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ALGEBRA & FORMAL MODELS OF UNDERSTANDING

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ABSTRACT. When a transformation semigroup arises from a system (of states and events) to be understood, a covering of the transformation semigroup by a wreath product of simpler transformation semigroups can be considered a formal model of understanding. Such a wreath product decomposition yields a coordinate system appropriate to the original system. This viewpoint (due to Rhodes) is surveyed and extended here.

We present several examples from mathematics of the use of such coordinate systems. As shown by Noether and Rhodes (and extended herein), conservation laws and symmetries lead to refined understanding of physical systems, and this understanding may be formalized as global hierarchical coordinatization. Hierarchical object structuring in computer software provides another example.

Relational morphisms can be considered as metaphors, since they can be interpreted as partially successful attempts at understanding one structure using another. Kernel theorems for relational morphisms of transformation semigroups can then be employed as algebraic tools for the manipulation and construction of such models of understanding.

We suggest that developing computational tools for the implementation of feasible automatic discovery of these formal models of understanding as well as for their algebraic manipulation will extend the human notions of understanding, metaphor and analogy to a formal automated realm.

1. UNDERSTANDING

When we (informally) understand a system, this implies several properties of whatever it is that constitutes understanding:

(1) **Globality.** We have some sort of description of all essential characteristics of the system and can describe or predict how it will change with the occurrence of events possible for that system.
(2) **Hierarchy.** (Except in the simplest cases) the whole is broken down into component parts, which themselves may consist of other parts, and so on, resulting in a partial order that encodes the dependencies among the parts. Information from the higher levels of the hierarchy gives good approximate knowledge, while details are fleshed out at the lower levels.
(3) **Simple Components.** The smallest parts should by themselves be easy to understand.
(4) **Covering.** We have (implicity or explicitly) a knowledge of how to map our understanding to the system.

We shall take these properties as axiomatic for understanding. Note that such understanding of a system is something quite distinct from knowledge of how build or emulate it efficiently, nor is it the same as knowledge of how the system is really structured.

Nevertheless, if the system is to be maintained or modified later, then efficiency in performance, number of states, and so on, will be often be subordinate to the great inefficiency and cost of modifying a poorly understood system. Systems which happen to be structured according to the above principles should thus be more effectively maintainable. This is evidenced for example by the success of the object-oriented paradigm for computer software.[1]

We present first a formalization of "system" common in physics and computer science. Then, beginning from the above properties, we present a formalization for "model of understanding" of such a system.[2]

Our longterm goal is to present a notion of understanding which would facilitate *automatic* generation, manipulation, and synthesis of understanding using mathematical and computational techniques. We shall see below that for finite-state systems, methods already exist for generating a formal model of understanding for the given system and that one also has definitions for analogy and metaphors as relational mappings (morphisms in an appropriate setting) as well as characterizations of exactly what [extension] is required for an incomplete relation to be transformed into to an finished model [emulation].

Throughout the history of science and mathematics, great advances have often come when a good "coordinate system" appropriate for the domain in question is discovered. We contend that generally such useful coordinate systems can be considered as formal models of understanding of the domain via an emulation by the (generally larger) coordinate system, giving global hierarchical coordinates in the sense of emulation by a cascade (or wreath product) of small component parts. See below for precise definitions.

The idea that wreath product emulations provide models of understanding appears to be due to John L. Rhodes [13]. In a still unpublished book written in the late 1960s, what we call formal models of understanding here are motivated as *theories* which provide understanding of an *experiment*, i.e. a system. This paper is an attempt to promote and extend this viewpoint. Much of it closely follows and extends Rhodes [13]. Indeed, with the advant of higher performance computers, it is becoming conceivable to automate the construction of "theories" of finite-state "experiments". Moreover, kernel theorems of the author [9], Tilson [17], and Rhodes-Tilson [16] point the way toward automatically manipulating, relating, and synthesizing collections of formal models of understanding according to an algebra of formalized analogies and metaphors between them.

