\mathbb{P}^2$. For a rational map r of $\mathbb{P}^2$ given by $[z : w : t] \to [R_0 : R_1 : R_2]$, where R_i (i=0,1,2) are homogeneous polynomials of degree d without commom factor, $p_0 = [z_0 : w_0 : t_0]$ is a point of indeterminacy if $R_i(p_0) = 0$ $(i = 0, 1, 2)$. The set of all points of indeterminacy of r is denoted by $I(r)$.

When $I(r) \neq \emptyset$ we always mean, by $r(p) = q$, that $p \in \mathbb{P}^2 \setminus I(r)$ and $r(p) = q$. We also mean, by $r^{-1}(A)$ where $A \subset \mathbb{P}^2$, the set $\{p \in \mathbb{P}^2 \setminus I(r); r(p) \in A\}$. When we write $r(A)$, the set A is assumed to be $A \subset \mathbb{P}^2 \setminus I(r)$.

The iteration of r is the study of the orbit $\{r^n(p); n \in \mathbb{Z}, n \geq 0\}$ of a point $p \in \mathbb{P}^2$. When we have $r^n(p) \in I(r)$ for a point $p \in \mathbb{P}^2 \setminus I(r)$ and for some $n \geq 1$, we do not consider $r^m(p)$ for $m > n$. Set $E_1(r) = I(r)$. Inductively on n, we define

$$E_n(r) = E_{n-1}(r) \cup \{p \in \mathbb{P}^2 \setminus E_{n-1}(r); r^{n-1}(p) \in I(r)\}$$

for $n \geq 2$. Then, $E_n(r) \subset E_{n+1}(r)$. Let $E(r) = \bigcup_{n=1}^{\infty} E_n(r)$. Then, $E(r) = \{p \in \mathbb{P}^2; r^n(p) \in I(r)$ for some $n \geq 0\}$. We call the closure $\overline{E(r)}$ the extended indeterminacy set. A point p is said to belong to the Fatou set $\mathcal{F}(r)$ of r if there exists an open neighborhood U of p such that the family $\{r^n; n \geq 0\}$ is equicontinuous in $U \setminus E(r)$. The complement of $\mathcal{F}(r)$ is called the Julia set $\mathcal{J}(r)$ of r. By definition, the Fatou set is an open set and $\overline{\cup_{n=1}^{\infty} I(r^n)} \subset \mathcal{J}(r)$.

We want to deal with the birational polynomial quadratic maps of $\mathbb{P}^2$. We always identify the set $\{t \neq 0\} \subset \mathbb{P}^2$ with $\mathbb{C}^2$. Then, our maps are written in the following form:

$$r: \ z_1 = R_0, \ w_1 = R_1, \ t_1 = R_2 = t^2,$$

where R_0 and R_1 are homogeneous polynomials of degree= 2. Here the equation $R_2 = t^2$ corresponds to the assumption that the r is a polynomial map. We assume that R_0 and R_1 do not have the common factor t (that is, $t \nmid R_0$ or $t \nmid R_1$) and that r is birational.

We denote by i the number of the elements of the set $I(r)$, and by f the number of the fixed points of r located in the line at infinity $\{t = 0\}$, where a point $p \in \mathbb{P}^2$ is called a fixed point of r if $r(p) = p$.

Then, we have the following classification result.

Proposition 1.1 *According to i and f, the birational polynomial quadratic maps of $\mathbb{P}^2$ are classified into the following 4 classes A, B, C, and D. Considering the conjugation by projective linear transformations as the equivalence relation, the representatives of each class are given by the maps defined by the following (R_0, R_1, R_2).*

Class	*i and f*	*Representatives*
A	$i=2, f=1$	$(wt, w^2 - azw + bwt + ct^2, t^2)$ $(a \neq 0)$
B	$i=2, f=0$	$(azt + bt^2, zw + t^2, t^2)$ $(a \neq 0)$ $(azt + bt^2, zw, t^2)$ $(a \neq 0)$ $(wt + bt^2, zw + ct^2, t^2)$
C	$i=1, f=1$	$(wt, w^2 - \delta zt + \gamma t^2, t^2)$ $(\delta \neq 0)$
D	$i=1, f=0$	$(w^2 + a_1 zt, a_2 wt, t^2)$ $(a_1, a_2 \neq 0)$ $(w^2 + t^2 + zt, awt, t^2)$ $(a \neq 0)$ $(w^2 + azt, t^2 + awt, t^2)$ $(a \neq 0)$ $(w^2 + azt, t^2 + wt, t^2)$ $(a \neq 0, 1)$

The maps r in the class C are called the Hénon maps. The restriction $r_{|\mathbb{C}^2}$ of a map r to $\mathbb{C}^2$ is an automorphism of $\mathbb{C}^2$. There are already extensive studies of the iteration of the Hénon maps, or more gereral polynomial automorphisms of $\mathbb{C}^2$ from the point of view of complex analysis (for examples, [H],[HO],[FM],[FS1],[B],[BS1],[BS2],[BS3],[BS4],[BLS]).

The restriction $r_{|\mathbb{C}^2}$ of a map r of the class D to $\mathbb{C}^2$ is also an automorphism of $\mathbb{C}^2$. The maps $r_{|\mathbb{C}^2}$ in the class D belong to the class of the elementary maps in the sense of [FM] and were studied in [FM].

We are intend to study the maps in the classes A and B. In this note, we deal with the first family of maps in the class B. We always denote by φ the rational map

$$\varphi : [z : w : t] \to [azt + bt^2 : zw + t^2 : t^2] \tag{1.1}$$

and by ψ the inverse of φ given by

$$\psi : [z : w : t] \to [(z - bt)^2 : a^2(w - t)t : a(z - bt)t], \tag{1.2}$$

where a and b are complex numbers with $a \neq 0$.

In the $x = \frac{z}{t}, y = \frac{w}{t}$ coordinates in $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{t = 0\}$, we have $\varphi : (x, y) \to (ax + b, xy + 1)$. So, the family $\bigcup_c \{x = c\}$ is invariant under φ. Hence, the problem of studing the iteration of φ and ψ is rather simple. We can deal with some dynamical objects quite concretely.

2 Fundamental properties of φ and ψ

Let us state the fundamental properties of our maps φ in (1.1) and ψ in (1.2). Let $I_1 = J_1 = [0 : 1 : 0]$, $I_2 = [1 : 0 : 0]$ and $J_2 = [b : 1 : 1]$. We can easily see that $I(\varphi) = \{I_1, I_2\}$ and $I(\psi) = \{J_1, J_2\}$.

For a rational map r of $\mathbb{P}^2$ and for a curve C, that is, an irreducible algebraic subset of dimension 1 of $\mathbb{P}^2$ with finite number of points deleted, C is said an r−constant curve if $r(C)$ is a point.

There are two φ−constant curves $C_1 = \{t = 0\} \setminus \{I_1, I_2\}$ and $C_2 = \{z = 0\} \setminus \{I_1\}$. There are two ψ−constant curves $D_1 = \{z - bt = 0\} \setminus \{J_1, J_2\}$ and $D_2 = \{t = 0\} \setminus \{J_1\}$.

We have $\varphi^{-1}(D_1 \cup D_2) = \emptyset$ and $\psi^{-1}(C_1 \cup C_2) = \emptyset$, while $\varphi^{-1}(p) \neq \emptyset$ for any $p \in \mathbb{P}^2 \setminus (D_1 \cup D_2)$ and $\psi^{-1}(p) \neq \emptyset$ for any $p \in \mathbb{P}^2 \setminus (C_1 \cup C_2)$.

In the following proposition, we assume $a \neq 1$ and let $c = \frac{b}{1-a}$.

Proposition 2.1 *For φ, we have $E(\varphi) = \overline{E(\varphi)} = \{t = 0\}$. For ψ, we have*

$$E_n(\psi) = \cup_{k=1}^{n-1}\{z = (c - ca^k)t\} \cup \{\varphi^{n-1}(J_2)\} \quad \text{for} \quad n \geq 2, \tag{2.3}$$

$$E(\psi) = \cup_{n=1}^{\infty}\{z - (c - ca^n)t = 0\}.$$

Hence $\overline{E(\psi)} = E(\psi) \cup \{t = 0\}$ when $|a| > 1$, *and* $\overline{E(\psi)} = E(\psi) \cup \{z - ct = 0\}$ when $|a| < 1$.

