数理解析研究所講究録
961 巻 1996 年 73-80

Hata-Yamaguti's result on Takagi function and its applications to digital sum problems

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1 Introduction

Let $n \in \mathbf{N}$ and denote its binary expansion by $n = \sum_{k\geq 0} \alpha_k(n) 2^k$ with $\alpha_k(n) \in \{0,1\}$. We define

$$s(n) = \sum_{k\geq 0} \alpha_k(n) \qquad \text{(the binary digital sum)},$$

$$F(\xi, N) = \sum_{n=0}^{N-1} e^{\xi s(n)} \qquad \text{(the exponential sum)},$$

$$S_p(N) = \sum_{n=0}^{N-1} s(n)^p \qquad \text{(the power sum)}$$

for $N \in \mathbf{N}$ and $p, \xi \in \mathbf{R}$. We first review some fundamental results on these sums. If N is a power of 2, we immediately have $F(\xi, N) = N^{\log_2(1+e^{\xi})}$ and $S_1(N) = N \log_2 N/2$. However it is not so easy to obtain explicit formulas for arbitary $N \in \mathbf{N}$. In early times the asymptotic behavior of $S_1(N)$ was studied:

$$S_1(N) \sim N\frac{\log_2 N}{2} \quad (N \to \infty) \qquad \text{(Bush [1])},$$

$$S_1(N) = N\frac{\log_2 N}{2} + O(N) \quad (N \to \infty) \qquad \text{(Mirsky [8])}.$$

Finally, Trollope [18] obtained a precise formula for $S_1(N)$ and Delange [4] gave its elegant proof. Let F be a nowhere differenetiable continuous periodic function of period 1 given by

$$F(x) = 1 - x - 2^{1-x} T(\frac{1}{2^{1-x}}), \quad 0 \leq x \leq 1$$

with T the Takagi function. Then

$$S_1(N) = N\frac{\log_2 N}{2} + \frac{N}{2} F(\log_2 N) \qquad \text{(Trollope , Delange)}.$$

Coquet [3] obtained a precise formula for positive integer powers.

Theorem 1.1 (Coquet [3]) *There are periodic functions $G_{p,\tau} : \mathbf{R} \to \mathbf{R}$, $0 \leq \tau \leq p$, of period 1, such that*

$$S_p(N) = N(\frac{\log_2 N}{2})^p + N \sum_{0\leq \tau < p} (\log_2 N)^{\tau} G_{p,\tau}(\log_2 N).$$

for every integer $p \geq 1$. *Furthermore* $G_{p,\tau}$ *verify*

$$2^{-\tau}\binom{d}{\tau} + \sum_{\tau<p<d}\binom{d}{p}G_{p,\tau} = 2^{1-d}\binom{d}{\tau} + \sum_{\tau<q<d} 2\binom{q}{\tau}G_{d,q} \tag{1}$$

for $d \geq 2$ *and* $\tau \leq d-2$.

Theorem 1.2 (Coquet [3])

$$S_2(N) = N(\frac{\log_2 N}{2})^2 + N\frac{\log_2 N}{2}\{\frac{1}{2} + F(\log_2 N)\} + NG(\log_2 N),$$

where G *is a nowhere differentiable continuous periodic function of period* 1.

An explicit form of the function G is stated in Osbaldestin [13]. However, for $p \geq 3$, we cannot get such an explicit formula via induction formulae (1) and the continuity of $G_{p,\tau}$ is unknown.