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[1] Indeed, object-oriented design can be regarded as a special case of wreath product decomposition over a partial order, and global semigroup theory then provides a rigorous algebraic foundation for the object-oriented paradigm [10]. Hierarchically specified objects and classes in the sense of object-oriented programming can be considered as formal models of understanding in the sense of this paper.

[2] We remark that the model need not be in any way *symbolic*, although it is not forbidden from being symbolically representational. Thus the standard mutual objections of the "artificial intelligence" symbolic vs. connectionist debate are *non sequitur* for our notion of formal understanding. Both symbolic and connectionist approaches (as well as others) could employ our formalization of understanding, albeit in different manners.

2. EXAMPLES

Cartesian coordinates on Euclidean n-space: The n "simple" components used to coordinatize Euclidean space are copies of the real numbers under addition. These copies are partially ordered by the empty partial order (no dependencies between them). In this case, there is a one-to-one correspondence from coordinates for points to points of the space, and points (or vectors) are added component-wise without regard to other components.
Decimal Expansion of Real Numbers: Here a real number is (perhaps non-uniquely!) coordinatized by specifying countably many components from the additive group of modulo 10 integers. Thus the small components are copies of the group $(\mathbb{Z}_{10}, +)$ and they are arranged hierarchically along a total order of type $(\mathbb{Z}, <)$: to add a real number to one in decimal coordinates, it is necessary to consider the values of the coordinates of lower magnitude positions (but never higher order positions) in order to decide which element of $\mathbb{Z}_{10}$ to add in a given position.

In the case of the decimal expansion of integers, the order type is that of the natural numbers, and each integer has exactly one coordinatization. [3]

However, there are many choices of the coordinates that do not correspond to any real or integer number. Indeed if we use base p (prime) rather than 10 for the coordinate expansion of the integers, then the p-adic numbers arise naturally by allowing coordinates to take all possible values in $\mathbb{Z}_p$.

We remark also that the (positive) real numbers arise naturally from the decimal (or other base) expansion of the (positive) rationals by allowing all possible coordinate values to be taken for coordinate positions of lower magnitude than some maximal non-zero coordinate.

Thus we have two examples in which global hierarchical coordinates naturally lead to interesting extensions of the original system (from $\mathbb{Q}$ to $\mathbb{R}$ and from $\mathbb{Z}$ to the p-adic numbers). This suggests that introducing a formal model of understanding on a given system may lead to interesting concepts that expand the original system.

Just few examples of many traditional uses of global hierarchical coordinatizations in mathematics include the Jordan canonical form of linear transformations on finite-dimensional vector spaces, power series under $+$, power series under $\cdot$, and Taylor series. Indeed, series methods are often employed in trying to understand possible solutions to a differential equation exactly because they give useful global hierarchical information about the behaviour of a solution (first-order, second-order, etc. approximations).

3. DEFINITIONS

3.1. Wreath Products and Coverings. *This section contains standard definitions and may be skipped by the reader and consulted later as necessary.*
A *transformation semigroup* (X, S) is a set X that a semigroup S acts on the right of: $x \cdot (ss') = (x \cdot s) \cdot s'$. Given two transformation semigroups (X, S) and (Y, T), we define the *wreath product* $(Y, T) \wr (X, S)$ to be the transformation semigroup with set

[3] In the case of the integers under addition (and other groups), the possible global hierarchical coordinate systems essentially (i.e. subject to "gluing" of actions) correspond to chains of subgroups. See [8].

$Y \times X$ and action semigroup $T^X \rtimes S$ with action $(y, x) \cdot (f, s) = (y \cdot f(x), x \cdot s)$. This defines a transformation semigroup with the evident multiplicative structure on its action semigroup. Moreover the wreath operation on the class of transformation semigroups is associative.[4]