Proof. The assertion on $E(\varphi)$ is obvious. By definition, $E_1(\psi) = I(\psi) = \{J_1, J_2\}$. We have

$$\begin{aligned} E_2(\psi) &= E_1(\psi) \cup \{p \in \mathbb{P}^2 \setminus E_1(\psi); \psi(p) \in I_1(\psi)\} \\ &= \{J_1, J_2\} \cup (\{z = (c - ca)t\} \setminus \{J_1, J_2\}) \cup \{\varphi(J_2)\} \\ &= \{z = (c - ca)t\} \cup \{\varphi(J_2)\}. \end{aligned}$$

Inductively, the assertion for $E_n(\psi)$ is proved. Then, the remaining assertions follow immediately. □

In general, let U be an open neighborhood of a point p in $\mathbb{C}^2$ and let $h : U \to \mathbb{C}^2$ be a holomorphic map with a fixed point p. The problem of canonical form of the map h is to seek for a neighborhood V of the origin in $\mathbb{C}^2$ and an injective holomorphic map $S : V \to U$ with $h(0) = p$ such that $S^{-1}hS : V \to \mathbb{C}^2$ is described as simple as possible. The map S is called a conjugation map.

Let λ, μ be two eigenvalues of the differential $dh(p)$ at p. In this note, we are specially interested in the canonical form around p of the following types of fixed point. The canonical form of (1) or (2) was decided by Lattès [LAT]. Let us denote by $\mathbb{N}$ the set of positive integers.

Definition 2.2 *(1) $0 < |\lambda| < 1$, $0 < |\mu| < 1$ and $\lambda \neq \mu^n$, $\mu \neq \lambda^n$ for all $n \in \mathbb{N}$. In this case, there is a conjugation map S such that $S^{-1}hS(\tau, \sigma) = (\lambda\tau, \mu\sigma)$.*

(2) $0 < |\lambda| < 1$, $0 < |\mu| < 1$ and $\mu = \lambda^N$ for some $N \in \mathbb{N}$. In this case, there is a conjugation map S such that either $S^{-1}hS(\tau, \sigma) = (\lambda\tau, \mu\sigma)$ or $S^{-1}hS(\tau, \sigma) = (\lambda\tau, \mu\sigma + \tau^N)$. We will call the former of type (2-1) and the latter of type (2-2).

(3) $0 < |\lambda| < 1$ and $\mu = 0$. It seems that the problem of the canonical form has not yet been solved for this type of the fixed point. So, we can not refer to any general result.
The fixed point p of type (1) or (2) is called attracting and p of type (3) is called semi super attracting.

Now we return to our maps φ and ψ and we suppose, say, that $|a| > 1$. Then we see that, for "almost" all points p of $\mathbb{P}^2$, $\varphi^n(p)$ tend to the point I_1. So, despite $I_1 \in I(\varphi)$ is not an attracting fixed point of φ, it behaves like such. This motivates us to consider the blowing up $\pi : M \to \mathbb{P}^2$ centered at the point I_1. We consider the lifts $\tilde{\varphi} : M \to M$ and $\tilde{\psi} : M \to M$ of φ and ψ. The strict transform $\overline{\pi^{-1}(\{z = \alpha t\} \setminus \{I_1\})}$ of $\{z = \alpha t\}$ is denoted by B_α for $\alpha \in \mathbb{C}$ and the strict transform of $\{t = 0\}$ by B_∞.

Definition 2.3 *In order to fix the notation, we set* $\Omega_1 = M \setminus B_0 \cong \mathbb{C}(\xi) \times \mathbb{P}^1(\eta)$ *where* η *is an inhomogeneous coordinate of* $\mathbb{P}^1$ *and the* π *restricted to* Ω_1 *is given by* $\frac{t}{z} = \xi, \frac{w}{z} = \frac{1}{\eta}$. *We set* $\Omega_2 = M \setminus B_\infty \cong \mathbb{C}(x) \times \mathbb{P}^1(y)$, *where we regard* y *as an inhomogeneous coordinate of* $\mathbb{P}^1$. *Then,* $\mathbb{C}^*(\xi) \times \mathbb{P}^1(\eta) \cong \Omega_1 \cap \Omega_2 \cong \mathbb{C}^*(x) \times \mathbb{P}^1(y)$, *where the transformation of two coordinate systems* (x, y) *and* (ξ, η) *is given by* $x = \frac{1}{\xi}, y = \frac{1}{\xi\eta}$.

Let $A = \pi^{-1}(I_1)$ be the exceptional set. Then, $\tilde{I}_3 := (x = 0, y = \infty)$ is the unique point of indeterminacy in A of $\tilde{\varphi}$, and $\tilde{J}_3 := (\xi = 0, \eta = 0)$ is the unique point of indeterminacy in A of $\tilde{\psi}$.

Let us suppose that $a \neq 1$ and let $c = \frac{b}{1-a}$ as in Proposition 2.1. We also suppose that $c \neq 1$ and $c \neq 0$. Then, in the whole M, $\tilde{\varphi}$ have three distinct fixed points $\tilde{F} = (x = c, y = \infty)$, $\tilde{P} = (x = c, y = \frac{1}{1-c})$ and $\tilde{J}_3 = (\xi = 0, \eta = 0)$ at each point of which the two eigenvalues of the differential $d\tilde{\varphi}$ are $\{a, \frac{1}{c}\}$,$\{a, c\}$ and $\{\frac{1}{a}, 0\}$ respectively. In the whole M, $\tilde{\psi}$ have three distinct fixed points $\tilde{F}$, $\tilde{P}$, and $\tilde{I}_2 = (\xi = 0, \eta = \infty)$ at each point of which the two eigenvalues of the differential $d\tilde{\psi}$ are $\{\frac{1}{a}, c\}$,$\{\frac{1}{a}, \frac{1}{c}\}$ and $\{a, 0\}$ respectively. We set $\tilde{J}_2 = \pi^{-1}(J_2) = (x = b, y = 1)$.

In this note we only deal with the maps φ and ψ with generic parameter values (a, c). We divide our description into the 4 cases and treat them in 2 sections of the rest of this note: §3, $|a| < 1, 0 < |c| < 1$ and $|a| < 1, |c| > 1$, §4, $|a| > 1, 0 < |c| < 1$ and $|a| > 1, |c| > 1$.

In each of these cases, each of $\tilde{\varphi}$ and $\tilde{\psi}$ has only one fixed point of the types in Definition 2.2 among the points $\tilde{F}$, $\tilde{P}$, $\tilde{J}_3$ and $\tilde{I}_2$. We seek for the canonical form and the global conjugation mapping for this fixed point. By falling down on $\mathbb{P}^2$, we can decide concretely the Julia sets of φ and ψ.

Our results in §3 and §4 are summarized in the following table.

	$\mathcal{J}(\varphi)$	$\mathcal{J}(\psi)$
$\|a\| < 1, 0 < \|c\| < 1$	$\overline{E(\varphi)}$	$\overline{E(\psi)}$
$\|a\| < 1, \|c\| > 1$	$\overline{W^s(\varphi, P)} \cup \overline{E(\varphi)}$	
$\|a\| > 1, 0 < \|c\| < 1$	$\overline{W^u(\psi, I_2)} \cup \overline{\bigcup_{n=1}^{\infty} \varphi^{-n}(J_2)}$ when $a^m c = 1$ by some $m \in \mathbb{N}$ $\overline{W^u(\psi, I_2)}$ when $a^m c \neq 1$ for all $m \in \mathbb{N}$	$\overline{W^s(\psi, P)}$
$\|a\| > 1, \|c\| > 1$	$\overline{W^u(\psi, I_2)}$	$\overline{E(\psi)}$

Here, $W^s(\varphi, P)$ is the stable curve of φ at the fixed point $P = \pi(\tilde{P})$ of saddle type, $W^s(\psi, P)$ is the stable curve of ψ at P, and $W^u(\psi, I_2)$ is the unstable curve of ψ at I_2. Since C_2 is the $\varphi-$constant curve with $\varphi(C_2) = \{J_2\}$, each $\varphi^{-n}(J_2)$ is a φ^n-constant curve.

3 φ and ψ when $0 < |a| < 1$

We assume that $0 < |a| < 1$ throughout this section. Though we deal with the cases $0 < |c| < 1$ and $|c| > 1$, some lemmas and propositions in this section hold under the weaker assumption on c.

