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HARDY CLASS OF FUNCTIONS DEFINED BY SALAGEAN OPERATOR

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ABSTRACT. The object of the present paper is to derive some properties for Hardy class of analytic functions defined by Salagean operator.

1. INTRODUCTION

Let A be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

that are analytic in the open unit disk $U=\{z:|z|<1\}$.

For $f(z)\in A$, the Salagean operator D^n (cf. [6]) is defined by

$$D^0f(z)=f(z), \tag{1.2}$$

$$D^1f(z)=Df(z)=zf'(z), \tag{1.3}$$

$$D^nf(z)=D(D^{n-1}f(z)) \quad (n\in\mathbb{N}=\{1,2,3,\cdots\}). \tag{1.4}$$

A function $f(z)$ belonging to A is said to be starlike of order α if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \quad (z\in U) \tag{1.5}$$

for some $\alpha\,(0\leq\alpha<1)$. We denote by $S^*(\alpha)$ the subclass of A consisting of functions which are starlike of order α in U.

A function $f(z)\in A$ is said to be convex of order α if it satisfies

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \quad (z\in U) \tag{1.6}$$

for some $\alpha\,(0\leq\alpha<1)$. Also we denote by $K(\alpha)$ the subclass of A consisting of all such functions. Note that $f(z)\in K(\alpha)$ if and only if $zf'(z)\in S^*(\alpha)$ for $0\leq\alpha<1$.

Let H^p $(0<p\leq\infty)$ be the class of all analytic functions in U such that

$$||f||_p=\lim_{r\to1^-}\{M_p(r,f)\}<\infty, \tag{1.7}$$

where

$$M_p(r,f)=\begin{cases}\left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}|f(re^{i\theta})|^pd\theta\right)^{\frac{1}{p}} & (0<p<\infty)\\ \max\limits_{|z|\leq r}|f(z)| & (p=\infty)\end{cases} \quad \text{(cf. [1]).} \tag{1.8}$$

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2. Some Lemmas

To discuss our problems for Hardy class H^p of functions, we need the following lemmas.

Lemma 1 ([7]). If $f(z) \in K(\alpha)$, then $f(z) \in S^*(\beta)$, where

$$\beta = \beta(\alpha) = \begin{cases} \dfrac{1-2\alpha}{2(2^{1-2\alpha}-1)} & (\alpha \neq \frac{1}{2}) \\ \dfrac{1}{2\log 2} & (\alpha = \frac{1}{2}). \end{cases} \tag{2.1}$$

This result is sharp.

Lemma 2 ([2]). If $f(z) \in S^*(\alpha)$ and is not of the form

$$f(z) = \frac{z}{(1-ze^{it})^{2(1-\alpha)}}, \tag{2.2}$$

then there exists $\delta = \delta(f) > 0$ such that $\dfrac{f(z)}{z} \in H^{\delta + \frac{1}{2(1-\alpha)}}$.

Lemma 3 ([5]). If $p(z)$ is analytic in U with $p(0) = 1$ and

$$\mathrm{Re}(p(z) + zp'(z)) > \frac{1-2\log 2}{2(1-\log 2)} \quad (z \in U), \tag{2.3}$$

then $\mathrm{Re}(p(z)) > 0\,(z \in U)$.

Remark. We see that

$$\frac{1-2\log 2}{2(1-\log 2)} \fallingdotseq -0.629\cdots .$$

Lemma 4 ([1]). Every analytic function $p(z)$ with positive real part in U is in the class H^p for all $0 < p < 1$.

Lemma 5 ([4]). If $f(z) \in A$ satisfies $z^r f(z) \in H^p\,(0 < p < \infty)$ for a real r, then $f(z) \in H^p\,(0 < p < \infty)$.

Lemma 6 ([1]). If $f'(z) \in H^p$ for some $p\,(0 < p < 1)$, then $f(z) \in H^q\,(q = p/(1-p))$.

Lemma 7 ([3]). Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r\,(0 \leq r < 1)$ at a point z_0, then we can write

$$z_0 w'(z_0) = k w(z_0),$$

where k is real and $k \geq 1$.

3. Hardy Class of Functions

Our first result for Hardy class is contained in

Theorem 1. *Let $f(z) \in A$ satisfy*

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \alpha_0 \quad (z \in U) \tag{3.1}$$

for some α_0 $(0 \leq \alpha_0 < 1)$, and let

$$\alpha_j = \begin{cases} \dfrac{1-2\alpha_{j-1}}{2(2^{1-2\alpha_{j-1}}-1)} & (\alpha_{j-1} \neq \frac{1}{2}) \\ \dfrac{1}{2\log 2} & (\alpha_{j-1} = \frac{1}{2}) \end{cases} \tag{3.2}$$

for $j = 1, 2, \cdots, n$. If $D^{n-j}f(z)$ is not of the form

$$D^{n-j}f(z) = \frac{z}{(1-ze^{it})^{2(1-\alpha_j)}}, \tag{3.3}$$

then there exists $\delta > 0$ such that $D^{n-j}f(z) \in H^{\delta+\frac{1}{2(1-\alpha_j)}}$.