A *relational morphism* of $\varphi : (X, S) \lhd (Y, T)$ is a subtransformation semigroup (Q, R) of $(X, S) \times (Y, T)$, thus,

$$(x, y) \in Q \text{ and } (s, t) \in T \text{ implies } (x \cdot s, y \cdot t) \in Q,$$

and Q and R project fully onto X and S, respectively. A relational morphism is a *morphism* if each Q and R are [graphs of] functions, an *embedding* if Q and R are injective functions, or is an *approximation* [or *simulation*] of (X, S) by (Y, T) if it is a surjective morphism. It is an *emulation* or *covering* of (X, S) by (Y, T) if Q and R are injective relations:

$$((x, y) \in Q \text{ and } (x', y) \in Q) \Rightarrow x = x',$$

and similarly for R. If (Y, T) covers (X, S) we write $(X, S) \prec (Y, T)$, and often say that (X, S) *divides* (Y, T). Also in the case of semigroups, we write $S \prec T$ (S divides T) if S is a homomorphic image of a subsemigroup of T.

In the case of covering, one can use (Y, T) in place of (X, S): given $x \in X$ and $s \in S$, we choose $(x, y) \in Q$ and $(s, t) \in T$. The elements y and t are called *lifts* of x and of s, respectively. Now the state $x \cdot s$ is uniquely determined from $y \cdot t$ by the injectivity condition. If (Y, T) has a nice form, say as a wreath product of simpler transformation semigroups, then this covering provides a global hierarchical coordinate system on (X, S), which may be more pleasant to manipulate and provide insight into the original (X, S).

A transformation semigroup (X, S) is *faithful* if $x \cdot s = x \cdot s'$ for all $x \in X$ implies $s = s'$. One can always make a transformation semigroup faithful by identifying the s and s' which both map X exactly the same way. The wreath product of faithful transformation semigroups is easily seen to be faithful.

3.2. Systems & Formal Models of Understanding. A *system* (X, A, λ) consists of a state space X, inputs A, and a transition function $\lambda : X \times A \to X$. Traditional physics considers (or hopes) that physical phenomenon are faithfully modelled as such systems: knowing the current state x and what happens a, one can determine the resulting state as $\lambda(x, a)$. Note that for a sequence of events $a_1, \ldots, a_{n+1}$, one has a *recursive* description of the behaviour of the system (as λ induces an action of the free semigroup A^+ on X):

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[4] More generally, transformation semigroups (X_α, S_α) generically combined with dependencies coded by an [irreflexive] partial order $\mu = (V, <)$ yield $(X, S) = \int_{\alpha \in V} (X_\alpha, S_\alpha)\, d\mu$, with states $X = \prod X_\alpha$ and semigroup elements $f : X \to X$ with $(xf)_\alpha = x_\alpha \cdot \overline{f}(x_{<\alpha})$, where $\overline{f}(x_{<\alpha}) \in S_\alpha$ and $x_{<\alpha}$ is the projection of x that forgets all components except the x_β with $\beta < \alpha$. The transformation semigroup (X, S) is called the *cascade integral* of the (X_α, S_α) over μ.

The usual wreath product of n transformation semigroups is just a cascade integral over a finite total order. Since the hierarchies we allow for formal models of understanding permit components to be combined according to a partial ordering, the above generalized form of wreath product is very useful. For computational application, one would evidently most often restrict to finite partial orders (as for Cartesian coordinates) or finitely many coordinates of an infinite partial order (as for the decimal expansion of the integers). See [7] for applications and its appendix for properties of the cascade integral.

$$x \cdot a_1 \ldots a_n a_{n+1} = \lambda(x \cdot a_1 \ldots a_n, a_{n+1})$$

Crucially, such a description of the system is *not* a model of understanding but rather only a starting point for analysis.[5] Indeed, A may consist of tiny intervals of time and λ describe the evolution of a physical situation according to a set of differential equations, e.g. determining the position and momentum of a set of point masses according to Newtonian mechanics. Such recursive descriptions including descriptions in terms of differential equations are often the beginning of analysis of a physical system and precede formal understanding.

From (X, A, λ) one derives a transformation semigroup (X, S) by making that induced action of A^+ faithful:

$$w \equiv w' \iff \forall x \in X, x \cdot w = x \cdot w'.$$

Here $S = A^+/\equiv$.

A *formal model of understanding* for a system (X, A, λ) is a covering of the induced transformation semigroup (X, S) by a wreath product of "simpler" transformation semigroups over a partial ordering (see footnote 4 above).