Lemma 3.1 *Let $c \neq 0$.*

(1) The radius of convergence of the following power series are ∞ and so the functions $K(\zeta)$, $B(\zeta)$ and $Y(\zeta)$ are entire functions:

$$K(\zeta) = 1 + \textstyle\sum_{m=1}^{\infty} \frac{(-1)^m a^{m(m-1)/2} c^m}{(a-1)\cdots(a^m-1)} \zeta^m,$$

$$B(\zeta) = \textstyle\sum_{m=1}^{\infty} \frac{(-1)^{m-1} a^{m(m-1)/2}}{(a^m-c)c^{m-1}(a-1)\cdots(a^{m-1}-1)} \zeta^m,$$

$$Y(\zeta) = \textstyle\sum_{m=1, m\neq N}^{\infty} \frac{(-1)^{m-1} a^{m(m-1)/2}}{(a^m-c)c^{m-1}(a-1)\cdots(a^{m-1}-1)} \zeta^m,$$

where we assume $a^m \neq c$ for all $m \in \mathbb{N}$ for $B(\zeta)$, and $c = a^N$ by some $N \in \mathbb{N}$ for $Y(\zeta)$.

(2) The functions $K(\zeta)$, $B(\zeta)$ and $Y(\zeta)$ satisfy the following functional equations:

$$K(\zeta) = (1 + c\zeta)K(a\zeta), \tag{3.4}$$

$$B(a\zeta) - cB(\zeta) - \zeta K(c^{-2}a\zeta) = 0, \tag{3.5}$$

$$Y(a\zeta) - cY(\zeta) - \zeta K(c^{-2}a\zeta) = \tfrac{-(-1)^{N-1} a^{N(N-1)/2}}{c^{N-1}(a-1)\cdots(a^{N-1}-1)} \zeta^N, \tag{3.6}$$

where we assume $a^m \neq c$ for all $m \in \mathbb{N}$ in (3.5) and $c = a^N$ by some $N \in \mathbb{N}$ in (3.6).

Proof. Let ρ_m be the coefficient of the ζ^m term of the power series of $K(\zeta)$. Then, $\frac{|\rho_m|}{|\rho_{m-1}|} = \frac{|a^{m-1}c|}{|a^m-1|} \to 0$ as $m \to 0$, so the radius of convergence of $K(\zeta)$ is ∞. Similarly, the radii of convergence of the other series are ∞.

The equation (3.5) is proved as

$$\begin{aligned} cB(\zeta) - B(a\zeta) &= \textstyle\sum_{m=1}^{\infty} \frac{(c-a^m)(-1)^{m-1} a^{m(m-1)/2}}{(a^m-c)c^{m-1}(a-1)\cdots(a^{m-1}-1)} \zeta^m \\ &= \zeta \textstyle\sum_{m=1}^{\infty} \frac{-(-1)^{m-1} a^{m(m-1)/2}}{c^{m-1}(a-1)\cdots(a^{m-1}-1)} \zeta^{m-1} \\ &= -\zeta \textstyle\sum_{m=0}^{\infty} \frac{(-1)^m a^{m(m-1)/2}}{c^m(a-1)\cdots(a^m-1)} a^m \zeta^m = -\zeta K(c^{-2}a\zeta). \end{aligned}$$

We can verify the other functional equations quite similarly. □

Lemma 3.2 *Let $c \neq 0$. Around the origin, the function $\frac{1}{K(\zeta)}$ has the power series expansion*

$$\frac{1}{K(\zeta)} = 1 + \sum_{m=1}^{\infty} \frac{c^m}{(a-1)\cdots(a^m-1)} \zeta^m$$

whose radius of convergence is equal to $\frac{1}{|c|}$. The zeros of $K(\zeta)$ are $\{\frac{-1}{ca^n}; n \in \mathbb{Z}, n \geq 0\}$ and they are all simple zeros.

Proof. It is easy to check the first statement. Therefore $K(\zeta)$ has no zero in $\{|\zeta| < \frac{1}{|c|}\}$. In view of the equation (3.4), in $\{|\zeta| < \frac{1}{|ac|}\}$, $K(\zeta)$ has the unique zero $\frac{-1}{c}$ which is simple. Inductively, the zeros of $K(\zeta)$ in $\{|\zeta| < \frac{1}{|c||a|^n}\}$ are $\{\frac{-1}{ca^m}; 0 \leq m \leq n-1\}$ and they are all simple. □

First we study the map $\tilde{\varphi}$ around the point $\tilde{P}$. The eigenvalues of $d\tilde{\varphi}(\tilde{P})$ are $\{a, c\}$ hence, under the assumption $0 < |c| < 1$, $\tilde{P}$ is attracting. According to a and c, $\tilde{P}$ is either of type (1) or (2) in the sense of Definition 2.2. Let us seek for the conjugation map S concretely. Using the notation in Definition 2.3, set $p = x - c$ and $q = y - \frac{1}{1-c}$. Then $U_P := \Omega_2 \cong \mathbb{C}(p) \times \mathbb{P}^1(q)$. We set $V_P = (\mathbb{C}(p) \setminus \bigcup_{n=0}^{\infty} \{p = \frac{-c}{a^n}\}) \times \mathbb{P}^1(q)$. Note that $\tilde{J}_2 = (p = -ac, q = \frac{-c}{1-c}) \in V_P$. In the (p,q) coordinates, the restriction of $\tilde{\varphi}$ to V_P defines a holomorphic map $\tilde{\varphi} : V_P \to V_P$ given by $\tilde{\varphi} : p_1 = ap, q_1 = cq + \frac{p}{1-c} + pq$, and it satisfies $\pi \circ \tilde{\varphi} = \varphi \circ \pi$ on V_P.

First, we consider the case where $\tilde{P}$ is of type (1) or of type (2) with $a = c^M$ by some $M \in \mathbb{N}$ in Definition 2.2. We remark here that, when $a = c^M$, $\tilde{\varphi}$ does not have the canonical form $(\tau, \sigma) \to (a\tau + \sigma^M, c\sigma)$, since $\{p = 0\}$ is an invariant curve in the direction of the eigenvalue c of $d\tilde{\varphi}$ while the map $(\tau, \sigma) \to (a\tau + \sigma^M, c\sigma)$ does not have an invariant curve in the direction of c. We also remark that, in the next proposition, we do not assume $0 < |c| < 1$. Set $W_P = (\mathbb{C}(\tau) \setminus \bigcup_{n=0}^{\infty} \{\tau = \frac{-c}{a^n}\}) \times \mathbb{P}^1(\sigma)$.

Proposition 3.3 *Suppose $c \neq 0, 1$ and $a^m \neq c$ for all $m \in \mathbb{N}$, and define $S : W_P \to V_P$ by*

$$S : p = \tau, q = \sigma K(c^{-2}\tau)^{-1} + \tfrac{1}{1-c} B(\tau) K(c^{-2}\tau)^{-1}.$$

Then S is a surjective biholomorphic map and $S^{-1}\tilde{\varphi}S$ is of the form $\tau_1 = a\tau, \sigma_1 = c\sigma$.

Proof. Using the equations (3.4) and (3.5),

$$\begin{aligned}
\sigma_1 &= q_1 K(c^{-2}p_1) - \tfrac{1}{1-c} B(p_1) \\
&= cqK(c^{-2}p) + \tfrac{1}{1-c} pK(c^{-2}ap) - \tfrac{1}{1-c} B(ap) \\
&= \sigma c + \tfrac{c}{1-c} B(\tau) + \tfrac{1}{1-c} \tau K(c^{-2}a\tau) - \tfrac{1}{1-c} B(a\tau) = c\sigma.
\end{aligned}$$

By Lemma 3.2, $K(c^{-2}\tau) \neq 0$ in W_P, which shows that S is biholomorphic. □

Next, we consider the case where $\tilde{P}$ is of type (2) with $c = a^N$ by some $N \in \mathbb{N}$. The next proposition shows that $\tilde{P}$ is in fact of type (2-2). We set

$$C_N = \frac{(-1)^{N-1}}{a^{N(N-1)/2}(a-1)\cdots(a^{N-1}-1)}. \tag{3.7}$$

Proposition 3.4 *Suppose that $c = a^N$ by some $N \in \mathbb{N}$ and define $S : W_P \to V_P$ by*

$$S : p = \tau, q = \tfrac{C_N}{1-c}\sigma K(c^{-2}\tau)^{-1} + \tfrac{1}{1-c}Y(\tau)K(c^{-2}\tau)^{-1}.$$

Then S is a surjective biholomorphic map and $S^{-1}\tilde{\varphi}S$ is of the form $\tau_1 = a\tau, \sigma_1 = c\sigma + \tau^N$.