Proof. Note that

$$\begin{aligned} D^{n+1}f(z) &= D(D^n f(z)) \\ &= z(D^n f(z))' \\ &= z(D^{n-1}f(z))' + z^2(D^{n-1}f(z))'' \end{aligned} \tag{3.4}$$

and

$$D^n f(z) = z(D^{n-1}f(z))'. \tag{3.5}$$

This implies that

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} = \mathrm{Re}\left\{1 + \frac{z(D^{n-1}f(z))''}{(D^{n-1}f(z))'}\right\} > \alpha_0, \tag{3.6}$$

so that, $D^{n-1}f(z) \in K(\alpha_0)$. Therefore, an application of Lemma 1 leads to

$$\begin{aligned} D^{n-1}f(z) \in K(\alpha_0) &\Longrightarrow D^{n-1}f(z) \in S^*(\alpha_1) \\ &\Longleftrightarrow D^{n-2}f(z) \in K(\alpha_1) \\ &\Longrightarrow D^{n-2}f(z) \in S^*(\alpha_2) \\ &\cdots \\ &\Longleftrightarrow D^{n-j}f(z) \in K(\alpha_{j-1}) \\ &\Longrightarrow D^{n-j}f(z) \in S^*(\alpha_j). \end{aligned}$$

Further, by using Lemma 2 and Lemma 5, we know that there exists $\delta > 0$ such that $D^{n-j}f(z) \in H^{\delta+\frac{1}{2(1-\alpha_j)}}$. ∎

Taking $j = n$ in Theorem 1, we have

Corollary 1. *Let $f(z) \in A$ satisfy* (3.1) *for some* α_0 $(0 \leq \alpha_0 < 1)$, *and let*

$$\alpha_n = \begin{cases} \dfrac{1-2\alpha_{n-1}}{2(2^{1-2\alpha_{n-1}}-1)} & (\alpha_{n-1} \neq \frac{1}{2}) \\ \dfrac{1}{2\log 2} & (\alpha_{n-1} = \frac{1}{2}). \end{cases}$$

If $f(z)$ is not of the form (3.3), *then there exists $\delta > 0$ such that* $f(z) \in H^{\delta+\frac{1}{2(1-\alpha_n)}}$.

Next, we derive

Theorem 2. *Let $f(z) \in A$ satisfy*

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{z}\right\} > \frac{1-2\log 2}{2(1-\log 2)} \quad (z \in U). \tag{3.7}$$

Then there exists p_j $(j = 1, 2, \cdots, n+1)$ such that $D^{n-j+1}f(z) \in H^{p_j}$, where

$$p_k < \frac{1}{j-k+1} \quad (k = 1, 2, \cdots, j). \tag{3.8}$$

Proof. Define the function $p(z)$ by

$$p(z) = \frac{D^n f(z)}{z}. \tag{3.9}$$

Then $p(z)$ is analytic in U and $p(0) = 1$. Since

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{z}\right\} = \mathrm{Re}(p(z) + zp'(z)) > \frac{1-2\log 2}{2(1-\log 2)}, \tag{3.10}$$

Lemma 3 gives that

$$\mathrm{Re}(p(z)) = \mathrm{Re}\left\{\frac{D^n f(z)}{z}\right\} > 0 \quad (z \in U). \tag{3.11}$$

Noting that

$$\frac{D^n f(z)}{z} = (D^{n-1}f(z))',$$

an application of Lemma 4 implies that $(D^{n-1}f(z))' \in H^{p_1}$, so by Lemma 6,

$$D^{n-1}f(z) \in H^{p_2} \quad (p_2 = \frac{p_1}{1-p_1}).$$

Further, since $D^{n-1}f(z) = z(D^{n-2}f(z))'$, using Lemma 5, we obtain $(D^{n-2}f(z))' \in H^{p_2}$. Taking this process again and again, we conclude that $D^{n-j+2}f(z) \in H^{p_{j-1}}$ and $0 < p_{j-1} < 1/2$. Thus, finally we have $D^{n-j+1}f(z) \in H^{p_j}$ $(0 < p_j < 1)$. This completes the proof of Theorem 2. ∎