4. Noether-Rhodes Coordinates from Conservation Laws & Symmetry

Emmy Noether found a correspondence between certain conservation laws in physics and certain one-parameter groups of automorphisms [11, 12]. A generalization of this idea is that a homomorphism defined on the system leads to a conservation law. This statement has an explicit expression due to Rhodes [13] in our language as the fact that a conserved quantity can serve has a highest level coordinate in a formal model of understanding for the system.

4.1. Conserved Quantities. Given our system (X, A, λ), one identifies a formal invariant $e : X \longrightarrow Y$, where $e(X) \subseteq Y$ is a set of "formally conserved values": That is, $e(x \cdot a)$ is determined by $e(x)$ and a without dependence on x, so that one may define $e(x) \cdot a := e(x \cdot a)$.

Examples of such conserved invariants e include energy, angular momentum, mass, etc. To determine the new energy, angular momentum, mass or what have you, it is only necessary to know a and the value of e before the system was hit by a.

Now one can coordinatize as follows:

$$x \longmapsto (rest, e(x)),$$

where the *rest* is anything which uniquely determines x among all $x' \in X$ with $e(x) = e(x')$.

Then we have:

$$(rest, e(x)) \cdot a = (rest \cdot f(e(x), a), e(x \cdot a)),$$

[5]However, such a system, if modelling real world phenomena, *abstracts* essential features of state and events while ignoring others. This abstraction *from the real world* is one property of understanding that our formalism does not address. We shall always assume the system (X, A, λ) to be available before beginning any formalization for understanding it or we shall construct it from other, already available, systems.

where $f(e(x), a)$ maps *rest* to the *rest'* appropriate for $x \cdot a$. For $f(e(x), a)$ to be well-defined it is also necessary that $f(e(x), a) = f(e(x'), a)$ whenever $e(x) = e(x')$.

How good this coordinatization is of course will depend on how simple the component where $f(e(x), a)$ is computing is, and on how simple the action $e(X) \times A \longrightarrow e(X)$ is.

The component for computing with the conserved quantity is $(e(X), S/\sim)$, with S made faithful on $e(X)$, and we have a surjective morphism of transformation semigroups $(X, A^+/\equiv) \twoheadrightarrow (e(X), S/\sim)$. By use of the covering lemma[6] for transformation semigroups [9] (see section 6 below), one can characterize exactly the solutions for component (Z, T) [computing with *rest* and transformations $f(e(x), a)$] for which $(X, A^+/\equiv)$ is covered by $(Z, T) \wr (e(X), S/\sim)$, thus extending to an emulation the original homomorphism from the system to the conserved quantity semigroup.

If each transformation $\cdot a$ of X is invertible (as is the case in reversible physics!), the kernel theorem [9] can be used to show that (Z, T) can be chosen to be a transformation group. Thus symmetry arises from a homomorphism on a reversible physical system.

4.2. Understanding via Symmetries. Given our same system $(X, A, \lambda : X \times A \longrightarrow X)$ and induced transformation semigroup (X, S), we consider the symmetries of the system. Here a permutation $\pi : X \longrightarrow X$ is called a *symmetry* if $\pi(x \cdot a) = \pi(x) \cdot a$ for all events $a \in A$.

Denote by G any group of symmetries of $(X, A^+/\equiv)$. It is easy to check that $[x] \cdot a = [x \cdot a]$ is well-defined and so induces an action of S on X/G. Then we establish

$$(X, S) \prec (G, G) \wr (X/G, T),$$

where $(X/G, T)$ is the faithfulization of the action of S induced on X/G:

Choose orbit representatives $\overline{x} \in [x] \in X/G$. Now for $x \in [x]$ there is a π with $\pi(\overline{x}) = x$. So we coordinatize by

$$x \longmapsto (\pi, [x]).$$

Taking all such $(\pi, [x])$ as related to x defines a injective relation, since π and $[x]$ determine x uniquely. We let $(\pi, [x]) \cdot a = (\pi \cdot f_a[x], [x \cdot a])$, where $f_a[x]$ is an element of G with $f_a[x](\overline{x \cdot a}) = \overline{x} \cdot a$. ($f_a[x]$ must exist since $[\overline{x \cdot a}] = [\overline{x} \cdot a]$.)