Concerning $F(\xi, N)$, Stolarsky [17] proved that $F(\log 2, N)/N^{\log_2 3}$ is not well-behaved asymptotically. Harborth [6] obtained the following estimates:

$$\limsup_{N\to\infty} \frac{F(\log 2, N)}{N^{\log_2 3}} = 1,$$

$$0.812556 < \liminf_{N\to\infty} \frac{F(\log 2, N)}{N^{\log_2 3}} < 0.812557. \tag{2}$$

We now introduce a functin G_ξ by

$$G_\xi(\log_2 N) = \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}}.$$

Coquet [3] and Stein [16] investigated the properties of G_ξ. Stein proved that G_ξ is a continuous periodic function of period 1 by giving a formula of F. However, it is unknown if G_ξ is differentiable. In this note, we get a simple explicit formula of $F(\xi, N)$ by the use of the connection between $s(n)$ and the binomial measure μ_r. And using the results obtained in Hata-Yamaguti [7] and Sekiguchi-Shiota [15], we derive explicit formulas of the power sum $S_p(N)$. We notice that the higher order derivatives of the distribution function of μ_r with respect to r play an improtant rule in the explicit formula of $S_p(N)$. The results in this note can be extended to the sum of q-adic digits by the use of multinomial measures (see M-O-S-S [9]).

2 Hata-Yamaguti's result

Let $I = I_{0,0} = [0,1]$ and

$$I_{n,j} = [\frac{j}{2^n}, \frac{j+1}{2^n}), \quad j = 0, 1, \ldots, 2^n - 2, \qquad I_{n,2^n-1} = [\frac{2^n-1}{2^n}, 1]$$

for $n = 1, 2, 3, \ldots$

Define the binomial measure μ_r $(0 < r < 1)$ by a probability measure on I such that

$$\mu_r(I_{n+1,2j}) = r\mu_r(I_{n,j}), \quad \mu_r(I_{n+1,2j+1}) = (1-r)\mu_r(I_{n,j})$$

for $n = 0, 1, 2, \cdots, j = 0, 1, \cdots, 2^n - 1$.

We denote the distribution function of μ_r by L:

$$L(r, x) = \mu_r([0, x]).$$

It is well-known that $L(r, .)$ is a strictly increasing continuous and singular function except for $r = 1/2$ (see Salem [14]). It immediately follows that $L(r, .)$ satisfies the system of infinitely many difference equations:

$$\begin{cases} L(r, \frac{2j+1}{2^{n+1}}) - (1-r)L(r, \frac{j}{2^n}) - rL(r, \frac{j+1}{2^n}) = 0, \\ L(r, 0) = 0, \ L(r, 1) = 1, \\ n = 0, 1, 2, \ldots, \ j = 0, 1, \ldots, 2^n - 1. \end{cases}$$

This system is equivalent to the following functional equation:

$$L(r, x) = \begin{cases} rL(r, 2x), & 0 \le x \le \frac{1}{2}, \\ (1-r)L(r, 2x-1) + r, & \frac{1}{2} \le x \le 1. \end{cases}$$

Let

$$R(x) = 1_{I_{1,0}}(x) - 1_{I_{1,1}}(x), \quad 0 \le x \le 1,$$

$$\phi(x) = \begin{cases} 2x & 0 \le x < 1/2, \\ 2x - 1, & 1/2 \le x \le 1, \end{cases}$$

$$\psi(x) = \int_0^x 2R(t)dt, \quad 0 \le x \le 1.$$

The Takagi function T is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \psi(\phi^n(x)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \psi^n(x), \quad 0 \le x \le 1.$$

It is well-known that T is a nowhere differentiable continuous function. And T satisfies the system of infinitely many difference equations:

$$\begin{cases} T(\frac{2j+1}{2^{n+1}}) - \frac{1}{2}T(\frac{j}{2^n}) - \frac{1}{2}T(\frac{j+1}{2^n}) = \frac{1}{2^{n+1}}, \\ T(0) = 0, \ T(1) = 0, \\ n = 0, 1, 2, \ldots, \ j = 0, 1, \ldots, 2^n - 1. \end{cases}$$

Hata–Yamaguti [7] have obtained the following formula which connects the Takagi function T with the function L.