Proof. By Lemma 3.2, it is proved that S is biholomorphic. The form $S^{-1}\tilde{\varphi}S$ can be proved directly by using the equations (3.4) and (3.6). In the rest of this paper, we exhibit many "canonical forms" and conjugation maps. The verification of these assertion are quite straightforward. So, we only indicate the lemmas or propositions which are used and omit the detailed computation. □

We will continue to study the map $\tilde{\varphi}$ around the point $\tilde{P}$. Now we assume that $|c| > 1$, so $\tilde{P}$ is a saddle point. Since we have $a^m \neq c$ for all $m \in \mathbb{N}$, we can apply Proposition 3.3. Let $\tilde{\gamma}_1$ be a curve in U_P defined by

$$q = \tfrac{1}{1-c}B(p)K(c^{-2}p)^{-1}. \tag{3.8}$$

Then Proposition 3.3 shows that, in the neighborhood V_P, $\tilde{\gamma}_1$ is the local stable curve of $\tilde{\varphi}$ at $\tilde{P}$. We will show that $\tilde{J}_2 \notin \tilde{\gamma}_1$.

Lemma 3.5 *Suppose $c \neq 0$, and $a^m \neq c$ for all $m \in \mathbb{N}$, and let*

$$j(\zeta) = 1 + \textstyle\sum_{m=1}^{\infty} \frac{1}{(a-c)\cdots(a^m-c)}\zeta^m.$$

Then the radius of convergence of $j(\zeta)$ is equal to $|c|$. Set $\beta = j(-ca) + c - 1$. Then, we have $\beta \neq 0$.

Proof. Since the first statement is easy, we will only show the second statement. First, we can see easily that

$$\beta = j(-ca) + c - 1 = \tfrac{(-1)^{n-1}c^n}{(a-c)\cdots(a^{n-1}-c)} + \textstyle\sum_{m=n}^{\infty} \frac{(-ca)^m}{(a-c)\cdots(a^m-c)}.$$

Set $\mu_n = \frac{(-1)^{n-1}c^n}{(a-c)\cdots(a^{n-1}-c)}$. Then, $\mu_n = c(1 + \frac{-a}{a-c})\cdots(1 + \frac{-a^{n-1}}{a^{n-1}-c})$. Since the series $\sum_n |\frac{a^n}{a^n-c}|$ converges, $\beta = \lim_{n\to\infty} \mu_n \neq 0$. □

Lemma 3.6 *Suppose $c \neq 0$, and $a^m \neq c$ for all $m \in \mathbb{N}$. Then, it holds*

$$B(\zeta) = (j(\zeta) - 1)K(c^{-2}\zeta) \text{ in } \{|\zeta| < |c|\}.$$

Proof. This can be proved by comparing the power series expansions around the origin of both sides. □

Proposition 3.7 *Suppose that $|c| > 1$. Then $\tilde{J}_2 \notin \tilde{\gamma}_1$.*

Proof. Since $\tilde{J}_2 = (p = -ca, q = \frac{-c}{1-c})$, and that $K(\frac{-a}{c}) \neq 0$ by Lemma 3.2, we have

$$\tilde{J}_2 \in \tilde{\gamma}_1 \text{ iff } B(-ca) + cK(\tfrac{-a}{c}) = 0.$$

On the other hand, by Lemma 3.6, $B(-ca) = (j(-ca) - 1)K(\frac{-a}{c})$. Therefore, by Lemma 3.5,

$$B(-ca) + cK(\tfrac{-a}{c}) = (\beta - c)K(\tfrac{-a}{c}) + cK(\tfrac{-a}{c}) = \beta K(\tfrac{-a}{c}) \neq 0. \ \square$$

Next we will study the map $\tilde{\varphi}$ around the fixed point $\tilde{F}$. The eigenvalues of $d\tilde{\varphi}(\tilde{F})$ are $\{a, \frac{1}{c}\}$, hence $\tilde{F}$ is attracting when $|c| > 1$. Let us seek for the conjugation map S concretely. Using the notation in Definition 2.3, set $f = x-c$ and $g = \frac{1}{y}$. Then $U_F := \Omega_2 \cong \mathbb{C}(f)\times\mathbb{P}(g)$. We set $V_F = U_F \backslash (\bigcup_{n=0}^{\infty}\{f = \frac{-c}{a^n}\})$. Note that $\tilde{J}_2 = (f = -ca, g = 1) \in V_F$. In the (f,g) coordinates, the restriction of $\tilde{\varphi}$ to V_F defines a holomorphic map $\varphi : V_F \to V_F$ given by $\tilde{\varphi} : f_1 = af, g_1 = \frac{g}{f+g+c}$, and it satisfies $\pi \circ \tilde{\varphi} = \varphi \circ \pi$ on V_F.

We remark here that, when $\tilde{F}$ is of type (2), that is, when $ac^n = 1$ or $a^n c = 1$ by some $n \in \mathbb{N}$, only the type (2-1) can occur since $\tilde{\psi}$ has two invarinat curves $\{f = 0\}$ and $\{g = 0\}$ through $\tilde{F}$. This fact is also proved by the next proposition. When $|c| > 1$, the assumption of the next proposition is fulfilled. Set $W_F = \mathbb{C}(\tau) \backslash \bigcup_{n=0}^{\infty}\{\tau = \frac{-c}{a^n}\}) \times \mathbb{P}^1(\sigma)$.

Proposition 3.8 *Suppose $c \neq 0$ and $a^m \neq c$ for all $m \in \mathbb{N}$, and define $S : W_F \to V_F$ by*

$$S : f = \tau, g = \frac{\sigma(1-c)K(c^{-2}\tau)}{\sigma(B(\tau)+K(c^{-2}\tau))-(1-c)}.$$

Then, S is a surjective holomorphic map and $S^{-1}\tilde{\varphi}S$ is of the form $\tau_1 = a\tau, \sigma_1 = \frac{\sigma}{c}$.

Proof. We use Lemma 3.2 in order to show that S is biholomorphic. The verification of the canonical form is performed by using the equations (3.4) and (3.5). $\square$

Now, we will turn to consider the map $\tilde{\psi}$ and treat with the problem of the canonical form around the point $\tilde{I}_2$. Using the notation in Definition 2.3, set $u = \frac{a^2\xi}{1-c\xi}$, $v = \frac{1}{\eta}$. Then we have

$$U_I := M \backslash (B_c \cup B_0) \cong (\mathbb{C}(u) \backslash \{u = \tfrac{-a^2}{c}\}) \times \mathbb{P}^1(v),$$

where B_c is the strict transform by $\pi : M \to \mathbb{P}^2$ of $\{z - ct = 0\}$. We set

$$V_I = (\mathbb{C}(u) \backslash \textstyle\bigcup_{n=-2}^{\infty}\{u = \frac{-1}{ca^n}\}) \times \mathbb{P}^1(v) \backslash \{(u = 0, v = \infty)\}.$$

In the (u,v) coordinates, the restriction of $\tilde{\psi}$ to V_I defines a holomorphic map $\tilde{\psi} : V_I \to V_I$ given by $u_1 = au, v_1 = \frac{(va^2+cvu-u)u}{(a+cu)^2}$ and it satisfies $\pi \circ \tilde{\psi} = \psi \circ \pi$ on V_I.

Since the eigenvalues of $d\tilde{\psi}(\tilde{I}_2)$ is a and 0, $\tilde{I}_2$ is a fixed point of type (3) in Definition 2.2. It turns out that, though it is possible to take the canonical form $(\tau, \sigma) \to (\tau, \tau\sigma)$ by the conjugation map in the the formal power series category, this series does not have a positive radius of convergence. So, in the following proposition, we select more complicated "canonical" form. Set $W_I = (\mathbb{C}(\tau) \backslash \bigcup_{n=-2}^{\infty}\{\tau = \frac{-1}{ca^n}\}) \times \mathbb{P}^1(\sigma) \backslash \{(\tau = 0, \sigma = \infty)\}$.

Proposition 3.9 *Suppose $c \neq 0$ and define $S : W_I \to V_I$ by*

$$S : u = \tau, v = \frac{\sigma K(\tau/a)}{a^2 + c\tau}.$$

Then S is surjective biholomorphic and $S^{-1}\tilde{\psi}S$ is of the form $\tau_1 = a\tau, \sigma_1 = \sigma\tau - \tau^2 K(\frac{\tau}{a})^{-1}$.

Proof. By Lemma 3.2, it is proved that S is biholomorphic. Using the equation (3.4), we can verify the canonical form. □

Now, we will determine the Julia set of φ and ψ in $\mathbb{P}^2$.