This is a covering as $(\pi, [x]) \cdot a = (\pi \cdot f_a[x], [x] \cdot a) = (\pi f_a[x], [x \cdot a])$ which is a coordinatization of $x \cdot a$ since $\pi f_a[x](\overline{x \cdot a}) = \pi(\overline{x} \cdot a) = \pi(\overline{x}) \cdot a = x \cdot a$.

The covering is an embedding in the case of transitive (X, S): Each π is a regular permutation, that is, it fixes no point or is the identity: if π fixes x_0, then by transitivity we can write any x as $x_0 \cdot a_1 \ldots a_n$ and then $\pi(x) = \pi(x_0) \cdot a_1 \ldots a_n = x_0 \cdot a_1 \ldots a_n = x$. Now it is easy to see the π taking $\overline{x}$ to x as well as the $f_a[x]$ are unique. And so $(\pi, [x])$ would be the unique coordinatization (lift) for x and ($f_a[]$, image of a in T) would be the unique lift for the action of a.

Thus the symmetries of a system provide a "most detailed" coordinate in a formal model for understanding the system. This is in contrast to conservation quantities which provide a top-level (least detailed or most vague) coordinate in a formal model of understanding for the system. Moreover, with the action of a symmetry

[6] Covering lemmata in various settings are also called kernel theorems.

group G as above, the component $(X/G,T)$ computes a conserved quantity (just take $e(x) = [x]$), invariant under the symmetries in G.

An interesting open problem is to re-cast Noether's original results [11] or their modern generalization [12] explicitly into the wreath product form.

5. Krohn-Rhodes & Lagrange Coordinates

For a given (faithful) (X,S), say with X finite, one can can ask, *What are the formal models for understanding* (X,S)*? Indeed, Do any non-trivial ones exist?* The answer to the latter question is given by the celebrated

Krohn-Rhodes Theorem. Let (X,S) be a finite faithful semigroup, then

$$(X,S) \prec (X_n,S_n) \wr \cdots \wr (X_1,S_1),$$

where each (X_i,S_i) is either (G,G) where G is a finite simple group or $(X_i,S_i) =$ the flip-flop, the faithful transformation semigroup with two states, two resets and an identity. In any such decomposition each finite simple group divisor of S must occur among the S_i. Moreover, the group S_i may be chosen to be divisors of S.

This gives the existence result for global hierarchical coordinate systems on arbitrary finite S. Note that the order type of the hierarchy is a total order. More generally partial orderings of components may be possible and provide better models of understanding (cf. the Cartesian coordinates on vector spaces).

If (X,S) is a transformation group, then in fact embedding in the wreath product is possible.

Lagrange Coordinatization Theorem. Let (X,S) be a finite, transitive faithful transformation group. Then

$$(X,S) \leq (G_n,G_n) \wr \cdots \wr (G_1,G_1),$$

where each G_i is a simple divisor of S.

A maximal subnormal chain in G yields such a decomposition, and results of the author [8] imply that each such decomposition corresponds essentially to such a subnormal chain.

The above results refer to finite (X,S). Several infinite versions of the Krohn-Rhodes theorem have now been published.[7] Also the results of the author for groups extend to infinite groups, using wreath products over arbitrary total orders (cascade integrals) rather than finite total orders.

In the finite case, with computer implementation in mind, it will useful to develop more computationally feasible versions of the Krohn-Rhodes Theorem, especially versions in terms of finite-state machines while reducing the full semigroups that must be calculated. Indeed, for $|X| = n$ and faithfully acting S, the number of elements in S may be as high as n^n, so one would not want to *explicitly* represent S

[7] Infinite versions of the Krohn-Rhodes theorem include the Infinite Iterative Matrix Semigroup coverings of Rhodes [14], the Allen-Birget-Rhodes Synthesis Theorem [1, 15], the Eilenberg-Elston-Henckell-Lazurus-Nehaniv-Rhodes-Zeiger Holonomy Embedding [2, 3, 4, 18], and the Cascade Covering Theorem [7].

or a cover in computer memory. Rather, one could seek to represent lifts of a small set of generators for S, e.g. lifts of just the events A, to a covering transformation semigroup with global hierarchical coordinates.