Theorem 2.1 (Hata–Yamaguti [7]) *We have*

$$\frac{1}{2} \frac{\partial}{\partial r} L(r, x) \Big|_{r=\frac{1}{2}} = T(x).$$

Remark 2.1 *We have*

$$\begin{cases} \frac{\partial}{\partial r}L(r,\frac{2j+1}{2^{n+1}}) - (1-r)\frac{\partial}{\partial r}L(r,\frac{j}{2^n}) - r\frac{\partial}{\partial r}L(r,\frac{j+1}{2^n}) = L(r,\frac{j+1}{2^n}) - L(r,\frac{j}{2^n}), \\ \frac{\partial}{\partial r}L(r,0) = 0, \ \frac{\partial}{\partial r}L(r,1) = 0, \\ n = 0,1,2,\dots, \ j = 0,1,\dots,2^n - 1. \end{cases}$$

Above system has a unique continuous solution (Hata-Yamaguti [7] , S-S [15]).

More generally we have the following:

Theorem 2.2 (S-S [15]) $L(r,x)$ *is a continuous function valued analytic function of* $r \in I$ *and the equality*

$$\left.\frac{\partial^k L(r,x)}{\partial r^k}\right|_{r=\frac{1}{2}} = k!T_{\frac{1}{2},k}(x)$$

holds for $k = 1,2,3,\dots$ *Here*

$$T_{\frac{1}{2},1}(x) = 2T(x),$$

$$T_{\frac{1}{2},k}(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} R(2^n x) T_{\frac{1}{2},k-1}(2^{n+1}x).$$

Remark 2.2 *It also follows that* $T_{r,k}$ *satisfies the system of infinitely many difference equations:*

$$\begin{cases} T_{r,k}(\frac{2j+1}{2^{n+1}}) - (1-r)T_{r,k}(\frac{j}{2^n}) - rT_{r,k}(\frac{j+1}{2^n}) = T_{r,k-1}(\frac{j+1}{2^n}) - T_{r,k-1}(\frac{j}{2^n}), \\ T_{r,k}(0) = 0, \ T_{r,k}(1) = 0, \\ n = 0,1,2,\dots, \ j = 0,1,\dots,2^n - 1. \end{cases}$$

3 An explicit formula of exponential sums

We first give a lemma which suggests a close connection between the distribution functin L and digital sums. Set $t = \log_2 N$ for $N \in \mathbf{N}$ and denote by $[t]$ its integer part and by $\{t\}$ its decimal part.

Lemma 3.1 *We have*

(3) $$L(r, \frac{1}{2^{1-\{t\}}}) = \sum_{n=0}^{N-1} r^{[t]+1-s(n)}(1-r)^{s(n)}.$$

Taking $r = \frac{1}{1+e^{\xi}}$ in (3) , we immediately have next theorem.

Theorem 3.1 *We have*

(4) $$F(\xi, N) = N^{\log_2(1+e^{\xi})} 2^{(1-\{t\})\log_2(1+e^{\xi})} L(\frac{1}{1+e^{\xi}}, \frac{1}{2^{1-\{t\}}})$$

for $\xi \in \mathbf{R}$.

Remark 3.1 *By* (4), *we know that* G_{ξ} *is differentiable for almost every* $t \in \mathbf{R}_{+}$.

4 Asymptotic behavior of $F(\xi, N)/N^{\log_2(1+e^\xi)}$

Theorem 4.1 *We have*

$$\limsup_{N\to\infty} \frac{F(\xi,N)}{N^{\log_2(1+e^\xi)}} = \max_{\frac{1}{2}\le x\le 1} x^{\log_2 r} L(r,x),$$
$$\liminf_{N\to\infty} \frac{F(\xi,N)}{N^{\log_2(1+e^\xi)}} = \min_{\frac{1}{2}\le x\le 1} x^{\log_2 r} L(r,x).$$

Hence, to obtain the precise values on the left-hand sides of these equations, it suffices to estimate the function $g(x) = x^{\log_2 r} L(r,x)$. However, it is very hard to get the maximum and the minimum of the function g.

