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Partitions of the set of positive integers, nonperiodic sequences, and transcendence

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§1 Partitions of the set of positive integers I

Throughout the paper, we identify a set $\{s_n;\ n\in\mathbb{N}\}\subset\mathbb{N}$ such that $s_1<s_2<s_3<\ldots$ with a sequence $\{s_n\}_{n=1,2,3,\ldots}$. It is well-known that for positive numbers α and β, two Beatty sequences (or sets) $\{[\alpha n]\}_{n=1,2,3,\ldots}$ and $\{[\beta n]\}_{n=1,2,3,\ldots}$ make a partition of the set $\mathbb{N}$ into two parts iff α, β are real irrationals satisfying $1/\alpha+1/\beta=1$, where $[x]$ $(x\in\mathbb{R})$ is the largest integer not exceeding x, and $\mathbb{N}$ (resp. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_+$) denotes the set of the positive integers (resp. the integers, the rational numbers, the real numbers, the positive numbers). This fact can be written in an equivalent form as

$$\dot{\bigcup_{(\gamma_0,\gamma_1)\in A}}\{[\gamma_0 n]+[\gamma_1 n];\ n\in\mathbb{N}\} = \mathbb{N},$$
$$A := \{(1,\alpha),\ \alpha^{-1}(1,\alpha)\},\ \alpha>0 \tag{1}$$

iff α is an irrational, where $\dot{\bigcup}$ indicates a disjoint union. Proposition 1 is a generalization of (1), which gives a partition of $\mathbb{N}$ into s+1 parts by specific sums of Beatty sequences, cf. [T5], Theorem 1.

We denote by αS, $S+\alpha$, $S+T$, and ST the set $\{\alpha s;\ s\in S\}$, $\{s+\alpha;\ s\in S\}$, $\{s+t;\ s\in S,\ t\in T\}$, and $\{st;\ s\in S,\ t\in T\}$, respectively, for given sets $S, T\subset\mathbb{R}$, and a number $\alpha\in\mathbb{R}$; by $\langle x\rangle$, the fractional part of $x\in\mathbb{R}$, i.e., $\langle x\rangle:=x-[x]$.

<u>Proposition 1</u>. Let s be a positive integer, and $\alpha_i>0$, β_i $(0\leqq i\leqq s)$ be real numbers. Then the condition

$$(\alpha_i^{-1}\mathbb{Z}-\alpha_i^{-1}\beta_i)\cap(\alpha_j^{-1}\mathbb{Z}-\alpha_j^{-1}\beta_j)\cap\mathbb{R}_+ = \phi \quad \text{for all } i\neq j \tag{2}$$

is necessary and sufficient to have a partition

$$\dot{\bigcup_{(\underline{\gamma},\underline{\delta})\in B}}\{\sum_{0\leqq j\leqq s}[\gamma_j n+\delta_j];\ n\in\mathbb{N}\} = \mathbb{N}, \tag{3}$$

where $(\underline{\gamma},\underline{\delta})=(\gamma_0,\gamma_1,\dots,\gamma_s,\delta_0,\delta_1,\dots,\delta_s)$, and

$$B := \{(\underline{\alpha},\underline{\beta});\ \underline{\alpha}=\alpha_i^{-1}(\alpha_0,\alpha_1,\dots,\alpha_s),$$
$$\underline{\beta}=-\alpha_i^{-1}\langle\beta_i\rangle(\alpha_0,\alpha_1,\dots,\alpha_s)+(\langle\beta_0\rangle,\langle\beta_1\rangle,\dots,\langle\beta_s\rangle),\ 0\leqq i\leqq s\}.$$

Setting $\alpha_0=1$, $\beta_i=0$ for all i, we have

Corollary 1. Let $s\in\mathbb{N}$, $\alpha_0=1$, $\alpha_i\in\mathbb{R}_+$ $(1\leqq i\leqq s)$. Then the condition $\alpha_i\notin\mathbb{Q}$, and $\alpha_i/\alpha_j\notin\mathbb{Q}$ for all $1\leqq i<j\leqq s$ implies

$$\dot{\bigcup_{(\gamma_0,\dots,\gamma_s)\in C}}\ \{\sum_{0\leqq j\leqq s}[\gamma_j n];\ n\in\mathbb{N}\} = \mathbb{N},$$

and vice versa, where $C:=\{\alpha_i^{-1}(\alpha_0,\dots,\alpha_s);\ 0\leqq i\leqq s\}$.

If we take s=1 in Corollary 1, we obtain (1). We remark that we may choose $\alpha_0=1$, $\beta_0=0$ in Theorem 1 without changing the form of the components of the partition. Related to the condition (2), we can show the implications (7)⇒(6)⇒(5)⇒(2)⇒(4) in the case of $\alpha_0=1$, $\beta_0=0$, where (4)-(7) are the following conditions:

$-\beta_i\notin\alpha_i\mathbb{N}+\mathbb{Z}$ for all $1\leqq i\leqq s$; (4)

$\alpha_i\beta_j-\alpha_j\beta_i\notin\alpha_i\mathbb{Z}+\alpha_j\mathbb{Z}$ for all $0\leqq i<j\leqq s$; (5)

1, α_i, β_i are linearly independent over $\mathbb{Q}$ for each $1\leqq i\leqq s$, and $(\alpha_i(\mathbb{Z}+\beta_i)\mathbb{Q})\cap(\alpha_j(\mathbb{Z}+\beta_j)\mathbb{Q})=\{0\}$ for all $0\leqq i<j\leqq s$; (6)

2s+1 numbers 1, and α_i, $\alpha_i\beta_i$ $(1\leqq i\leqq s)$ are linearly independent over $\mathbb{Q}$. (7)

We remark that a result obtained by J. V. Uspensky [Us] says the impossibility of having a partition into t parts by Beatty sequences for $t\geqq 3$. Proposition 2 is a generalization of Proposition 1 (cf. [T5], Theorem 3).

Proposition 2. Let $f_i:\mathbb{R}_+\cup\{0\}\to\mathbb{R}$ $(0\leqq i\leqq s,\ 1\leqq s\in\mathbb{N})$ be continuous, strictly monotone increasing functions with $\lim_{x\to\infty} f_i(x) = \infty$ for all i. Then the condition

$$f_i^{-1}(\mathbb{Z})\cap f_j^{-1}(\mathbb{Z})\cap\mathbb{R}_+ = \phi \text{ for all } i\neq j \qquad (8)$$

is necessary and sufficient to have partition

$$\dot{\bigcup_{0\leq i\leq s}} \{ \sum_{0\leq j\leq s} ([f_j(f_i^{-1}(n+[f_i(0)]))]-[f_j(0)]);\ n\in\mathbb{N} \} = \mathbb{N}. \qquad (9)$$

The property $\lim_{x\to\infty} f_i(x) = \infty$ can be omitted from Proposition 2, at most, for s indices i. In that case, some of the components of the partition (9) turn out to be a finite set. We remark that in this sense, any partition of $\mathbb{N}$ into s+1 parts can be given by (9) under a suitable choice of the functions f_i (without loss of generality, we may assume that all the f_i are of $\mathscr{C}^\infty$ class with $f_0(x)=x$), that will be clear by the following argument.