6. Creating Understanding from Failed Emulation

A (for simplicity) surjective morphism or relational morphism

$$(Q, R) : (X, S) \lhd (Y, T)$$

can be regarded as a failed attempt to cover (X, S) with (Y, T). It is suggestive to call a relational morphism a *metaphor* or *partial model for understanding* (X, S).

Furthermore, it is possible to characterize exactly what must be "added" to this attempt in order to obtain a real emulation. That is, we want to determine how and for which (Z, U) one can extend the relation to obtain an emulation

$$(X, S) \prec (Z, U) \circ (Y, T).$$

The answer is that the the *kernel* of the relation morphism (Q, R) must be computed by (Z, U). The reader can find the exact definition of the kernel and theorems on extending a relational morphism to an emulation (or embedding) in [9].[8]

This situation is entirely analogous with the case of group theory in which any group K' that is divided by ("computes") the kernel K of a group homomorphism $\varphi : G \twoheadrightarrow H$ may be wreathed onto H to obtain an emulation: $G \prec K' \wr H$, projecting onto φ, and no K' for which $K \not\prec K'$ will work.

If (Y, T) is already in global hierarchical coordinate form, the relational morphism can be regarded as an *analogical mapping*, that is as the suggestion of a full analogy facilitating the re-use of an already available hierarchical decomposition.

Using the above-mentioned kernel theorem [9], one could find exactly what else (what (Z, U)) is needed to "make to the analogy work".

This provides a formal algebraic approach to moving from metaphors and analogical mappings to formal models for understanding (X, A, λ) or (X, S). In fact, an automated computational system could carry out decomposition algorithms for systems, look for relational morphisms between them, and generate new global hierarchical coordinate systems according to the kernel theorem. This suggests an algebraic computational basis for finding formal models of understanding using metaphor and analogy. Resolving partially successful metaphors (relational morphisms) with the covering lemma as a constraining guide, we seek to automatically generate new understanding from old. Of course, a global hierarchical coordinatization for formal understanding of a system that an automated computational process finds and uses need not be one that a human would create.

[8] As a sketch, the kernel consists of the maximal subsets X_y of X related to fixed elements of Y and arrows consisting of mappings between two such (formally disjoint) subsets X_y to $X_{y'}$ induced by the action of S (with some arrow identifications). To compute the kernel a [possibly relational] labelling of the subsets and mappings by elements of Z and U must be compatible with the mappings of the kernel, and satisfy certain separation conditions. Moreover, the converse holds for faithful (X, S). Full details and proofs are in [9].

Some Acknowledgements & Remarks

I owe a great intellectual debt to my friend and teacher Professor John Rhodes for the concept that the theory of an experiment can be formalized by a wreath product covering. Further motivation and a more complete discussion of the relations to physics can be found in his book [13]. Prof. T.L. Kunii encouraged me to apply ideas from semigroup theory to computer systems. Thanks to Dr. Bret Tilson for proving the first kernel theorems for semigroups, which inspired similar considerations for systems modelled as transformation semigroups [9]. Also I thank Prof. Masami Ito for his kind invitation to present these ideas at RIMS.

The notion (due to the author) that relational morphisms can be considered as analogies between formal models of understanding appears to be original (first announced in [6]). It first occurred to me while reading S. Ryan Johansson's article [5] which argues that metaphors provide a kind of software for the human mind by offering suggestions or commands to attempt to consistently relate two systems and thus force the mind to construct understanding by trying to re-solve ambiguities and contradictions of the resulting mapping.

The programme for automatic manipulation of formal models of understanding via algebra can itself be understood as an attempt to resolve the meta-metaphor: *Relational morphisms are metaphors* and *Kernel theorems describe the creation of meaning from resolving the failure of metaphors to work perfectly.*

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