Theorem 3.10 *Suppose that $0 < |c| < 1$. Then, $\mathcal{J}(\varphi) = \{t = 0\} = \overline{E(\varphi)}$.*

Proof. Since the iteration sequence of the canonial forms of Propositions 3.3 and 3.4 converges uniformly on every compact of $\mathbb{C}(\tau) \times \mathbb{C}(\sigma)$ to the constant map 0, $\{\tilde{\varphi}^n\}$ converges uniformly on every compact in $V_P \backslash \{q = \infty\}$ to the constant map $\tilde{P}$. So, $\pi(V_P \backslash \{q = \infty\}) \subset \mathcal{F}(\varphi)$. Let us consider the remaining set

$$\mathbb{P}^2 \backslash \pi(V_P \backslash \{q = \infty\}) = \bigcup_{n=0}^{\infty} \{z - (c - \tfrac{c}{a^n})t = 0\} \cup \{t = 0\}.$$

Note that $J_2 \in \mathcal{F}(\varphi)$ since $\tilde{J}_2 \in V_P$. Then, since $\varphi^{n+1}(\{z - (c - \frac{c}{a^n})t = 0\} \backslash \{I_1\}) = J_2$, it is easy to see that $\bigcup_{n=0}^{\infty}(\{z - (c - \frac{c}{a^n})t = 0\} \backslash \{I_1\}) \subset \mathcal{F}(\varphi)$.

Finally, we study $\{t = 0\}$. Let $U \subset \mathbb{P}^2 \setminus \{I_1, I_2\}$ be an open set such that $U \cap \{t = 0\} \neq \emptyset$. Since $\tilde{\varphi}(\pi^{-1}(\{t = 0\} \setminus \{I_1, I_2\})) = \tilde{J}_3$ and $\tilde{J}_3$ is a fixed point of $\tilde{\varphi}$, $\tilde{\varphi}^n(\pi^{-1}(U))$ contains a point near $\tilde{P}$ and a point near $\tilde{J}_3$ for sufficiently large n. Therefore, $\{\varphi^n\}$ is not equicontinuous in $U \setminus E(\varphi)$, which shows that $\{t = 0\} \subset \mathcal{J}(\varphi)$. □

Next, we suppose $|c| > 1$ and set $\gamma_1 = \pi(U_P \cap \tilde{\gamma}_1)$ in $\mathbb{P}^2$.

Theorem 3.11 *Suppose that $|c| > 1$. Then, $\mathcal{J}(\varphi) = \overline{\gamma_1} = \gamma_1 \cup \{t = 0\} = \gamma_1 \cup \overline{E(\varphi)}$.*

Proof. By Proposition 3.8, $\pi(V_F \backslash \tilde{\gamma}_1) \subset \mathcal{F}(\varphi)$ because $\tilde{\gamma}_1 = S(\{\sigma = \infty\})$ in V_F. Let us study the remaining set $\mathbb{P}^2 \backslash \pi(V_F \backslash \tilde{\gamma}_1) = \gamma_1 \cup \bigcup_{n=1}^{\infty}\{z = (c + \frac{-c}{a^{n-1}})t\} \cup \{t = 0\}$.

By Proposition 3.7, $\varphi^n(\{z = (c + \frac{-c}{a^{n-1}})t\}) = J_2 \in \mathcal{F}(\varphi)$ when $n \geq 1$, hence $\bigcup_{n=1}^{\infty}\{z = (c + \frac{-c}{a^{n-1}})t\} \subset \mathcal{F}(\varphi)$. It is clear that $\{t = 0\} \subset \overline{\gamma_1} \subset \mathcal{J}(\varphi)$. □

Let $|c| > 1$. By a sufficiently small open neighborhood Ω of $P \in \mathbb{P}^2$, $\bigcup_{n=1}^{\infty} \varphi^{-n}(\gamma_1 \cap \Omega)$ is called the stable curve of φ at P and denoted by $W^s(\varphi, P)$.

Theorem 3.12 *Suppose $|c| > 1$. Then, we have $W^s(\varphi, P) = \gamma_1 \setminus \{I_1\}$, $J_2 \notin W^s(\varphi, P)$ and $(\bigcup_{n=1}^{\infty} \varphi^{-n}(J_2)) \cap W^s(\varphi, P) = \emptyset$.*

Proof. The first statement follows from the definition of $W^s(\varphi, P)$. The second follows from Proposition 3.7. Finally, since $\varphi^{-n}(J_2) = \{z = (c + \frac{-c}{a^{n-1}})t\} \setminus \{I_1\} \subset \mathcal{F}(\varphi)$ for $n \geq 1$, we have $\varphi^{-n}(J_2) \cap W^s(\varphi, P) = \emptyset$ for $n \geq 1$. □

Finally, we will determine the Julia set of the map ψ.

Theorem 3.13 *Suppose $c \neq 0$. Then, $\mathcal{J}(\psi) = \bigcup_{n=1}^{\infty}\{z = (c - ca^n)t\} \cup \{z = c\} = \overline{E(\psi)}$.*

Proof. Let $W = (\mathbb{C}(\tau)\backslash \bigcup_{n=-1}^{\infty}\{\tau = \frac{-1}{ca^n}\}) \times \mathbb{C}^1(\sigma)$.Then, since $K(\tau/a) \neq 0$ in W, it is easy to see that the iteration of the "canonical" form in Proposition 3.9 converges uniformly on every compact of W to the constant map 0. So, by the same Proposition, $\{\tilde{\psi}^n\}$ converges uniformly to the constant map $\tilde{I}_2$ on every compact on $V_I\backslash\{v = \infty\}$. Hence we have $\pi(V_I\backslash\{v = \infty\}) \subset \mathcal{F}(\psi)$. Let us examine the remaining set

$$\mathbb{P}^2\backslash\pi(V_I\backslash\{v = \infty\}) = \bigcup_{n=-2}^{\infty}\{z - (c - ca^{n+2})t = 0\} \cup \{z = c\}.$$

Since

$$\tilde{\psi}^{n+2}(\{u = \tfrac{-1}{ca^n}\}\backslash\{\tilde{\varphi}^{n+1}(\tilde{J}_2)\}) = \tilde{I}_3 \in \{v = \infty\} \text{ for } n \geq -1$$

and

$$\tilde{\psi}^{n+m}(\{u = \tfrac{-1}{ca^n}\}\backslash\{\tilde{\varphi}^{n+1}(\tilde{J}_2)\}) \in \{v = \infty\} \text{ for } m \geq 2,$$

$\{\tilde{\psi}^n\}$ is not equicontinuous around a point of $\{u = \frac{-1}{ca^n}\}\backslash\{\tilde{\varphi}^{n+1}(\tilde{J}_2)\}$. Therefore, we have $\bigcup_{n=-1}^{\infty}\{z - (c - ca^{n+2})t = 0\} \subset \mathcal{J}(\psi)$.On the other hand, since $\psi(\{z = 0\}\backslash\{J_1\}) \subset \mathcal{F}(\psi)$, it is easy to see that $\{z = 0\}\backslash\{J_1\} \subset \mathcal{F}(\psi)$. Finally, it is clear that $\{z = c\} \subset \mathcal{J}(\psi)$. □

4 φ and ψ when $|a| > 1$

We assume that $|a| > 1$ throughout this section. Though we deal with the cases $0 < |c| < 1$ and $|c| > 1$, some lemmas and propositions in this section hold under the weaker assumption on c.

When $c = a^N$ by some $N \in \mathbb{N}$, we use the constant C_N defined in (3.7).

Lemma 4.1 *Let $c \neq 0$.*

(1) The radius of convergence of the following power series are ∞ and so the functions $k(\zeta)$, $h(\zeta)$, $j(\zeta)$ and $i(\zeta)$ are entire functions:

$$k(\zeta) = 1 + \sum_{m=1}^{\infty} \frac{c^m}{(a-1)\cdots(a^m-1)}\zeta^m,$$

$$h(\zeta) = 1 + \sum_{m=1}^{\infty}(a^{-(m^2+3m)/2}\sum_{k=0}^{m} a^{(k^2+k)/2}\frac{c^k}{(a-1)\cdots(a^k-1)})\zeta^m,$$

$$j(\zeta) = 1 + \sum_{m=1}^{\infty} \frac{1}{(a-c)\cdots(a^m-c)}\zeta^m,$$

$$i(\zeta) = \sum_{m=1}^{N-1} \frac{(-1)^m}{a^{m(m+1)/2}(a^{N-1}-1)\cdots(a^{N-m}-1)}\zeta^m$$

$$-\sum_{m=N}^{\infty} \frac{C_N a^{N(N-1)}}{a^{Nm}(a-1)\cdots(a^{m-N}-1)} \times \{(m-1) - \sum_{k=1}^{N-1} \frac{a^k}{a^k-1} + \sum_{k=1}^{m-N} \frac{1}{a^k-1}\}\zeta^m,$$

where we assume $a^m \neq c$ for all $m \in \mathbb{N}$ for $j(\zeta)$ and $c = a^N$ by some $N \in \mathbb{N}$ for $i(\zeta)$.