The idea of the proof of the propositions is very simple. First, we refer the fact that if an infinite word (to the right) $\omega=\omega_1\omega_2\omega_3\ldots$ strictly over an alphabet $S_s:=\{a_0,a_1,\ldots,a_s\}$ (i.e., every symbol a_i eventually occurs in ω) is given, then it gives rise to a partition of $\mathbb{N}$ into s+1 parts:

$$\dot{\bigcup_{0\leq i\leq s}} \chi(\omega;a_i) = \mathbb{N}, \qquad (10)$$

which will be referred to as the partition corresponding to the ω, and vice versa, where $\chi(\omega;a)$ is the characteristic set of ω with respect to a:

$$\chi(\omega;a) := \{n\in\mathbb{N};\ \omega_n=a\},\ a\in S_s.$$

We denote by Π; $\Pi_i\subset\mathbb{R}^{s+1}$ the set of hyperplanes defined by

$$\Pi_i:=\{(x_0,\ldots,x_i,\ldots x_s);\ x_j\in\mathbb{R}\ (j\neq i),\ x_i\in\mathbb{Z}\}\ (0\leq i\leq s),\ \Pi:=\bigcup_{0\leq i\leq s}\Pi_i,$$

by $K\subset\mathbb{R}^{s+1}$ the curve

$$K := \{\underline{f}(x)=(f_0(x),\ldots,f_s(x));\ x\in\mathbb{R}_+\}.$$

Secondly, we consider an infinite word $\omega=\omega(K)=\omega_1\omega_2\omega_3\ldots$ given by

$$\omega_n=a_i \text{ if } \underline{f}(x_n)\in\Pi_i,$$

where the sequence $\{x_n\}_{n=1,2,3,\ldots}$ is defined by

$$\{\underline{f}(x_n);\ 0<x_1<x_2<\ldots<x_n<\ldots\} := K\cap\Pi.$$

Note that the sequence $\{x_n\}_{n=1,2,3,\ldots}$ is well-defined, since the set $K\cap\Pi$ is a discrete one in $\mathbb{R}^{s+1}$ if the functions f_i are continuous, and strictly monotone increasing; and that the word ω is well-defined by the condition (8).

Under the assumption that the functions f_i are continuous, strictly monotone increasing, we can calculate the nth term of the sequence (or the set) $\chi(\omega; a_i)$ by using the intermediate value theorem, and we can obtain Proposition 2. Taking K to be a half-line L, we get Proposition 1. For further details of the proof, see [T5].

Proposition 1 has some conection with higher dimensional billiards: Let I^{s+1} ($I:=[0,1]$) be the unit cube of dimension s+1 with the faces

$$\{(x_1,\dots,x_i,\dots x_t);\ x_j\in I\ (\forall j\neq i),\ x_i=0,\ \text{or}\ 1\}\quad (0\leqq i\leqq s)$$

labelled by a_i. Let a particle start at a point $\underline{\beta}\in[0,1)^{s+1}$ along a vecter $\underline{\alpha}\in \mathbb{R}_+^{s+1}$ with the condition for α_i, β_j stated in Proposition 1, and be reflected at each face of I^{s+1} specularly. Then the word $\omega(L)$ ($L:=\{\underline{\alpha}t+\underline{\beta};\ t\in\mathbb{R}_+\}$ defined above coincides with a word obtained by writing down the label a_i of the faces which the particle hits in order of collision. The complexity $p(n)=p(n;w)$ of an infinite word $w=w_1w_2w_3\dots$ is a function $p:\mathbb{N}\to\mathbb{N}$ defined to be the number of subwords of length n of w:

$$p(n;w) := \#\{w_m w_{m+1}\dots w_{m+n-1};\ m\in\mathbb{N}\}\quad (n>0,\ p(0;w):=1).$$

An infinite word w (or a sequence) is called sturmian (on s+1 letters) if $p(n;w)=n+s$ for every n. It is known that if w is not an ultimately periodic word strictly over the alphabet S_s, then $p(n;w)\geqq n+s$, cf. [He-Mo, F-M]. It is a classical result that for s=1, $\omega(L)$ is sturmian on 2 letters provided that ω is not periodic. A conjecture of G. Rauzy says that $p(n;\omega(L))=n^2+n+1$ for s=2 when $\alpha_0,\alpha_1,\alpha_2$ are linearly independent over $\mathbb{Q}$, which was proved affirmatively in [A-M-S-T1], cf. [R1, R2, A-M-S-T2]. An exact formula for $p(n;\omega(L))=p(n,s;\omega(L))$ as a function of n and s in the case where $\alpha_0,\dots,\alpha_s$ are linearly independent over $\mathbb{Q}$ was conjectured in [A-M-S-T1]:

$$p(n,s) = \sum_{0\leqq i\leqq \min\{n,s\}} n!\cdot s!/((n-i)!\cdot i!\cdot(s-i)!)$$

and proved affirmatively by Yu. Baryshnikov [B]. Consequently, $p(n,s)=p(s,n)$, and $p(n;3)=n^3+2n+1$ holds, that was one of my conjectures, from which P. Arnoux and Ch. Mauduit derived the exact formula under some minor hypotheses. It will be an interesting question, which was posed by Ch. Mauduit, that asks for a direct (or combinatorial) proof of the symmetry; it still remains mysterious why $p(n,s)$ is a symmetric function.

Quite recently, a remarkable result was obtained by S. Ferenczi and Ch. Mauduit [F-M], which asserts that the numbers having a sturmian sequence consisting of any number of letters $\in \{n\in \mathbb{N};\ 0\leqq n\leqq h\}$ as their expansion in some base $g(\geqq h+1)$ are transcendental, that was conjectured by Ch. Mauduit for himself in 1989. They gave further results on transcendency of numbers having a sequence (or an infinite word) with low complexity as their expansion in base g.

We say that the partition (10) is nonperiodic (resp. totally nonperiodic) if ω (resp. $\partial\chi(\omega;a_i)$ for all i) is not an ultimately periodic word (or sequence), and vice versa, where we mean by ∂C the sequence $\{c_{n+1}-c_n\}_{n=1,2,3...}$ for a given sequence $C:=\{c_n\}_{n=1,2,3...}$. We may assume that $\alpha_0=1$ in Proposition 1 as we have already seen. In that case, the partition (3) is nonperiodic if one of the α_i $(1\leqq i\leqq s)$ is irrational, since the irrationality of $\check{p}(a_i)/\check{p}(a_o)$ implies the nonperiodicity of $\omega(L)$, where $\check{p}(a_i)$ is the frequency, in the limit sense, of a symbol a_i appearing in the word $\omega(L)$ corresponding to the partition (3).

In what follows, we shall give some classes of nonperiodic partitions of $\mathbb{N}$ (or some classes of nonperiodic infinite words), and some results and problems related to transcendence and complexity.

§2 Partition of the set of positive integers II

By $S=S_s$ we mean the alphabet $\{a_0,a_1,...,a_s\}$ $(s\geqq 1)$ as in Section 1. We denote by S^* the set of all finite words over the S, S^* is a free monoid generated by the S with the operation of concatenation and the empty word λ as its unit. S^∞ denotes the set of all infinite word (to the right) over S. A substitution σ (over S) is a monoid endomorphism σ on S^* extended to S^∞ defined by $\sigma(w) := \sigma(w_1)\sigma(w_2)\sigma(w_3)\ldots$ for $w = w_1 w_2 w_3 \ldots \in S^\infty$. A fixed point of σ is an infinite word $\omega \in S^\infty$ satisfying $\sigma(\omega) = \omega$. Any substitution of the form

$$\sigma(a) = au \ (a \in S,\ u \neq \lambda),\quad \sigma(x) \neq \lambda \ (\forall x \in S)$$

has a unique fixed point w prefixed by a, i.e., $w = au\sigma(u)\sigma^2(u)\sigma^3(u)\ldots$, where σ^n is an n-fold iteration of σ (σ^0 is an identity map on $S^* \cup S^\infty$).