Letting $j = n+1$ in Theorem 2, we have

Corollary 2. *Let $f(z) \in A$ satisfy* (3.7). *Then there exists p_{n+1} such that $f(z) \in H^{p_{n+1}}$, where*

$$p_k < \frac{1}{n-k+2} \quad (k = 1, 2, \cdots, n+1).$$

4. Hardy Class of Bounded Functions

Next our theorem for Hardy class of bounded functions is contained in

Theorem 3. *Let* $f(z) \in A$ *satisfy*

$$\left| \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1 \right| < \frac{5\alpha_0 - 2\alpha_0^2 - 1}{2\alpha_0} \quad (z \in U) \tag{4.1}$$

for some α_0 $(1/3 \leq \alpha_0 \leq 1/2)$, *or*

$$\left| \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1 \right| < \frac{\alpha_0 - 2\alpha_0^2 + 1}{2\alpha_0} \quad (z \in U) \tag{4.2}$$

for some α_0 $(1/2 \leq \alpha_0 < 1)$. *If* $D^{n-j}f(z)$ *is not of the form* (3.3), *then there exists* $\delta > 0$ *such that* $D^{n-j}f(z) \in H^{\delta + \frac{1}{2(1-\alpha_j)}}$ $(j = 1, 2, \cdots, n)$, *where* α_j *is given by* (3.2).

Proof. Define the function $w(z)$ by

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1 + (1 - 2\alpha_0)w(z)}{1 - w(z)} \quad (w(z) \neq 1). \tag{4.3}$$

Then $w(z)$ is analytic in U and $w(0) = 0$. It follows from (4.3) that

$$\begin{aligned} &\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1 \\ &= \left(\frac{w(z)}{1 - w(z)} \right) \left(2(1 - \alpha_0) + \frac{zw'(z)}{w(z)} + \frac{(1 - 2\alpha_0)(1 - w(z))}{1 + (1 - 2\alpha_0)w(z)} \left(\frac{zw'(z)}{w(z)} \right) \right). \end{aligned} \tag{4.4}$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq 1).$$

Then Lemma 7 leads us to $w(z_0) = e^{i\theta}$ and

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1).$$

Therefore, we have

(4.5)

$$\begin{aligned} &\left| \frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 1 \right| \\ &= \left| \frac{w(z_0)}{1 - w(z_0)} \right| \left| 2(1 - \alpha_0) + \frac{zw'(z_0)}{w(z_0)} + \frac{(1 - 2\alpha_0)(1 - w(z_0))}{1 + (1 - 2\alpha_0)w(z_0)} \left(\frac{zw'(z_0)}{w(z_0)} \right) \right| \\ &= \left| \frac{e^{i\theta}}{1 - e^{i\theta}} \right| \left| 2(1 - \alpha_0) + k + k\frac{(1 - 2\alpha_0)(1 - e^{i\theta})}{1 + (1 - 2\alpha_0)e^{i\theta}} \right| \\ &\geq \frac{2(1 - \alpha_0) + k}{|1 - e^{i\theta}|} - \frac{k|1 - 2\alpha_0|}{|1 + (1 - 2\alpha_0)e^{i\theta}|} \\ &\geq \frac{2(1 - \alpha_0) + k}{2} - \frac{k|1 - 2\alpha_0|}{2\alpha_0}. \end{aligned}$$

For $1/3 \leq \alpha_0 \leq 1/2$, we have

$$\left|\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 1\right| \geq \frac{5\alpha_0 - 2\alpha_0^2 - 1}{2\alpha_0} \tag{4.6}$$

and for $1/2 \leq \alpha_0 < 1$, we have

$$\left|\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 1\right| \geq \frac{\alpha_0 - 2\alpha_0^2 + 1}{2\alpha_0}. \tag{4.7}$$

Since the above contradicts our conditions (4.1) and (4.2) of the theorem, we conclude that $|w(z)| < 1$ for all $z \in U$. This implies that

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \alpha_0 \quad (z \in U). \tag{4.8}$$

Noting that (4.8) is equivalent to $D^n f(z) \in S^*(\alpha_0)$. Using the same manner in the proof of Theorem 1, we conclude that $D^{n-j}f(z) \in S^*(\alpha_j)$. Thus, applying Lemma 2 and Lemma 5, we can prove Theorem 3. ∎

If we put $j = n$ in Theorem 3, then we have

Corollary 3. *Let $f(z) \in A$ satisfy the condition* (4.1) *for some $\alpha_0\,(1/3 \leq \alpha_0 \leq 1/2)$ or* (4.2) *for some $\alpha_0\,(1/2 \leq \alpha_0 < 1)$. If $f(z)$ is not of the form* (3.3), *then there exists $\delta > 0$ such that $f(z) \in H^{\delta + \frac{1}{2(1-\alpha_n)}}$, where α_n is given by* (3.2).

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