(2) The functions $k(\zeta)$, $h(\zeta)$, $j(\zeta)$ and $i(\zeta)$ satisfy the following functional equations:

$$k(a\zeta) = (1 + c\zeta)k(\zeta), \tag{4.9}$$

$$h(a\zeta) = k(\zeta) + a^{-1}\zeta h(\zeta), \tag{4.10}$$
$$j(a\zeta) = (\zeta + c)j(\zeta) + 1 - c, \tag{4.11}$$
$$(\zeta + a^N)i(\zeta) + \zeta - i(a\zeta) = C_N\zeta^N k(a^{-2N+1}\zeta), \tag{4.12}$$

where we assume $a^m \neq c$ for all $m \in \mathbb{N}$ for (4.11) and $c = a^N$ by some $N \in \mathbb{N}$ for (4.12).

Lemma 4.2 *Let $c \neq 0$. Around the origin, the function $\frac{1}{k(\zeta)}$ has the power series expansion*

$$\frac{1}{k(\zeta)} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m a^{m(m-1)/2} c^m}{(a-1)\cdots(a^m-1)} \zeta^m$$

whose radius of convergence is equal to $\frac{|a|}{|c|}$. The zeros of $k(\zeta)$ are $\{\frac{-a^n}{c}; n = 1, 2\cdots\}$ and they are all simple zeros.

First we study the map $\tilde{\varphi}$ around the fixed point $\tilde{J}_3$. The eigenvalues of $d\tilde{\varphi}(\tilde{J}_3)$ are $\{\frac{1}{a}, 0\}$, hence $\tilde{J}_3$ is of type (3) in Definition 2.2.

It turns out that, the canonical form $(\tau, \sigma) \to (\frac{\tau}{a}, \sigma\tau)$ is achieved. Using the notation in Definition 2.3, set $r = \frac{a\xi}{1-c\xi}$, $s = \eta$. Then,

$$U_J := \Omega_1 \cap \Omega_2 \cong (\mathbb{C}(r) \backslash \{r = \tfrac{-a}{c}\}) \times \mathbb{P}^1(s),$$

where B_c is the strict transform by $\pi : M \to \mathbb{P}^2$ of $\{z - ct = 0\}$. We set

$$V_J = U_J \backslash (\cup_{n=2}^{\infty} \{r = \tfrac{-a^n}{c}\} \cup \{\tilde{I}_2\}).$$

Note that $\tilde{J}_2 \in V_J$. In the (r, s) coordinates, the restriction to V_J of $\tilde{\varphi}$ defines a holomorphic map $\tilde{\varphi} : \tilde{\varphi} : V_J \to V_J$ given by $\tilde{\varphi} : r_1 = \frac{r}{a}, s_1 = \frac{(a^2+cr)sr}{(a+rc)^2+sr^2}$ and it satisfies $\pi \circ \tilde{\varphi} = \varphi \circ \pi$ on V_J. Set

$$W_J = (\mathbb{C}(\tau) \setminus \cup_{n=1}^{\infty} \tau = \tfrac{-a^n}{c}) \times \mathbb{P}^1(\sigma) \setminus \{(\tau = 0, \sigma = \infty)\}.$$

Proposition 4.3 *Suppose $c \neq 0$, and define $S : W_J \to V_J$ by*

$$S : r = \tau, s = \frac{\sigma k(\tau)(a + c\tau)}{1 - \sigma a^{-1}\tau^2 h(\tau)}. \tag{4.13}$$

Then, S is a surjective biholomorphic map and $S^{-1}\tilde{\varphi}S$ is of the form $\tau_1 = \frac{\tau}{a}, \sigma_1 = \sigma\tau$.

Proof. By Lemma 4.2, it is shown that S is biholomorphic. Using the equations (4.9) and (4.10), we can verify the canonical form. □

Now we turn to study the map $\tilde{\psi}$. First we study it around the fixed point $\tilde{F}$. The eigenvalues of $d\tilde{\psi}(\tilde{F})$ are $\{\frac{1}{a}, c\}$. Hence, $\tilde{F}$ is attracting when $|c| < 1$. Take the (f, g) coordinates in U_F defined before Proposition 3.8. Set $V_F' = (\mathbb{C}(f) \backslash (\cup_{n=1}^{\infty} \{f = -ca^n\}) \times \mathbb{P}^1(g)$. Then, the restriction of $\tilde{\psi}$ to V_F' defines a holomorphic map $\tilde{\psi} : V_F' \to V_F'$ given by $f_1 = \frac{f}{a}, g_1 = \frac{acg+fg}{a(1-g)}$, and it satisfies $\pi \circ \tilde{\psi} = \psi \circ \pi$ on V_F'.

We remark here that, when $\tilde{F}$ is of type (2), that is, when $ac^n = 1$ or $a^n c = 1$ by some $n \in \mathbb{N}$, only the type (2-1) can occur since $\tilde{\psi}$ has two invarinat curves $\{f = 0\}$ and $\{g = 0\}$ through $\tilde{F}$. This fact is also proved by the next proposition. When $|c| < 1$, the assumption of the next proposition is fulfilled. Set $W_F' = (\mathbb{C}(\tau) \backslash (\cup_{n=1}^{\infty} \{\tau = -ca^n\}) \times \mathbb{P}^1(\sigma)$.

Proposition 4.4 *Suppose $c \neq 0$ and $c \neq a^m$ for all $m \in \mathbb{N}$ and define $S : W_F' \to V_F'$ by*

$$S : f = \tau, g = \frac{(1-c)\sigma}{\sigma j(\tau) - (1-c)k(\tau/c^2)}.$$

Then, S is a surjective biholomorphic map and $S^{-1}\tilde{\psi}S$ is of the form $\tau_1 = \frac{\tau}{a}, \sigma_1 = c\sigma$.

Proof. By Lemma 4.2, S is biholomorphic. Using the equations (4.9) and (4.11), we can verify the canonical form. □

Next we consider the map $\tilde{\psi}$ around the fixed point $\tilde{P}$. The eigenvalues of $d\tilde{\psi}(\tilde{P})$ are $\{\frac{1}{a}, \frac{1}{c}\}$, hence $\tilde{P}$ is attracting when $|c| > 1$. According to a and c, $\tilde{P}$ is either of type (1) or (2) in Definition 2.2. Let us seek for the conjugation map S concretely. Using the notation in Definition 2.3, set $p = x - c$ and $q = y - \frac{1}{1-c}$. Then $U_P := \Omega_2 \cong \mathbb{C}(p) \times \mathbb{P}^1(q)$. We set $V_P' = (\mathbb{C}(p) \setminus \bigcup_{n=1}^{\infty}\{p = -ca^n\}) \times \mathbb{P}^1(q)$. In the (p, q) coordinates, the restriction of $\tilde{\psi}$ to V_P' defines a holomorphic map $\tilde{\psi} : V_P' \to V_P'$ given by $\tilde{\psi} : p_1 = \frac{p}{a}, q_1 = \frac{aq(1-c)-p}{(1-c)(p+ca)}$, and it satisfies $\pi \circ \tilde{\psi} = \psi \circ \pi$ on V_P'.

First, we consider the case where $\tilde{P}$ is of type (1) or of type (2) with $a = c^M$ by some $M \in \mathbb{N}$ in the sense of Definition 2.2. We remark here that, when $a = c^M$, $\tilde{\psi}$ does not have the canonical form $(\tau, \sigma) \to (\frac{\tau}{a} + \sigma^M, \frac{\sigma}{c})$, since $\{p = 0\}$ is an invariant curve in the direction of the eigenvalue $\frac{1}{c}$ of $d\tilde{\psi}$ while the map $(\tau, \sigma) \to (\frac{\tau}{a} + \sigma^M, \frac{\sigma}{c})$ does not have an invariant curve in the direction of $\frac{1}{c}$. We also remark that, in the next proposition, we do not assume $|c| > 1$. Set $W_P' = (\mathbb{C}(\tau) \setminus \bigcup_{n=1}^{\infty}\{\tau = -ca^n\}) \times \mathbb{P}^1(\sigma)$.