We denote by $|w|$ the length of a finite word w, and by $|w|_a$ the number of occurrences of a symbol $a \in S$ appearing in a word $w \in S^*$. For a given sequence $C = \{c_n\}_{n=1,2,3,\ldots}$, $\int_i C$ indicates the sequence

$$\int_i C := \{i + \sum_{1 \leq m \leq n-1} c_m\}_{n=1,2,3,\ldots}.$$

Then we can show the following

Proposition 3. Let σ be a substitution over the S defined by

$$\sigma(a_j) := a_0^{k_{s-j}} a_{j+1} \quad (0 \leqq j \leqq s-1),\quad \sigma(a_s) := a_0,$$

where k_i $(0 \leqq i \leqq s)$ are integers satisfying $k_s \geqq k_{s-1} \geqq \ldots \geqq k_0 = 1$. Let L_j be the set $\{|\sigma^{j-1}(a_0)|,\ |\sigma^{j-1}(a_0)| + |\sigma^{j-1}(a_1)|,\ \ldots,\ |\sigma^{j-1}(a_0)| + |\sigma^{j-1}(a_s)|\}$, and let $\tau_j : S^* \to L_j^*$ be a monoid morphism defined by

$$\tau_j(a_i) := (|\sigma^{j-1}(a_0)|)^{k_{s-i}} (||\sigma^{j-1}(a_0)| + |\sigma^{j-1}(a_i)| \quad (0 \leqq i \leqq s-1),$$

$$\tau_j(a_s) := |\sigma^{j-1}(a_0)|.$$

Then

$$\dot{\bigcup_{0 \leqq j \leqq s}} \int_{|\sigma^{j-1}(a_0)|} \tau_j(\omega) = \mathbb{N}, \tag{11}$$

where ω is the fixed point of σ.

It is clear that (11) follows from $\chi(\omega; a_j) = \int_{|\sigma^{j-1}(a_0)|} \tau_j(\omega)$, which is Theorem 4 in [T4].

Note that the partition (11) is a totally nonperiodic one for all $s \geqq 1$, and all $k_i \in \mathbb{Z}$ satisfying $k_s \geqq k_{s-1} \geqq \ldots \geqq k_0 = 1$, that follows from [T4], Lemma 11:

$$\lim_{n \to \infty} |\sigma^n(a_0)|_{a_i} \ / \ |\sigma^n(a_0)| = \alpha^{s-i} / (\alpha^s + \alpha^{s-1} + \cdots + \alpha + 1), \tag{12}$$

where $\alpha>1$ is an algebraic number with minimal polynomial $f(x):=x^{s+1}-\sum_{0\leq i\leq s} k_i x^i$; the minimality follows from [T4], Lemma 10.

We remark that in general, the partition (11) can not be the partion of the form (3). For instance, suppose that (11) with $s=2$, $k_1=k_2=1$ coincides with (3) corresponding to some infinite word $\omega=\omega(L)$ ($L=\{t(1,\alpha_1,\alpha_2)+\underline{\beta};\ t\in\mathbb{R}_+\}$, then (12) implies that $\alpha_i=\alpha^{-i}$ ($i=1,2$). The minimality of $f(x)$ implies that 1, α_1, α_2 are linearly independent over $\mathbb{Q}$. Hence, $p(n;\omega)=n^2+n+1$, which contradicts that $p(n)=2n+1$ is the complexity of the fixed point of the substitution σ with $s=2$, $k_1=k_2=1$ (the fixed point is an Arnoux-Rauzy sequence, cf. [A-R], [F-M]). On the other hand, in the case of $s=1$, the partition (11) turns out to be the partition (1), that will be seen by the following argument: Proposition 3 with $s=1$ implies $f(x)=x^2-kx-1$ ($k:=k_1$), so that $\alpha=(k+(k^2+4))^{1/2}/2$. Setting $\chi(\omega;a_i)=\{t_1^{(i)}<t_2^{(i)}<\dots<t_n^{(i)}<\dots\}$ ($i=0,1$), we get by Proposition 3

$$t_n^{(1)}=kn+t_n^{(0)}, \qquad \chi(\omega;a_0)\mathbin{\dot\cup}\chi(\omega;a_1)=\mathbb{N}. \tag{13}$$

Noting that the sets $\chi(\omega;a_i)$ are uniquely determined by (13), and

$$[\eta_1 n]=kn+[\eta_0 n],\quad 1/\eta_0+1/\eta_1=1 \quad (\eta_0:=1+1/\alpha,\ \eta_1:=1+\alpha),$$

we obtain $\chi(\omega;a_i)=\{[\eta_i n];\ n\in\mathbb{N}\}$ ($i=0,1$).

Let $\tau: S^*\to G^*$ ($G=G_g:=\{0,1,\dots,g-1\}^*$, $2\leqq g\in\mathbb{N}$) be a monoid morphism such that $\tau(a)\neq\lambda$ for all $a\in S$. We denote by ${}_g0.\tau(w)$ ($w=w_1w_2w_3\dots\in S^\infty$, $w_i\in S$) the number defined by $\sum_{i\geqq 1}\tau(w_i)/g^i$. We say ω is transcendental if ${}_g0.\tau(w)$ is transcendental for an integer g and a morphism τ. The fixed point ω is not only totally nonperiodic, but also transcendental:

<u>Proposition 4</u> ([T4], Theorem 3). Let ω be as in Proposition 3, $g\geqq 2$ an integer, τ a monoid morphism such that $\tau(a)\neq\lambda$ for all $a\in S$, and

$$\mathrm{rank}\ (|\tau(a_i)|_j)_{0\leq i\leq s,\ 0\leq j\leq g-1} > 1.$$

Then the number ${}_g0.\tau(\omega)$ is transcendental.

The key for the proof of Propositon 4 is to show that the ω has a prefix which is $(2+\varepsilon)$-power of a nonempty word (cf. Proposition 9 below, and [T4], Lemma 13); that can be connected with Roth's theorem. A stronger argument works in [F-M], where S. Ferenczi and Ch. Mauduit made use of a theorem of Ridout ([Mah], pp. 147-148) instead of Roth's theorem. We shall mention their results in the following sections.

§3 Partition of the set of positive integers III

Let $D\,(\ni 1)$ be a subset of $\mathbb{N}$. In some cases, we can show that there exists a subset Γ such that

$$\dot{\bigcup_{d \in D}} d\Gamma = \mathbb{N} \quad (D \neq \phi,\ \{1\}) \tag{14}$$

Such a partition will be referred to as a similis partition (of $\mathbb{N}$ with respect to D). We gave some results on similis partitions in [T1]. I would like to mention that a simple example of similis partitions came from a linguistic phenomena in Hungarian and Japanese language that are probably well-known to linguists, cf. [Ta, To]: Numerals one, two, three, four, ... in Hungarian (resp. Japanese) are egy, kető, három, négy, ... (hi, fu, mi, yo,...). So, we can make the following diagram, where in each language, underlined consonants of two numerals in each row are common, or they have a resemblance (e.g., n and ny= palatalized n in the 3rd stage of the diagram); and simultaneously, in each row, the number corresponding to the right group is exactly the two times of the left:

	Γ		2Γ	
1):	1	egy (hi←fi←pi)	2	kettő (fu←pu)
2):	3	három (mi)	6	hat (mu)
3):	4	négy (yo)	8	nyolc (ya)
4):	5	öt (itsu←itu)	10	tíz (to)

Here, among the numerals in Japanese language that are written in parentheses, for instanse, itsu←itu indicates that contemporary Japanese word itsu comes from old Japanese word itu, that is a kind of palatalization. If we look at the numerals of older Japanese in parentheses, the consonants correspondence turns out to be an exact one. (Related to vowels, see, e.g., [Ha]; vowel harmony is also common to Hungarian and old Japanese.)

