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On Distribution of Idempotents of Semigroup

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Abstract Many attempts have been made to study the structure of a semigroups. One of the fundamental study of semigroup has been based on that of idempotents, each of which plays the role of identity elements in local. In this paper, two kinds of relations induced by translations on semigroup will be introduced, which are closely related with the distribution of idempotents and regular elements.

1. Basic Definitions and Results

If S is any semigroup and $a,b \in S$, a mapping f on S is called a left translation if $f(xy)=f(x)y$ for any elements $x,y \in S$ and a mapping g on S is called a right translation if $g(xy)=xg(y)$ for any elements $x,y \in S$. The set $\Psi(S)$, of all left translations of S is the (left translation) semigroup under the ordinary composition of mappings. Similarly, the set $\Phi(S)$, of all right translations of S is the (right translation) semigroup under the composition.

Definition 1. Let $\Psi(S)$ and $\Phi(S)$ be the left translation semigroup of S and the right translation semigroup of S, respectively, which have identity mappings. Then, the quasi-order relation $<_L$ and the equivalence relation L on S are defined as follows:

x $<_L$ y if and only if for any f, g $\in \Psi(S)$, f(x)=g(x) implies f(y)=g(y),

x L y if and only if x $<_L$ y and y $<_L$ x.

Similarly, $\Phi(S)$ leads us to the definitions of the quasi-order relation $<_R$ and the equivalence relation R on S , which are defined as follows:

x $<_R$ y if and only if for any f, g $\in \Phi(S)$, f(x)=g(x) implies f(y)=g(y),

x R y if and only if x $<_R$ y and y $<_R$ x.

Let L and R be the equivalence relations on S defined above and let us denote the intersection of L and R by H.

Definition 2. Now $S = \cup(L(a) : a \in \Lambda)$, $S = \cup(R(b) : b \in \Gamma)$, and $S = \cup(H(a,b) = L(a) \cap R(b) : a \in \Lambda, b \in \Gamma)$ stand for the partitions induced by the equivalence relations L, R and H, and let us call $L(a)$, $R(b)$ and $H(a,b)$ a L-class (containing an element a), a R-class (containing an element b) and a H-class (the intersection of $L(a)$ and $R(b)$), respectively.

We shall use the fundamental results shown bellow.

Let x and y be arbitrary elements of a semigroup S , and let $\Psi(S)$ and $\Phi(S)$ be the left translation semigroup and the right translation semigroup, respectively, we have the following lemmas.

Lemma 1.

(1) If $x \lhd_{\mathcal{L}} y$, then $f(x) \lhd_{\mathcal{L}} f(x)$ for all $f \in \Psi(S)$;

(2) If $x \lhd_{\mathcal{R}} y$, then $g(x) \lhd_{\mathcal{R}} g(x)$ for all $g \in \Phi(S)$.

Lemma 2.

(1) If $x \in L(y)$, then $f(x) \in L(f(y))$ for all $f \in \Psi(S)$;

(2) If $x \in R(y)$, then $g(x) \in R(g(y))$ for all $g \in \Phi(S)$.

Lemma 3.

(1) $x \lhd_{\mathcal{L}} xu$ for all $u \in S$;

(2) $x \lhd_{\mathcal{R}} vx$ for all $v \in S$.

From above lemmas, we have that L is a left congruence relation and R is a right congruence relation.
Let ϕ and ϕ be subsemigroups of $\Psi(S)$ and $\Phi(S)$ respectively, we have restrictions of the relations L and R to the subsemigroups ϕ and ϕ, which will be denoted by $L(\phi)$ and $R(\phi)$, respectively. Then we have following lemmas:

Lemma 4.

(1) $\phi' \subseteq \phi'' \subseteq \Psi(S)$ implies $L \subseteq L(\phi'') \subseteq L(\phi')$;

(2) $\phi' \subseteq \phi'' \subseteq \Phi(S)$ implies $R \subseteq R(\phi'') \subseteq R(\phi')$.

Lemma 5.

(1) $L(\phi' \cap \phi'') = L(\phi') \cup L(\phi'')$ and $R(\phi' \cap \phi'') = R(\phi') \cup R(\phi'')$;

(2) $L(\phi' \cup \phi'') = L(\phi') \cap L(\phi'')$ and $R(\phi' \cup \phi'') = R(\phi') \cap L(\phi'')$.

From above lemmas, we have

$\mathbb{R} \subseteq L \subseteq L(\phi')$, $\mathbb{L} \subseteq R \subseteq R(\phi')$ and $\mathbb{H} \subseteq H \subseteq H(\phi', \phi') = L(\phi') \cap R(\phi')$ for the Green's relations $\mathbb{R}$, $\mathbb{L}$ and $\mathbb{H}$.

2. Distribution of Idempotents

Let S be an semigroup, then we have the following lemmas.

Lemma 6. Let **e** be any element of S, then

(1)**e** is an idempotent if and only if **et** = t for all $t \in L(e)$;

(2)**e** is an idempotent if and only if **se** = s for all $s \in R(e)$.

(Proof) (1) It is trivial that **ee** = e, since $e \in L(e)$. Conversely, assume that e is an idempotent, that is, **ee** = **e**, then from the definition of the left translation, **ee** = $f_e(e) = I(e)$ implies that **e**t= $f_e(t) = I(t) = t$ for all $t \in L(e)$, where I is the identity mapping. (2) is shown similarly.