Proposition 4.1 *we have*

$$\limsup_{N\to\infty} \frac{F(\xi,N)}{N^{\log_2(1+e^\xi)}} = 1 \quad for\ \xi > 0,$$
$$\liminf_{N\to\infty} \frac{F(\xi,N)}{N^{\log_2(1+e^\xi)}} = 1 \quad for\ \xi < 0.$$

These estimations are obtained by Stein [16].

Proposition 4.2 *For $k = 1,2,\ldots,2^{n-1}-1, n = 1,2,\ldots,$ we have*

$$g\left(\frac{2k+1}{2^{n+1}}\right) > \min\left\{g\left(\frac{4k+1}{2^{n+2}}\right), g\left(\frac{4k+3}{2^{n+2}}\right)\right\}, \quad if\ \ 0 < r < \frac{1}{2},$$
$$g\left(\frac{2k+1}{2^{n+1}}\right) < \max\left\{g\left(\frac{4k+1}{2^{n+2}}\right), g\left(\frac{4k+3}{2^{n+2}}\right)\right\}, \quad if\ \ \frac{1}{2} < r < 1.$$

Remark 4.1 The above inequalities are essential in Harborth's algorithm concerned with the lower bound of the function $F(\log 2, N)/N^{\log_2 3}$. Harborth's algorithm is that, by starting with $n_0 = 1$ and $n_{r+1} = 2n_r \pm 1$ where $+$ or $-$ is chosen so that $q_{r+1} = F(\log 2, n_{r+1})/n_{r+1}^{\log_2 3}$ becomes minimal. Then $\{q_r\}$ is strictly decreasing and $q = \lim_{n\to\infty} q_r < 0.812556\ldots$ (c.f. (2)). The question whether $q = \lim_{n\to\infty} q_r$ gives a true lower bound is still unknown.

5 From exponential sums to power sums

We set

$$E(r,t) = 2^{1-\{t\}} L\left(r, \frac{1}{2^{1-\{t\}}}\right), \quad 0 < r < 1, t \in \mathbf{R}.$$

Evidently $E(r,0) = 2r$, $E(r,1-) = 1$, $E(\frac{1}{2},t) = 1$, and E is continuous except for $t \in \mathbf{Z}$ and periodic of period 1 as a function of t. Furthermore E is analytic in $r \in (0,1)$. By use of $E(r,t)$,

$$F(\xi,N) = (1+e^\xi)^t \left(\frac{1+e^\xi}{2}\right)^{1-\{t\}} E\left(\frac{1}{1+e^\xi}, t\right).$$

On the other hand, evidently the equality

$$S_k(N) = \frac{\partial^k}{\partial \xi^k} F(\xi, N)\Big|_{\xi=0}$$

holds for $k = 1, 2, 3, \ldots$ Hence we can directly derive explicit formulas of power sums of lower order from these equations. We set

$$E^{(k)}(\frac{1}{2}, t) = \frac{\partial^k}{\partial r^k} E(r,t)\big|_{r=\frac{1}{2}}.$$

Then we have

$$S_1(N) = N\Big(\frac{t}{2} + \frac{1-\{t\}}{2} - \frac{1}{4}E^{(1)}(\frac{1}{2}, t)\Big) \quad \text{(Trollope [18], Delange [4])},$$

$$S_2(N) = N\Big(\big(\frac{t}{2}\big)^2 + H_{2,1}(t)\frac{t}{2} + H_{2,0}(t)\Big) \quad \text{(Coquet [3], Osbaldestin [13])}$$

where

$$H_{2,1} = \frac{1}{2} + 1 - \{t\} - \frac{1}{2}E^{(1)}(\frac{1}{2}, t),$$
$$H_{2,0} = \frac{2 - 3\{t\} + \{t\}^2}{4} - \frac{1-\{t\}}{4}E^{(1)}(\frac{1}{2}, t) + \frac{1}{16}E^{(2)}(\frac{1}{2}, t),$$

$$S_3(N) = N\Big(\big(\frac{t}{2}\big)^3 + H_{3,2}(t)\big(\frac{t}{2}\big)^2 + H_{3,1}(t)\frac{t}{2} + H_{3,0}(t)\Big)$$