Considering what will happen, apart from numerals in natural language, when we formally prolong the diagram downwards, we get a similis partition (14) with $D=\{1,2\}$, which is uniquely determined. In fact, it is clear that $\gamma_1:=1\in\Gamma$, so that $2\gamma_1\in 2\Gamma$, which gives the first stage 1) of the diagram. Now, consider the smallest positive integer γ_2 among the numbers that have not appeared in the stage 1). Then the minimality of γ_2 implies $\gamma_2\in\Gamma$, othewise $\gamma_2\in 2\Gamma$, so that $\gamma_2>\gamma_2/2\in\Gamma$, i.e., the second stage is of the form $\gamma_2/2\in\Gamma$, $\gamma_2\in\Gamma$, which contradicts the minimality of γ_2. (Forget that $\gamma_2\in\Gamma$ follows from that $\gamma_2=3$ is odd; we shall see that $\gamma_3(=4)$ is even in the folowing argument.) Suppose that we have obtained a diagram with stages 1)-n). Considering the number γ_{n+1} defined to be the smallest positive integer that differs from all the numbers appearing in the stages 1)-n). Then $\gamma_{n+1}\in\Gamma$ follows from its minimality. We can continue the process, and we must have $\Gamma=\{\gamma_1,\gamma_2,\gamma_3,\ldots\}$ as far as all the numbers $d\gamma_{n+1}$ $(d\in D)$ are different from the numbers $d\gamma_m$ $(d\in D,\ 1\leqq m\leqq n)$. Hence, noting that the argument given above is vaid for any nonempty finite, or infinite subset $D\subset\mathbb{N}$, we obtain

<u>Proposition 5</u>. If there exists a similis partition (14) for a given nonempty subset D of $\mathbb{N}$, then the partition is unuquely determined by the set D.

On the other hand, it is clear that a similis partition (14) for $D=\{1,2\}$ exists, since $\Gamma=\{2^{2j}m;\ j\geqq 0,\ m\geqq 1,\ m \text{ is odd}\}$ satisfies (14), that will be refferred to as the H.-J.(Hungarian-Japanese) partition. The H.-J. partition can be easily generalized as

<u>Proposition 6</u>. Let $D=D(k;q_1,\ldots,q_k;e_1,\ldots,e_k)$ be a set defined by

$$D := \{\prod_{1\leqq i\leqq k} q_i^{j_i};\ 0\leqq j_i\leqq e_i\ (1\leqq i\leqq k)\},$$
$$k\geqq 1,\ q_i\geqq 2,\ e_i\geqq 1\ (1\leqq i\leqq k),\ \text{G.C.D.}(q_i,q_j)=1 \text{ for all } i\neq j. \qquad (15)$$

Then

$$\Gamma=\{\prod_{1\leq i\leq k} q_i^{(e_i+1)j_i} m;\ j_i\geqq 0,\ m\geqq 1,\ \text{G.C.D.}(m,q_1\cdots q_k)=1\}$$

satisfies (14), which is uniquely determined by D.

One of my old conjecture says that if a similis partition (14) is a partition of $\mathbb{N}$ into finite components, then there exist numbers k, and $q_1, \ldots, q_k, e_1, \ldots, e_k$ satisfying (15); that is probably still open. It is easily seen that there are no partitions (14) for some explicitely given D which are not of the form (15), cf. [T1], Theorem 12. For example, if we take $D=\{1,2,3\}$, and trace the uniqueness proof of (14) above, we see $2\gamma_4=12=3\gamma_2$, which contradicts that (14) is a disjoint union. Proposition 6 can be extended to infinite partitions with respect to D given by (15) with $0\leqq j_i$ for some indices i instead of $0\leqq j_i\leqq e_i$ $(1\leqq i\leqq k)$:

$$D:=\{\prod_{1\leq i\leq k-h} q_i^{j_i}\cdot\prod_{k-h+1\leq i\leq k} q_i^{j_i};\ 0\leqq j_i\leqq e_i\ (1\leqq i\leqq k-h),\ 0\leqq j_i\ (k-h+1\leqq i\leqq k)\}$$

$k\geqq 1$, $k\geqq h\geqq 1$, $q_i\geqq 2$ $(1\leqq i\leqq k)$, $e_i\geqq 1$ $(1\leqq i\leqq k-h)$, $\text{G.C.D.}(q_i,q_j)=1$ for all $i\neq j$.

For D given above, we can show

$$\Gamma=\{\prod_{1\leq i\leq k-h} q_i^{(e_i+1)j_i} m;\ j_i\geqq 0,\ m\geqq 1,\ \text{G.C.D.}(m,q_1\cdots q_k)=1\}\ (k\geqq 2).$$

If $k=1$, then $\Gamma=\mathbb{N}\setminus q_1\mathbb{N}$, and the partition (14) is periodic (not interesting). We remark that for some infinite partitions (14), D is not always of the form above. For instance, if we take $D=\{p_i^{j_i};\ j_i\geqq 0\ (0\leqq i\leqq s)\}$ with prime numbers p_i $(p_0>p_1>\ldots>p_s,\ s\geqq 1)$, then

$$\Gamma=\{(p_0\cdots p_s)^i m;\ i\geqq 0,\ m\geqq 1,\ \text{G.C.D.}(m,p_0\cdots p_s)=1\}$$

satisfies (14). By the way, we remark that a sequence $\omega=\omega_1\omega_2\omega_3\ldots$ over S_s defined by

$$\omega_n:=a_i \text{ if } m_n\in\{p_i^j;\ j\geqq 0\}\ (\{1<m_1<m_2<\ldots<m_n<\ldots\}:=D=\{p_i^{j_i};\ j_i\geqq 0\ (0\leqq i\leqq s)\})$$

coincides with a word $\omega(L)$ defined by the billiard in I^{s+1} with $\underline{\alpha}=(\alpha_0,\ldots,\alpha_s)$, $\underline{\beta}=\underline{0}$, $\alpha_i=\log p_i/\log p_0$, cf. [R2].

Now, we return to the first example of (14), the H.-J. partition. We shall show that the word $\omega=\omega_1\omega_2\omega_3\ldots$ $(\omega_n\in S_1)$ corresponding to the H.-J. partition is a totally nonperiodic word, which is the fixed point of a substitution. We mean by UV the set $\{uv;\ u\in U,\ v\in V\}$, by U^* the set $\{u_1\dots u_n;\ u_i\in U\ (1\leqq i\leqq n),\ n\geqq 0\}$ for subsets U, V of a monoid, and by $\Gamma\ni 1$ the component of the H.-J. partition. Then $\gamma\in\Gamma$ iff $E_2(\gamma)=u0^{2n}(n\geqq 0,\ u\in\{0,1\}^*$ is a word having 1 as its prefix, and suffix), where $E_g(\gamma)$ denotes the base-g expansion of $\gamma\in\mathbb{N}\cup\{0\}$ $(E_g(0):=\lambda)$, and w^n $(w\in S_s{}^*)$ is the word obtained by concatenating n copies of w. So, $\gamma\in\Gamma$ iff $E_2(\gamma-1)=v1^{2n}$ $(v\in G_1{}^*=\{0,1\}^*,\ n\geqq 0)$. Hence the set $\{0\}^*\{E_2(\gamma-1);\ \gamma\in\Gamma\}$ coincides with the language accepted by an automaton M defined by

$$M:=(S_1,G_1,\delta,a_0,\{a_0\})$$

with a transition function δ

$$\delta(a_0,0):=a_0,\ \ \delta(a_0,1):=a_1,\ \ \delta(a_1,i):=a_0\ \ (i=0,1),$$

for the definition and notation related to automata, see [Ho-U]. Therefore, noting that $\omega=\delta(a_0,E_2(0))\dots\delta(a_0,E_2(n-1))\dots$, we see that ω is the fixed point of a substitution over S_1 given by