Proposition 4.5 *Suppose $c \neq 0, 1$ and $a^m \neq c$ for all $m \in \mathbb{N}$, and define $S : W_P' \to V_P'$ by*

$$S : p = \tau, q = \sigma k(c^{-2}\tau) + \frac{1}{1-c}(j(\tau) - 1).$$

Then S is a surjective biholomorphic map and $S^{-1}\tilde{\psi}S$ is of the form $\tau_1 = \frac{\tau}{a}, \sigma_1 = \frac{\sigma}{c}$.

Proof. By Lemma 4.2, S is biholomorphic. Using the equations (4.9) and (4.11), we can verify the canonical form. □

Next, we consider the case where $\tilde{P}$ is of type (2) with $c = a^N$ by some $N \in \mathbb{N}$. The next proposition shows that $\tilde{P}$ is in fact of type (2-2).

Proposition 4.6 *Suppose that $c = a^N$ by some $N \in \mathbb{N}$ and define $S : W_P' \to V_P'$ by*

$$S : p = \tau, q = \frac{C_N}{1-c}\sigma k(c^{-2}\tau) + \frac{1}{1-c}i(\tau).$$

Then S is a surjective biholomorphic map and $S^{-1}\tilde{\psi}S$ is of the form $\tau_1 = \frac{\tau}{a}, \sigma_1 = \frac{\sigma}{c} - \frac{\tau^N}{ca^N}$.

Proof. By Lemma 4.2, S is biholomorphic. Using the equations (4.9) and (4.12), we can verify the canonical form. □

We will continue to study the map $\tilde{\psi}$ around the point $\tilde{P}$. Now we assume that $|c| < 1$, so $\tilde{P}$ is a saddle point. Since we have $a^m \neq c$ for all $m \in \mathbb{N}$, we can apply Proposition 4.5. Let $\tilde{\gamma}_1'$ be a curve in U_P defined by

$$q = \frac{j(p)-1}{1-c}. \tag{4.14}$$

In view of Proposition 4.5, we can see that in the neighborhood V_P', $\tilde{\gamma}_1'$ is the local stable curve of $\tilde{\psi}$ at $\tilde{P}$. We will show that $\tilde{J}_2 \in \tilde{\gamma}_1'$.

Lemma 4.7 *Suppose* $0 < |c| < 1$. *Then,* $j(-ca) = 1 - c$.

Proof. By the same computation which we used in the proof of Lemma 3.5, we have

$$j(-ca) + c - 1 = \frac{(-1)^{n-1}c^n}{(a-c)\cdots(a^{n-1}-c)} + \sum_{m=n}^{\infty} \frac{(-ca)^m}{(a-c)\cdots(a^m-c)}.$$

As $n \to \infty$, the first term on the right side converges to 0 since $0 < |c| < 1$, and the second term converges to 0 since $j(\zeta)$ is an entire function. □

Proposition 4.8 *Suppose that* $0 < |c| < 1$. *Then,* $\tilde{J}_2 \in \tilde{\gamma}_1'$.

Proof. Note that $\tilde{J}_2 = (p = -ca, q = \frac{-c}{1-c})$. So, in view of the equation (4.14), we have $\tilde{J}_2 \in \tilde{\gamma}_1'$. □

Next, we study the map $\tilde{\psi}$ around the fixed point $\tilde{I}_2$ where the eigenvalues of $d\tilde{\psi}(\tilde{I}_2)$ are $\{a, 0\}$. Take the (r, s) coordinates on U_J defined before Proposition 4.3 and the (p, q) coordinates on U_P defined before Proposition 4.5.

In view of the equation (4.13), we can find the unstable curve of $\tilde{\psi}$ at $\tilde{I}_2$. Let $\tilde{\gamma}_2$ be the curve defined on $M \backslash B_c$ by

$$\begin{cases} s = \frac{-ak(r)(a+cr)}{r^2 h(r)} & \text{in } U_J \\ q = \frac{-h(a/p)}{pk(a/p)} - \frac{1}{1-c} & \text{in } U_P \backslash \{p = 0\}. \end{cases} \tag{4.15}$$

Proposition 4.9 *Let* $c \neq 0$. *Then,* $\tilde{\gamma}_2$ *is the local unstable curve of* $\tilde{\psi}$ *at* $\tilde{I}_2$.

Proof. We will work on V_J. Then, by the conjugation map S in Proposition 4.3, $S\tilde{\psi}S^{-1}$ is of the form $\tau_1 = a\tau, \sigma_1 = \frac{\sigma}{a\tau}$. So, $S(\tau, \infty)$ is the local unstable curve. □

We will study when $\tilde{J}_2 \in \tilde{\gamma}_2$.

Proposition 4.10 *Let* $c \neq 0$. *Then,* $\tilde{J}_2 \in \tilde{\gamma}_2$ *if and only if* $h(\frac{-a}{c}) = 0$.

Proof. Note that $\tilde{J}_2 = (r = \frac{-1}{c}, s = c - ac) \in V_J$. We remark that $k(\frac{-1}{c}) \neq 0$ by Lemma 4.2. So, we have $\tilde{J}_2 \in \tilde{\gamma}_2$ iff $k(\frac{-1}{c}) + \frac{-1}{ac}h(\frac{-1}{c}) = 0$.

On the other hand, by the equation (4.10), we have $h(\frac{-a}{c}) = k(\frac{-1}{c}) + \frac{-1}{ac}h(\frac{-1}{c})$. □

Lemma 4.11 *Let* $A(\zeta) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m(ac-1)\cdots(a^m c-1)}{a^{(m+3)m/2}}\zeta^m$.
If $a^M c = 1$ *for some* $M \in \mathbb{N}$, $A(\zeta)$ *is a polynomial of degree* $M - 1$.
If $a^m c \neq 1$ *for all* $m \in \mathbb{N}$, *the radius of convergence of* $A(\zeta)$ *is equal to* $\frac{|a|}{|c|}$.

Proof. Using the condition $|a| > 1$, the assertion can be easily verified. □

Proposition 4.12 *Let $c \neq 0$. We have $h(\zeta) = A(\zeta)k(\zeta)$ in the domain of $A(\zeta)$.*

Proof. This can be shown by comparing the power series expansion around 0 of the both sides.

Proposition 4.13 *If $a^m c \neq 1$ for all $M \in \mathbb{N}$, then $h(\frac{-a}{c}) \neq 0$.*
If $a^M c = 1$ by some $M \in \mathbb{N}$, then $h(\frac{-a}{c}) = 0$.

Proof. Suppose that $a^m c \neq 1$ for all $m \in \mathbb{N}$. Since the ratio of two entire functions $A(\zeta) = \frac{h(\zeta)}{k(\zeta)}$ has the power series expansion with radius of convergence equal to $\frac{|a|}{|c|}$ and that $\frac{-a}{c}$ is a unique simple zero of the function $k(\zeta)$ in $\{|\zeta| < \frac{|a|^2}{|c|}\}$ by Lemma 4.2, it follows that $h(\frac{-a}{c}) \neq 0$.

Now, suppose that $a^M c = 1$ for some $M \in \mathbb{N}$. Then, by Lemma 4.11, $A(\zeta)$ is a polynomial of degree $M-1$. Since $h(\zeta) = A(\zeta)k(\zeta)$ by Proposition 4.12 and $k(\frac{-a}{c}) = 0$ by Lemma 4.2, we have $h(\frac{-a}{c}) = 0$. □

The following is the immediate consequence of Propositions 4.13 and 4.10.

Theorem 4.14 *Let $c \neq 0$. Then, $\tilde{J}_2 \in \tilde{\gamma}_2$ iff $a^M c = 1$ by some $M \in \mathbb{N}$. Specially, we have $\tilde{J}_2 \notin \tilde{\gamma}_2$ when $|c| > 1$.*

Now, we will describe the Julia set $\mathcal{J}(\varphi)$. We set $\gamma_2 = \pi(\tilde{\gamma}_2) \subset \mathbb{P}^2$.

Theorem 4.15 *Let $c \neq 0$.*
(1) When $J_2 \notin \gamma_2$, $\mathcal{J}(\varphi) = \overline{\gamma_2}$.
(2) When $J_2 \in \gamma_2$, $\mathcal{J}(\varphi) = \overline{\gamma_2} \cup \bigcup_{n=1}^{\infty} \{z + (\frac{c}{a^{n-1}} - c)t = 0\} = \overline{\gamma_2} \cup \bigcup_{n=1}^{\infty} \{\varphi^{-n}(J_2)\}$.