(Grabner, Kirschenhofer, Prodinger and Tichy [5], O-S-S [10])

where

$$H_{3,2}(t) = -\frac{3}{4}E^{(1)}(\frac{1}{2}, t) - \frac{3\{t\} - 6}{2},$$
$$H_{3,1}(t) = \frac{3}{16}E^{(2)}(\frac{1}{2}, t) + \frac{6\{t\} - 9}{8}E^{(1)}(\frac{1}{2}, t) + \frac{3\{t\}^2 - 12\{t\} + 9}{4},$$
$$H_{3,0}(t) = -\frac{1}{64}E^{(3)}(\frac{1}{2}, t) - \frac{3\{t\} - 3}{32}E^{(2)}(\frac{1}{2}, t) - \frac{3\{t\}^2 - 9\{t\} + 4}{16}E^{(1)}(\frac{1}{2}, t) - \frac{\{t\}^3 - 6\{t\}^2 + 9\{t\} - 4}{8},$$

$$S_4(N) = N\Big(\big(\frac{t}{2}\big)^4 + H_{4,3}(t)\big(\frac{t}{2}\big)^3 + H_{4,2}(t)\big(\frac{t}{2}\big)^2 + H_{4,1}(t)\frac{t}{2} + H_{4,0}(t)\Big) \quad \text{(O-S-S [11])}$$

where

$$H_{4,3}(t) = -E^{(1)}(\frac{1}{2}, t) - 2\{t\} + 5,$$
$$H_{4,2}(t) = \frac{3}{8}E^{(2)}(\frac{1}{2}, t) + \frac{3\{t\} - 6}{2}E^{(1)}(\frac{1}{2}, t) + \frac{6\{t\}^2 - 30\{t\} + 27}{4},$$
$$H_{4,1}(t) = -\frac{1}{16}E^{(3)}(\frac{1}{2}, t) - \frac{6\{t\} - 9}{16}E^{(2)}(\frac{1}{2}, t) - \frac{3\{t\}^2 - 12\{t\} + 7}{4}E^{(1)}(\frac{1}{2}, t) - \frac{2\{t\}^3 - 15\{t\}^2 + 27\{t\} - 13}{4},$$
$$H_{4,0}(t) = \frac{1}{256}E^{(4)}(\frac{1}{2}, t) + \frac{\{t\} - 1}{32}E^{(3)}(\frac{1}{2}, t) + \frac{3\{t\}^2 - 9\{t\} + 2}{32}E^{(2)}(\frac{1}{2}, t) + \frac{\{t\}^3 - 6\{t\}^2 + 7\{t\} - 2}{8}E^{(1)}(\frac{1}{2}, t) + \frac{\{t\}^4 - 10\{t\}^3 + 27\{t\}^2 - 26\{t\} + 8}{16}.$$

We now extend Theorem 1.1 and 1.2 and get a precise formula of S_p.

Theorem 5.1 (O-S-S [10]) *We have*

$$S_k(N) = N\sum_{p=0}^{k} H_{k,p}(t)(\frac{t}{2})^p, \quad k = 0, 1, 2, \ldots$$

Here $H_{k,p}(t)$ is a periodic continuous function of period 1, defined inductively as follows:

$$H_{0,0}(t) = E(\frac{1}{2}, t) = 1,$$

$$\frac{(-2)^k}{k!} H_{k,0}(t) = \frac{1}{2^k k!} E^{(k)}(\frac{1}{2}, t) - \sum_{j=0}^{k-1} a(k, j, 1 - \{t\}) H_{j,0}(t),$$

$$\frac{(-2)^k}{k!} H_{k,p}(t) = -\sum_{j=0}^{k-1} \sum_{q=0\vee(p-j)}^{p\wedge(k-j)} \frac{2^q}{q!} a^{(q)}(k, j, 1 - \{t\}) H_{j,p-q}(t)$$
$$\text{for} \quad p = 1, 2, \ldots, k,$$

where $a(k, j, t)$ is defined by

$$\frac{2^{t-k}}{k!} \frac{\partial^k}{\partial r^k} \left(r^{t-s}(1-r)^s\right)\Big|_{r=\frac{1}{2}} = \sum_j a(k, j, t) s^j,$$