$$\sigma(a_0):=a_0a_1,\ \sigma(a_1):=a_0a_0. \tag{16}$$

Using this fact shown above, we can prove that the ω is a totally nonperiodic word by the following manner: We remark that so far as similis partitions are concerned, nonperiodicity implies total nonperiodicity. So, it suffices to show the nonperiodicity of ω. Suppose that ω is an ultimately periodic word, then $\theta:={}_20.\tau(\omega)\in\mathbb{Q}$ $(\tau(a_i):=i)$. We put $a=a_0$, $b=a_1$, $\theta_n={}_20.\tau(u_n)^*$ $(u_n=\sigma^n(a))$, where u^* denotes the periodic word uuu... for a nonempty word u. We write $u\neg v$ if v is a prefix of u. The binary relation $\neg$ is transitive. In view of (16), we get $u_2=abaa=u_1u_0{}^2$, so that $u_{n+2}=u_{n+1}u_n{}^2$ for all $n\geqq 0$, $|u_n|=2^n$, and $\omega\neg u_{n+1}\neg u_nu_{n-1}$. Hence, we obtain $|\theta-\theta_n|\leqq 2^{-3\cdot 2^{n-1}}$. For any $n\geqq 1$, we can put $\theta_n=E_2{}^{-1}(u)/(2^{2^n}-1)$ with certain $u\in 1G_1{}^*$. Let θ_n equal P_n/Q_n, G.C.D.$(P_n,Q_n)=1$. Then $|\theta-P_n/Q_n|\leqq Q_n{}^{-3/2}$, which together with $\theta\in\mathbb{Q}$ implies that $\{P_n/Q_n;\ n\geqq 0\}$ is a finite set. Therefore $\theta_i=\theta_{i+j}$ for some $i\geqq 0$ and $j\geqq 1$, so that $u_{i+j}=u_i{}^{2^j}$. Since $u_{j+1}=u_{i+j-1}u_{i+j-1}{}^2$, we get $u_{i+j-1}=u_i{}^{2^{j-1}}$, and inductively, $u_{i+1}=u_i{}^2$. By $u_{i+1}=u_iu_{i-1}{}^2$, we get $u_i=u_{i-1}{}^2$. Repeating the argument, we obtain $u_1=u_0{}^2$ which contradicts $u_1=ab\neq aa=u_0^2$.

By direct calculation, we see $\partial\Gamma=2112221121121122...$ for the H.-J. partition. We can show that the sequence (or word) $\partial\Gamma$ is the fixed point of a substitution over $\{1,2\}$ by the following manner: Let ω be the fixed point of the σ (16). Noting that bb does not occur in ω, we can factorize ω into two words A:=ab and B:=a, and we get a new word $\tilde{\omega}$ over $\{A,B\}$:

$$\omega = \text{ab a a ab ab ab a a ab a a ab a a ab ab} \ldots,$$

$$\tilde{\omega} = \text{A B B A A A B B A B B A B B A A} \ldots,$$

and noting that ω is the fixed point of σ, we see that $\tilde{\omega}$ is the fixed point of a substitution τ over $\{A,B\}$:

$$\tau : \begin{array}{l} A=ab \longrightarrow \sigma(ab)=abaa=ABB \\ B=a \longrightarrow \sigma(a)=ab=A. \end{array}$$

Since $\partial\Gamma=\zeta(\tilde{\omega})$ with $\zeta(A)=2$, $\zeta(B)=1$, $\partial\Gamma$ is the fixed point of the substitution $2\to 211$, $1\to 2$.

We can generalize all the statements given above for the H.-J. partition to those for the partition with $D=\{q^i;\ 0\leqq i\leqq e\}$ as in Proposition 7-8, by considering an automaton

$$M_{e,q} := (S_e, G_q, \delta, a_0, \{a_0\})$$

with a transition function $\delta=\delta_{e,q}$ defined by

$$\delta(a_i,j)=a_0,\ \ \delta(a_i,q-1)=a_{i+1}\ (0\leqq i\leqq e-1,\ 0\leqq j\leqq q-2),$$

$$\delta(a_s,j)=a_0\ (0\leqq j\leqq q-1).$$

<u>Proposition 7</u>. Let ω be the word corresponding to a similis partition (14) with respect to $D=\{q^i;\ 0\leqq i\leqq e\}$ $(e\geqq 1,\ q\geqq 2)$, then ω is totally nonperiodic word over S_e, which is the fixed point of a substitution over S_e defined by

$$\sigma(a_i):=a_0^{q-1}a_{i+1}\ (0\leqq i\leqq e-1),\ \ \sigma(a_e):=a_0^{q}. \tag{17}$$

<u>Proposition 8</u>. Let (14) be a similis partition with respect to D as in Proposition 7. Let $\tau: S_e^*\to S_e^*$, $\kappa: S_e^*\to\{1,2\}^*$ be morphisms defined by

$\tau(a_i):=a_0a_{i+1}$ $(0\leqq i\leqq e-2)$, $\tau(a_{e-1}):=a_0a_e^2$, $\tau(a_e):=a_0$,

$\kappa(a_i)=2$ $(0\leqq i\leqq e-1)$, $\kappa(a_e)=1$

for q=2, and

$\tau(a_0):=a_0^{q-2}a_1$, $\tau(a_i):=a_0^{q-2}a_1a_0^{q-2}a_{i+1}$ $(1\leqq i\leqq e-1)$, $\tau(a_e):=a_0^{q-2}a_1a_0^q$,

$\kappa(a_i)=1$ $(0\leqq i\leqq e-1)$, $\kappa(a_e)=2$

for q>2. Then $\partial\Gamma$ is a nonperiodic word over $\{1,2\}$, which is given by

$$\partial\Gamma=\kappa(\omega')$$

where ω' is the fixed point of τ.

Let ω be as in Proposition 7. Then, in view of a locally catenative formula $\sigma^{n+e}(a_0)=(\sigma^{n+e-1}(a_0))^{q-1}\ldots(\sigma^{n+1}(a_0))^{q-1}(\sigma^n(a_0))^q$, we can easily find that the frequency of a_i apearing in ω is rational for all i. This fact together with the nonperiodicity of ω implies that a similis partition with respect to D given by (15) can not be neither a partition (3) nor (11).

Let us consider a similis partition (14) with respect to $D=\{1,2,3,6\}$. Then $\Gamma=\{2^{2i}3^{2j}k;\ i\geqq0,\ j\geqq0,\ k\geqq1,\ \text{G.C.D.}(2\cdot3,k)=1\}$, which equals $\{2^{2i}k;\ i\geqq0,(2,k)=1\}\cap\{3^{2i}k;\ i\geqq0,\ (3,k)=1\}$. So, it is clear that

$$\Gamma=\chi(\omega(1,2);a_0)\cap\chi(\omega(1,3);a_0),$$

where $\omega(e,q)$ is the fixed point of a substitution over S_e defined by (17). In general, we can show, considering the languages accepted by the automata $M_{e_i,q_i,j_i}:=(S_{e_i},G_{q_i},\delta_{e_i,q_i},a_0,\{a_{j_i}\})$ $(0\leqq j_i\leqq e_i\ (1\leqq i\leqq k))$ that

$$q_1^{j_1}\cdots q_k^{j_k}\Gamma=\bigcap_{1\leqq i\leqq k}\chi(\omega(e_i,q_i);a_{j_i})\quad(0\leqq j_i\leqq e_i\ (1\leqq i\leqq k))\tag{18}$$

holds for any finite similis partition with respect to D given by (15). We denote by Ω the word (strictly over $\prod_{1\leqq i\leqq k}(e_i+1)$ letters) corresponding to a partition given by Proposition 6. Then, it folows from (18) that Ω is an interpretation of $\omega(e_i,q_i)$ (i.e., whenever the ith symbol counted from the beginning differs from the jth symbol in ω, then so does in Ω, cf. [Sa]). Hence, Ω is not an ultimately periodic word by Proposition 7. Therefore, any similis partition given by Proposition 6 is totally nonperiodic. Related to the transcendence of the word $\partial\Gamma$ and the word corresponding to a finite similis partition, the following result obtained by S. Ferenczi and Ch. Mauduit is usefull (a substitution σ over S is called primitive if $(|\sigma^n(a)|_b)_{a,b\in S}$ is a positive matrix for some n):

<u>Proposition 9</u> ([F-M], Proposition 5). If the expansion of θ in some base k is non-ultimately periodic fixed point of a primitive substitution, and does contain at least one word of the form $v^{2+\varepsilon}$ (that is, vvv' for nonempty word v and a prefix v' of v with $|v'|\geqq\varepsilon|v|$ $(\varepsilon>0)$), then θ is transcendental.