Proof. Since the canonical form of Proposition 4.3 converges uniformly on every compact of $\mathbb{C}(\tau) \times \mathbb{C}(\sigma)$ to the constant map 0, $\{\tilde{\varphi}^n\}$ converges uniformly on every compact in $V_J \setminus \tilde{\gamma}_2$ to the constant map $\tilde{J}_3$. So, $\pi(V_J \setminus \tilde{\gamma}_2) \subset \mathcal{F}(\varphi)$. Let us examine the remaining set $\mathbb{P}^2 \backslash \pi(V_J \setminus \tilde{\gamma}_2) = (\bigcup_{n=1}^{\infty} \{z + (\frac{c}{a^{n-1}} - c)t = 0\} \cup \{z - ct = 0\} \cup \gamma_2)$. Then it is clear that $(\{z - ct = 0\} \cup \gamma_2) \subset \mathcal{J}(\varphi)$. Note that $\varphi^n(\{z + (\frac{c}{a^{n-1}} - c)t = 0\}) \setminus \{I_1\} = \{J_2\}$. Then it is clear that:

(1) when $J_2 \notin \gamma_2$, we have $\bigcup_{n=1}^{\infty} (\{z + (\frac{c}{a^{n-1}} - c)t = 0\} \setminus \{I_1\}) \subset \mathcal{F}(\varphi)$, and

(2) when $J_2 \in \gamma_2$, we have $\bigcup_{n=1}^{\infty} \{z + (\frac{c}{a^{n-1}} - c)t = 0\} \subset \mathcal{J}(\varphi)$.

Here we would like to state some comments. Since the $\varphi-$constant curve $C_1 = \{t = 0\} \setminus \{I_1, I_2\}$ satisfies $\varphi(C_1) = I_1 \in I(\varphi)$, we do not consider $\varphi^n(p)$ for $p \in C_1$ and $n \geq 2$. However, for an open neighborhood $U \subset \mathbb{P}^2 \setminus \{I_1, I_2\}$ of a point $p \in C_1$, $\{\varphi^n\}$ is equicontinous in $U \setminus E(\varphi)$. In conclusion, we have $C_1 \subset \mathcal{F}(\varphi)$ though $\varphi(C_1) \in I(\varphi)$. □

By a sufficiently small open neighborhood Ω of $I_2 \in \mathbb{P}^2$, $\bigcup_{n=1}^{\infty} \psi^n(\gamma_2 \cap \Omega \setminus E_n(\psi))$ is called the unstable curve of ψ at I_2 and is denoted by $W^u(\psi, I_2)$.

Theorem 4.16 *(1) Suppose $J_2 \notin \gamma_2$. Then for $\alpha \in \mathbb{C}$, $\gamma_2 \cap \{z - \alpha t = 0\} = \{J_1\}$ iff $\alpha = c - \frac{c}{a^{n-1}}$ by some $n \in \mathbb{N}$. We have $W^u(\psi, I_2) = \gamma_2 \setminus \{J_1\}$.*

(2) Suppose $J_2 \in \gamma_2$. Then $J_1 \notin \gamma_2$ and $W^u(\psi, I_2) = \gamma_2 \setminus \bigcup_{n=1}^{\infty}\{z + (\frac{c}{a^{n-1}} - c)t = 0\}$.

Proof. By the equation (4.15), $\tilde{\gamma}_2 : q = \frac{-h(a/p)}{pk(a/p)}$ in $U_P \setminus \{p = 0\}$. By Lemma 4.2, the denominator is equal to 0 iff $p = \frac{-c}{a^{n-1}}$ by some $n \geq 1$.

(1)When $\tilde{J}_2 \notin \tilde{\gamma}_2$, by Proposition 4.10, $h(\frac{-a}{c}) \neq 0$. On the other hand, by the equation (4.10) and Lemma 4.2, we have $h(\frac{-a^{n+1}}{c}) = k(\frac{-a^n}{c}) + \frac{-a^{n-1}}{c}h(\frac{-a^n}{c}) = \frac{-a^{n-1}}{c}h(\frac{-a^n}{c})$ for $n \geq 1$.

Hence, inductively, we know that $h(\frac{-a^n}{c}) \neq 0$ for $n \geq 1$. Therefore, by Lemma 4.2, $\tilde{\gamma}_2 \cap \{p = \frac{-c}{a^{n-1}}\} \in \{q = \infty\}$. Now the first assertion of (1) is proved.

(2)By the above argument, $h(\frac{-a^n}{c}) = 0$ for $n \geq 1$. By Lemma 4.2, $k(\frac{-a^n}{c}) = 0$ and this is a simple zero. Hence $q \neq \infty$ at $p = \frac{-c}{a^{n-1}}$. This implies $J_1 \notin \gamma_2$.

Finally, by the equation (2.3), it is easy to see the assertion on $W^u(\psi, I_2)$. □

Next, we will describe the Julia set $\mathcal{J}(\psi)$.

Theorem 4.17 *Let $|c| > 1$. Then, we have $\mathcal{J}(\psi) = \bigcup_{n=1}^{\infty}\{z = (c - ca^n)t\} \cup \{t = 0\} = \overline{E(\psi)}$.*

Proof. Since the iteration sequences of the canonical forms of Propositions 4.5 and 4.6 converge uniformely on every compact of $\mathbb{C}^2(\tau, \sigma)$ to the constant map 0, $\{\tilde{\psi}^n\}$ converges uniformly on every compact in $V'_P \setminus \{q = \infty\}$ to the constant map $\tilde{P}$. So, $\pi(V'_P \setminus \{q = \infty\}) \subset \mathcal{F}(\psi)$. Let us consider the remaining set $\mathbb{P}^2 \setminus \pi(V'_P \setminus \{q = \infty\}) = \bigcup_{n=1}^{\infty}\{z - (c - ca^n)t = 0\} \cup \{t = 0\}$. Then, since

$$\tilde{\psi}^{n+1}(\{p = -ca^{n+1}\} \setminus \{\tilde{\varphi}^n(\tilde{J}_2)\}) = \tilde{I}_3 \in \{q = \infty\} \quad \text{for} \quad n \geq 0,$$

it is easy to see that $\bigcup_{n=1}^{\infty}(\{z - (c - ca^n)t = 0\}) \subset \mathcal{J}(\psi)$. On the other hand, we have clearly $\{t = 0\} \subset \mathcal{J}(\varphi)$. □

When $0 < |c| < 1$, we set $\gamma'_1 = \pi(\tilde{\gamma}'_1) \subset \mathbb{P}^2$.

Theorem 4.18 *Let $0 < |c| < 1$. Then, $\mathcal{J}(\psi) = \overline{\gamma'_1} = \gamma'_1 \cup \{t = 0\}$.*

Proof. Since the iteration sequence of the canonical forms of Proposition 4.4 converges uniformely on every compact of $\mathbb{C}^2(\tau, \sigma)$ to the constant map 0, $\{\tilde{\psi}^n\}$ converges uniformly on every compact in $V'_F \setminus \tilde{\gamma}'_1$ to the constant map $\tilde{F}$. So, $\pi(V'_F \setminus \tilde{\gamma}'_1) \subset \mathcal{F}(\psi)$. Let us consider the remaining set $\mathbb{P}^2 \setminus \pi(V'_F \setminus \tilde{\gamma}'_1) = \gamma'_1 \cup \bigcup_{n=1}^{\infty}\{z - (c - ca^n)t = 0\} \cup \{t = 0\}$. Then, since

$$\tilde{\psi}^{n+1}(\{p = -ca^{n+1}\} \setminus \{\tilde{\varphi}^n(\tilde{J}_2)\}) = \tilde{I}_3 \in \{q = \infty\} \quad \text{for} \quad n \geq 0,$$

it is easy to see that $\bigcup_{n=1}^{\infty}(\{z - (c - ca^n)t = 0\} \setminus \{\varphi^{n-1}(J_2)\}) \subset \mathcal{F}(\psi)$. On the other hand, we have clearly $\gamma'_1 \cup \{t = 0\} \subset \mathcal{J}(\psi)$. □

Let $0 < |c| < 1$. By a sufficiently small open neighborhood Ω of $P \in \mathbb{P}^2$, $\bigcup_{n=1}^{\infty} \psi^{-n}(\gamma'_1 \cap \Omega)$ is called the stable curve of ψ at P and denoted by $W^s(\psi, P)$.

Theorem 4.19 *Let $0 < |c| < 1$. Then, $W^s(\psi, P) = \gamma'_1 \setminus \bigcup_{n=0}^{\infty}\{\varphi(J_2)\}$.*

Proof. By Proposition 4.8, $J_2 \in \gamma'_1$. In view of the equation 4.14, $\gamma'_1 \subset \mathbb{C}^2(x, y)$. Hence, the assertion can be seen easily. □

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