Lemma 7. Let **e** be any idempotent of S, then

(1)For any left translation $f \in \Psi(S)$ and any element $s \in L(e)$, there exist an element $t \in L(f(s))$ such that $f(s) = ts$;

(2)For any right translation $g \in \Phi(S)$ and any element $s \in R(e)$, there exists an element $t \in R(g(s))$ such that $g(s) = st$.

(Proof) (1) It is obvious from the fact that $f(s) = f(es) = f(e)s$ for any element $s \in L(e)$ (from Lemma 6) and $t = f(e) \in L(f(s))$ (from Lemma 2). (2) is shown similarly.

Lemma 8. Let **e** and f be any idempotents of S.

(1) If $e \in L(f)$ then e is an inverse of f;

(2) If $e \in R(f)$ then e is an inverse of f.

(Proof) (1) $efe = e(fe) = e(e) = e$, from Lemma 6(1) and that f is an idempotent in $L(e)(=L(e))$. Similarly it is shown that $fef = f(ef) = f(f) = f$.

(2) is shown similarly.

From above lemmas, we have following results:

Lemma 9. For any elements $a, b \in S$, the H-class, $L(a) \cap R(b)$, cannot have more than one idempotent.

(Proof) Assume that e and f be idempotents in a H-class, $L(a) \cap R(b)$, that is, $e, f \in L(f) \cap R(f) = L(e) \cap R(e)$. Then $f = fef = f(ef) = ef = e$ from Lemma 6.

Lemma 10. For any elements $a, b \in S$, if $R(a) \cap L(b)$ contains an idempotent then $ab \in L(a) \cap R(b)$

(Proof) Assume that e is an idempotent in a H-class such that $e \in R(a) \cap L(b)$, that is, $a \in R(e)$ and $b \in L(e)$. Then from Lemma 2 and Lemma 6, we have that $g(a) \in R(g_b(e))$ for the inner right translation g_b such that $g_b(a) = ab$ and $g_b(e) = eb = b$, since $e \in L(b)$. Thus $ab \in R(b)$. Similarly, we have that $f_a(b) \in L(f_a(e))$ for the inner left translation f_a such that $f_a(b) = ab$ and $f_a(e) = ae = a$, since $e \in R(a)$. Thus $ab \in L(a)$.

Theorem 1. For any elements $a \in S$, the following conditions are equivalent:

(1) $R(a)$ contains an idempotent;

(2) For any right translation $g \in \Phi(S)$, there exist an element $t \in R(g(a))$ such that $g(a)=at$.

(Proof) (1) → (2): Let **e** be an idempotent in $R(a)$, then from Lemma 7, we have that for any right translation $g \in \Phi(S)$, there exists an element $t \in R(g(a))$ such that $g(a)=at$.

(2)→(1): Let I be the identity mapping, then there exists an element $t \in R(I(a)) = R(a)$ such that $I(a) = at$. From the fact that $t \in R(a)$, $at = g_t(a) = I(a)$ implies that $tt = g_t(t) = I(t) = t$. Thus t is an idempotent in $R(a)$.

Similarly, we also have

Theorem 1'. For any elements $b \in S$, the following conditions are equivalent:

(1) $L(b)$ contains an idempotent;

(2) For any left translation $f \in \Psi(S)$, there exist an element $t \in L(f(b))$ such that $f(b)=tb$.

The following theorem is a direct result from Lemma 6, Lemma 9 and Lemma 10.

Theorem 2. For any elements $a, b \in S$, $R(a) \cap L(b)$ contains an idempotent if and only if $R(a) \cap L(b)$ is a monoid.

3. Some Applications of the Relations

In this section, some special class of semigroups will be discussed, which have idempotent in every H-classes of them.

Lemma 11. Let $H(e)$ and $H(f)$ are any H-classes which contain idempotents e and f.

(1) If the H-classes, $H(e)$ and $H(f)$ are included in a same R-class, then there exists a homomorphism from $H(e)$ into $H(f)$;

(2) If the H-classes, $H(e)$ and $H(f)$ are included in a same L-class, then there exists a homomorphism from $H(e)$ into $H(f)$.

(Proof) (1) The mapping $\lambda : H(e) \rightarrow H(f)$ is defined by $\lambda(s) = fsf$, for $s \in H(e)$. Assume that $s, t \in H(e)$, then $\lambda(s)\lambda(t) = (fsf)(ftf) = fsfftf = fsftf = fstf = \lambda(st)$, since $s \in R(f)$. Since $\lambda(e) = fef = f$ for the idempotent e.

(2) is similarly shown.

From the above results we have the following theorems:

Theorem 2. Let $H(e)$ and $H(f)$ are any H-classes which have idempotents, e and f , respectively.

(1) If the H-classes, $H(a)$ and $H(b)$ are included in a same R-class, then $H(e)H(f) \subseteq H(e)$;

(2) If the H-classes, $H(a)$ and $H(b)$ are included in a same L-class, then $H(e)H(f) \subseteq H(f)$.

Theorem 3. The following conditions are equivalent:

(1) s is a regular element;

(2) There exists an element t such that st=e for some idempotent $e \in L(s)$;

(3) There exists an element t such that ts=f for some idempotent $f \in R(s)$.

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