If we apply Proposition 9 to the word ω in Proposition 7, and note that the transcendence of ω implies the tarnscendence of Ω, we obtain Proposition 10 (resp. Proposition 11) by Proposition 7 (resp. Proposition 8) as follows:

<u>Proposition 10</u>. Let (14) be a finite similis partition with respect to D given by (15) with the word Ω corresponding to (14). Then Ω is transcendental.

<u>Proposition 11</u>. Let Γ be a set satisfying (14) with D given by (15). Then $\partial\Gamma$ is transcendental.

§4 Log-fixed point

A word (or a sequence) ξ' over $\{1,2\}$ is referred to as a Kolakoski word if the word defined by its run-lengths is equal to ξ' itself:

$$\xi'=\underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{11}_{2}\,\underbrace{22}_{2}\,\underbrace{1}_{1}\,\underbrace{2}_{1}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{11}_{2}\ldots$$
$$2\;2\;1\;1\;2\;1\;2\;2\;1\;2\;2\;1\;1\;2\;1\;1\;2\;2\;1\;2\;1\;1\;2\;1\;2\;2\ldots=\xi',$$

where we mean by a run a maximal subword consisting of identical letters, cf. [Ko], [D2] with its references. The word $\xi:=1\xi'$ is the only other word having this property. It can be easily seen that ξ is not an ultimately periodic word, cf. [Ü]. Related to the complexity $p(n)=p(n;\xi)$, $n+1\leqq p(n)\leqq n^{7.2}$ has been shown by F. M. Dekking; his conjecture says $p(n) \asymp n^{\log 2/\log(3/2)}$, cf. [D1, D2].

Let S be an alphabet with $\# S \geqq 2$. We denote by $S^{\times}$ the set

$$S^{\times} := (S^{*} \cup S^{\infty}) \setminus \left(\bigcup_{a \in S} S^{*}\{a^{*}\} \right),$$

i.e., $S^{\times}$ is the set of all finite or infinite words that are different from all the words of the form ua^{*} ($u \in S^{*}$, $a \in S$). We shall write $S^{*}w$ instead of $S^{*}\{w\}$. For any word $\omega = \omega_1 \omega_2 \ldots \omega_n \ldots \in S^{\times}$, we can define two words $\log \omega$, and base ω by

$$\log \omega := e_1 e_2 e_3 \ldots, \quad \text{base } \omega := b_1 b_2 b_3 \ldots,$$

$$\text{if} \quad \omega = b_1^{e_1} b_2^{e_2} b_3^{e_3} \ldots \quad (e_i \geqq 1,\ b_i \in S,\ b_i \neq b_{i+1} \text{ for all } i \geqq 1).$$

In what follows, we take $S \subset \mathbb{N}$. A word $\omega \in S^{\times}$ satisfying $\omega = \log \omega$ will be referred to as a log-fixed point. The Kolakoski word ξ is defined to be a log-fixed point with base $\xi = (12)^{*}$. If $\# S = 2$, then $\#\{\omega \in S^{\infty};\ \omega = \log \omega\} = 2$; if $\# S \geqq 3$, then the set $\{\omega \in S^{\infty};\ \omega = \log \omega\}$ has continuum cardinality, since so does the set {base ω}. It can be easily seen by the similar manner to that given by [Ü] that all the log-fixed points are not ultimately periodic. Now, for instance, consider a log-fixed point ω with base $\omega = (26)^{*}$, and factorize it into the words of length 2:

$$\begin{aligned} \omega &= 22\ 66\ 22\ 22\ 22\ 66\ 66\ 66\ 22\ 66 \ldots \\ &= A\ B\ A\ A\ A\ B\ B\ B\ A\ B \ldots \\ &= W_1\ W_2\ W_3\ W_4\ W_5\ W_6\ W_7\ W_8\ W_9\ W_{10} \ldots . \end{aligned}$$

Then it is clear that W_i is $A := 22$ or $B := 66$ (since the length of the period of base ω is 2, which divides 2, $6 \in S$), and ω as a word over $\{A, B\}$ is invariant under a morphism

$$A = 22 \longrightarrow 2266 = AB$$

$$B = 66 \longrightarrow 222222666666 = AAABBB.$$

Note that such an argument does not work at all for the Kolakoski words, but it can be applied to some general cases:

<u>Proposition 12</u>. Let $s \geqq 1$ be an integer, $S = \{a_0, \ldots, a_s\} \subset \mathbb{N}$ such that s

divides a_i for all $1 \leqq i \leqq s$. Let σ be a substitution over $\{A_0, \ldots, A_s\}$ defined by $\sigma(A_i) = A_0^{a_i/s} \ldots A_s^{a_i/s}$ $(0 \leqq i \leqq s)$, and let Ω be its fixed point. Then the log-fixed point ω with base $\omega = (a_0 \ldots a_s)^*$ can be given by $\omega = \tau(\Omega)$, where τ is a morphism defined by $\tau(A_i) = a_i{}^s$ $(0 \leqq i \leqq s)$.

Now, we return to the Kolakoski word ξ'. Consider what is the σ in Proposition 1 for ξ' in formal sense. Then, the it becomes a "substitution" defined by

$$\sigma(A_0) = A_0 A_1, \quad \sigma(A_1) = A_0{}^{1/2} A_1{}^{1/2},$$

where we mean by $A^{1/2}$ a half of a symbol. If we define $(W_1 W_2 \cdots W_n)^{1/2}$ (each W_i is a symbol, or a half-symbol) to be a "word" $W_1 \ldots W_{[n/2]}$ (resp. $W_1 \cdots W_{[n/2]} W_{[n/2]+1}{}^{1/2}$, which is possibly a "word" containing a fourth of a symbol) for even n (resp. odd n), $\sigma(W^{1/2})$ to be a "word" $\sigma(W)^{1/2}$, and consider an infinite word $\Omega = \lim \sigma^n(A_0)$, then

$$A_0 \xrightarrow{\sigma} A_0 A_1 \xrightarrow{\sigma} A_0 A_1 A_0{}^{1/2} A_1{}^{1/2} \xrightarrow{\sigma} A_0 A_1 A_0{}^{1/2} A_1{}^{1/2} A_0 A_0{}^{1/2} \xrightarrow{\sigma} \ldots$$

$$\longrightarrow \Omega = A_0 A_1 A_0{}^{1/2} A_1{}^{1/2} A_0 A_0{}^{1/2} A_0 A_1 A_0{}^{1/2} A_0 A_1 A_0{}^{1/2} A_1{}^{1/2} A_0 \ldots.$$