$s \in \mathbf{R},\ t \in \mathbf{R},\ k, j \in \mathbf{Z},\ k \geq 0,$ *and* $a^{(p)}(k, j, t) = \partial^p a(k, j, t)/\partial t^p$.

Furthermore the functions $H_{k,p}(t)$ satisfies the induction formulas:

$$\sum_{j=p}^{k-1} \binom{k}{j} H_{j,p}(t) = 2^{p+1} \sum_{j=p+1}^{k} 2^{-j} \binom{j}{p} H_{k,j}(t)$$

for $k \geq 1,\ 0 \leq p \leq k - 1$.

References

[1] L. E. Bush, An asymptotic formula for the average sums of the digits of integers, *Amer. Math. Monthly*, **47**(1940), 154–156.

[2] J. Coquet, A summation formula related to the binary digits, *Invent. Math.*, **73**(1983), 107–115.

[3] J. Coquet, Power sums of digital sums, *J. Number Theory.*, **22**(1986), 161–176.

[4] H. Delange, Sur la fonction sommatoire de la fonction "somme des chiffres", *Enseign. Math.*(2), **21**(1975), 31–47.

[5] P. J. Grabner,P. Kirschenhofer,H. Prodinger and R. F. Tichy, On the moments of the sum-of-digits function,to appear in "Fibonacci Numbers and Applications," Vol.5, Kluwer,1993.

[6] H. Harborth, Number of odd binomial coeffcients, *Proc. Amer. Math. Soc.* **62**(1977), 19–22.

[7] M. Hata and M. Yamaguti, The Takagi function and its generalization, *Japan J. Appl. Math.*, **1**(1984), 183–199.

[8] L. Mirsky, A theorem on representations of integers in the scale of r, *Scripta. Math.*, **15**(1949), 11–12.

[9] K. Muramoto, T. Okada, T. Sekiguchi and Y. Shiota, Digital sum problems for the p-adic expansion of natural numbers, in preparation.

[10] T. Okada, T. Sekiguchi and Y. Shiota, Application of binomial measures to power sums of digital sums, *J. Number Thorey*(2),**52**(1995), 256-266.

[11] T. Okada, T. Sekiguchi and Y. Shiota, Explicit Formulas of exponential sums of digital sums ,*to appear in Japan J. Indust. Appl. Math.*.

[12] T. Okada, T. Sekiguchi and Y. Shiota, A Generalization of Hata-Yamaguti's results on the Takagi function II , *submited to Japan J. Indust. Appl. Math.* .

[13] A. H. Osbaldestin, Digital sum problems, *in* "Fractals in the Fundamental and Applied Sciences" Elsevier Science Publishers B. V., North-Holland, 1991.

[14] R. Salem, On some singular monotonic functions which are strictly increasing, *Trans. Amer. Math. Soc.*, **53**(1943), 427–439.

[15] T. Sekiguchi and Y. Shiota, A generalization of Hata–Yamaguti's results on the Takagi function, *Japan J. Indust. Appl. Math.*, **8**(1991), 203–219.

[16] A. H. Stein, Exponential sums of sum-of-digit functions, *Illinois J. Math.*, **30**(1986),660–675.

[17] K. B. Stolarsky, Power and exponential sums of digital sums related to binomial coefficient parity, *SIAM J. Appl. Math.*, **32**(1977), 717–730.

[18] J. R. Trollope, An explict expression for binary digital sums, *Math. Mag.*, **41**(1968), 21–25.