We can define the sequence Ω to be the fixed point of a substitution over an alphabet {a, b, c, d} in usual sense, where we identify $a = A_0$, $b = A_1$, $c = A_0{}^{1/2}$, $d = A_1{}^{1/2}$. Can we find any relation between ξ' and Ω? (Probably, no!; then find a better treatment for half-symbols.) It will be remarkable that the word ξ is a fixed point of the map

$$\Psi : \mathbb{N}^{\mathbb{N}} \to \{0,1,2\}^\infty, \quad \Psi(\omega) := B_3(B_{2\to 3}(\varphi(c(\omega))) + 1/2),$$

where c, φ, $B_{2\to 3}$, B_3 are maps defined as follows:

1°) $c : \mathbb{N}^\infty \cup (\mathbb{N}^* \setminus \mathbb{N}^* 1) \to I = [0,1]$, $c(a_1 a_2 a_3 \ldots) := [0; a_1, a_2, a_3, \ldots]$ for $a_1 a_2 a_3 \ldots \in \mathbb{N}^\infty \cup (\mathbb{N}^* \setminus \mathbb{N}^* 1)$, where the right-hand side denotes a continued fraction as usual;

2°) $\varphi : I \to I$ is the so called point-of-interrogation-function introduced by Minkowski determined by the following conditions:

(i) φ is continuous with $\varphi(0)=0$, $\varphi(1)=1$,

(ii) $\varphi((p+p')/(q+q')) = (\varphi(p/q) + \varphi(p'/q'))/2$ for all

$p, q, p', q' \in \mathbb{N} \cup \{0\}$ such that $p/q,\ p'/q' \in I,\ p'q-pq'=\pm 1$;

3°) $B_{2\to 3}: I \to [0, 1/2]$, $B_{2\to 3}({}_2 0.b_1b_2b_3\ldots) := {}_3 0.b_1b_2b_3\ldots$ for $b_1b_2b_3\ldots \in \{0,1\}^{\infty} \setminus \{0,1\}^* 0^*$ $(B_{2\to 3}(0):=0)$;

4°) $B_3: I \to \{0,1,2\}^{\infty}$, $B_3(x) = c_1c_2c_3\ldots$ for $x = {}_3 0.c_1c_2c_3\ldots$ with $c_1c_2c_3\ldots \in \{0,1,2\}^{\infty} \setminus \{0,1,2\}^* 0^*$ $(B_3(0) := 0^*)$

We can see that ξ is uniquely determined by $\Psi(\xi)=\xi$ by the fact

$$\varphi([0; a_1, a_2, a_3, \ldots]) = {}_2 0.0^{a_1} 1^{a_2} 0^{a_3} 1^{a_4} 0^{a_5} \ldots, \tag{19}$$

cf. [P]. Related to the existence of frequencies of words, it is known as Keane's problem which asks whether the frequency of 1 in ξ exists, and it equals 1/2, [Ke]. This is still open. If it does not exists (probably it does!), or if it equals 1/2 (probably it does), then it is easy to see that the words ω corresponding to the partition (3), (11), or (14) can not be the word ξ.

Instead of (19), we may ask for a number $x \in I$ satisfying

$$[0; a_1, a_2, a_3, \ldots] = {}_g 0.a_1a_2a_3\ldots (= x), \tag{20}$$

where

$$a_n \in \mathbb{Z},\ 1 \leqq a_n \leqq g-1\ (i \geqq 1). \tag{21}$$

Such a number x exists for a square number $g=h^2$ $(2 \leqq h \in \mathbb{Z})$, since $[0;h]=1/h=h/g={}_g o.h$; this is not interesting. Now, we ask for an irrational number $x \in I$ satisfying (20) with (21). If we take g=10, then by simple calculations, we can show that such a number does not exist. If we consider (20) with

$$a_1a_2a_3\ldots \in (\{0,\ldots,g-1\}^{\infty} \setminus \{0,\ldots,g-1\}^* 0^* \tag{22}$$

instead of (21), then it seems very likely that a number $x \in I$ satisfying (20) exists; a caluculation says that

$$[0; 3, 3, 5, 8, 3, 4, 7, \ldots] = {}_{11} 0.3358347\ldots,$$

where we mean, for example,

$$[0; 3, 3, 5, \ldots, 1, 1, 9, 10, \underbrace{0, \ldots, 0}_{\text{odd number of 0s}}, 2, 9, \ldots] = [0; 3, 3, 5, \ldots, 1, 1, 9, 10+2, 9, \ldots],$$

$$[0; 3, 3, 5, \ldots, 1, 1, 9, 10, \underbrace{0, \ldots, 0}_{\text{even number of 0s}}, 2, 9, \ldots] = [0; 3, 3, 5, \ldots, 1, 1, 9, 10, 2, 9, \ldots].$$

The difficulty of the proof of the existance of a number x satisfying (20) for $g=11$ comes from the possibility of a long run of 0s. Probably, the length of a run of 0s which begins by nth symbol counted from the begining is bounded by a function of n taking sufficiently small values; and probably, such a number x satisfying (20) with (22) exists for infinitely many g. It is clear that if an irrational number x satisfying (20) with (21) exists, then x is an irrational number being different from all the quadratic irrationals. Note that periodic, or nonperiodic infinite continued fraction with (22) can be a rational number, for instance, $[0;3,1,0,3,0,0,0,5,0,0,0,0,0,7,\ldots]=[0;3,\infty]=1/3$, $[0;3,1,0,7,0,7,0,7,\ldots]=1/3$.

§5 Problems

1°) We denote by $\psi_i(z)$ the analytic function on the unit disc defined by

$$\psi_i(z)=\psi_i(z;\omega):=\sum_{n\in\chi(\omega;a_i)} z^n \quad (0\leqq i\leqq s)$$

for $\omega\in S^\infty$, $S=\{a_0,\ldots a_s\}$, and we take ω to be the word $\omega(L)$ defined by the billiard as in Section 1 with $L=\{t\underline{\alpha}+\underline{\beta};\ t\in\mathbb{R}_+\}$, then

$$\psi_i(z)=\sum_{1\leqq n<\infty} z^{\sum_{0\leqq j\leqq s}[\alpha_i^{-1}\alpha_j n-\alpha_i^{-1}\langle\beta_i\rangle\alpha_j+\langle\beta_j\rangle]}$$

follows from Proposition 1. We suppose that $\alpha_0,\ldots,\alpha_s$ are linearly independent over $\mathbb{Q}$. Is the number $\Sigma c_i\cdot\psi_i(g^{-1})$ $(c_i, g\in\mathbb{Z},\ g\geqq 2)$ always transcendental except for the case where $c_i=c$ for all i? It follows from a result ([F-M], Proposition 2) that for $s=1$, $\Sigma c_i\cdot\psi_i(g^{-1})$ $(c_0\neq c_1)$ is transcendental since ω is sturmian for $s=1$, as we have mentioned in Section 1. It will be a difficult problem which asks for a proof of transcendence of words having complexity bounded by a polynomial of degree 2. (Note that it is difficult to show the transcendence of the number ${}_2 0.\theta$ for $\theta=10^1 10^2 10^3\ldots$, and that $\partial\{p(n;\theta)\}_{n=0,1,2,\ldots}=1^2 2^2 3^2\ldots$, i.e., $p(n;\theta)=n^2/4+n/2+9/8+(-1)^{n+1}/8$, cf. [TO]. θ is, in some sense, a simple word, i.e., $p^*(n;\theta)=n+1$, where $p^*(n;\omega)$ denotes the number of subwords w such that $|w|=n$ and w occurs infinitely often in ω.) Recalling that $p(n;\omega(L))=n^2+n+1$ for $s=2$, we see the difficulty to prove the transcendence of $\omega=\omega(L)$ for $s>1$. For a proof of the transcendence of $\omega=\omega(L)$, it suffices to show the transcendence of $\psi_i(g^{-1};s)$ for some i. Taking $s=2$, $\alpha_0=1$, $\beta_0=0$, $0<\beta_i<1$, we have

$$\psi_0(z)=\sum_{1\leqq n<\infty} z^{[(\alpha_1+1)n+\beta_1]+[\alpha_2 n+\beta_2]}. \tag{23}$$

Can we show the transcendence of the value $\psi_0(g^{-1})$? (Probably, yes; (23) is a simple expression similar to that in the case s=1.) Problems related to linear independence and transcendence for $\psi_i(z)$, see [T5], (i)-(v), p.213.

2°) It is difficult to show that there is no number x satisfying (20) with (22) for g=10. The difficulty comes from that, for example,

$$[0;2,0,2,1,0,0,9,0,8,\dots]=0.202100908\dots(\text{in base } 10)$$

may be a solution for (20).

We may ask for the existance of a number x satisfying (20) for irrational g, e.g.,

$$[0;3,2,4,6,9,8,2,\dots]=0.3246982\dots(\text{in base } \beta=((1+5^{1/2})/2)^5)$$

is possibly such a number.

It is easy to show that there exists a number $\beta=\beta(\omega)$ satifying

$$[0;a_1,a_2,a_3,\dots]=0.a_1a_2a_1\dots(\text{in base } \beta) \tag{24}$$

for any given $\omega=a_1a_2a_3\dots\in\{0,1,\dots,h\}^\infty\backslash\{0,1,\dots h\}^*0^*$. For instance, for the Kolakoski sequence ξ,

$$[0;1,2,2,1,1,2,1,2,2,\dots]={}_\beta 0.122112122\dots,\quad \beta=2.837559\dots;$$

for the fixed point of a substitution $1\rightarrow10$, $0\rightarrow1$,

$$[0;1,0,1,1,0,1,0,1,1,\dots]={}_\beta 0.101101011\dots,\quad \beta=2.729451\dots.$$

Can we show the transcendence of such a number $\beta(\omega)$ for a nonperiodic fixed point ω of a substitution? We give two conjectures:

(Conjecture 1) For any integer $g=h^2+h+i$ ($i=0,1$, $h\geqq3$), there exits an irrational number satisfying (20) with (22); such an irrational number is always transcendental.

(Conjecture 2) The number $\beta(\omega)$ defined by (24) is transcendental for any nonperiodic word ω.

3°) Let ω be the word corresponding to the partition (9), i.e., $\omega=\omega(K)$ for a curve $K=\{(f_0(x),\ldots,f_s(x);\ x\in\mathbb{R}_+\}$ for f_i as in Proposition 2. Suppose that $f_i(x)\in\mathbb{Q}(x)$ for all i. Then, can we show that $p(n;\omega(K))$ is bounded by a polynomial in n (resp. s) for fixed s (resp. n)?, cf. [T5], (vi, vii), p.214.

<u>References</u>

[A-M-S-T1] P. Arnoux, Ch. Mauduit, I. Shiokawa, and J. Tamura, Complexity of sequences defined by billiard in the cube, Bull. Soc. Math. France, <u>122</u> (1994), 1-12.

[A-M-S-T2] P. Arnoux, Ch. Mauduit, I. Shiokawa, and J. Tamura, Rauzy's conjecture on billiards in the cube, Tokyo J. of Math, <u>17</u> (1994), 211-218.

[A-R] P. Arnoux, G. Rauzy, Représentation géométrique de suites de complexité 2n+1, Bul. Soc. Math. France, <u>119</u> (1991), 194-198.

[B] Yu. Baryshnikov, Complexity of trajectories in rectangular billiards, preprint (Univ. Osnabrück; 49069 Osnabrück, Germany; May, 1994).

[D1] F. M. Dekking, On the structure of selfgenerating sequences, in Séminaire de Théorie des Nombres de Bordeaux, 1980-1981, pp. 101-106.

[D2] F. M. Dekking, What is the long range order in Kolakoski sequence?, preprint (Univ. Delft, Netherlands; Reports of the Faculty of Tech. Math. and Informatics, No. 95-100), to appear in Proc. Nato ASI held at Fields Inst.,Aug. 21-Sept. 1, 1995, Eds.: R. V. Mordy and J. Patera, Kluwer Acad. Publish.

[F-M] S. Ferenczi, Ch. Mauduit, Transcendency of numbers with a low complexity expansion, preprint (L.M.D., Univ. Marseille, France; December, 1995).

[Ha] S. Hashimoto (橋本 進吉), On Phonemics in Ancient Japanese Language (in Japanese), Meiseidô-shoten, 1942 (Iwanami-shoten, Tokyo, 1980), 古代国語の音韻について,

明世堂書店, 1942 (岩波書店, 1980).

[He-Mo] G. A. Hedlund, M. Morse, Symbolic dynamics II: Amer. J. Math. 62 (1940), 1-42.

[Ho-U] E. Hopcroft, J. D. Ullman, Introduction to Automata Theory, Languages, and computation, Addison-Wesley, 1969.

[Ke] M. S. Keane, Ergodic Theory and subshifts of finite type, in Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces, Eds.: T. Bedford, M. S. Keane, C. Series, Oxford Univ. Press, Oxford, 1991.

[Ko] W. Kolakoski, Problem 5304, Amer. Math. Monthly 72 (1965), 674.

[Mah] K. Mahler, g-adic numbers and Roth's theorem, in Lectures on Diophantine Approximations Part 1, Univ. Notre Dame, 1961.

[P] D. P. Parent, Exercices de Théorie des Nombres, Gauthier-Villars, Paris, 1978 = 数論問題ゼミ1,2 (村田玲音 訳), シュプリンガー・フェアラーク東京(株), Tokyo, 1987.

[R1] G. Rauzy, Suites à termes dans un alphabet fini, in Séminaire de Théorie des Nombres de Bordeaux, 25, 1982-1983, pp. 1-16.

[R2] G. Rauzy, Mots infinis en arithmétique, Automta on Infinite Words, in Lecture Notes in Comput. Sci. 192, Springer-Verlag, Berlin, 1985, pp. 165-171.

[Sa] A. Salomaa, Jewels of Formal Language Theory, Pitman, London, 1981.

[T0] J. Tamura, Complexity of words having long runs, manuscript, 1993.

[T1] J. Tamura, Some problems and results having their origin in the power series $\sum_{n=1}^{\infty} z^{[\alpha n]}$, in Proc. of the colloquium "Analytic Number Theory and its Circumference" held at Gakushûin Univ, Nov. 11-13, 1991; 研究集会報告集「解析数論とその周辺 1991年」(於 学習院大学), 学習院大学, 1992, pp. 190-212.

[T2] ________, Transcendental numbers having explicit g-adic and Jacobi-Perron expansions, Séminaire de Théorie des Nombres de Bordeaux, 4 (1992), 75-95.

[T3] ________, A class of transcendental numbers with explicit g-adic and the Jacobi-Perron algorithm, Acta Arith., 61 (1992), 51-67.

[T4] ________, A class of transcendental numbers having explicit g-adic and

Jacobi-Perron expansions of arbitrary dimension, Acta Arith. 71 (1995), 301-329.

[T5] ________, Certain sequences making a partition of the set of positive integers, Acta Math. Hungar., 70 (1996), 207-215.

[Ta] K. Tanaka (田中 克彦), Races and Nations from A Lingual Point of View (in Japanese), Iwanami-shoten, Tokyo; 言語からみた民族と国家, 岩波現代選書 13, 岩波書店, 1978.

[To] M. Tobe (戸部 実之), Introduction to Hungarian (in Japanese), Tairyûsha, Tokyo; マジャール語入門, 泰流社, 1988.

[Us] J. V. Uspensky, On a problem arising out of a certain game, Amer. Math. Monthly, 34 (1927), 516-521.

[Ü] N. Üçoluk, Solution for Problem 5304, Amer. Math. Monthly, 73 (1996), 681-682.

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