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On Certain Starlike Functions

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Abstract

Let $f(z)$ be analytic in $|z|<1$, $f(0)=f'(0)-1=0$ and suppose that

$$1+\mathrm{Re}(zf''(z)/f'(z))<3/2 \qquad in \quad |z|<1.$$

Then, R. Singh and S. Singh [Colloqium Mathematcum, 47, 309-314 (1982)] proved that $f(z)$ is starlike in $|z|<1$.

The authors proved that if $f(z)$ is analytic in $|z|<1$, $f(0)=f'(0)-1=0$ and suppose that

$$1+\mathrm{Re}(zf''(z)/f'(z))<1+(\alpha/2) \qquad in \quad |z|<1$$

for $0<\alpha\leq 1$, then we have

$$|arg(zf'(z)/f(z))|<(\pi\alpha)/2 \qquad in \quad |z|<1.$$

1 Introduction.

Let A denote the class of functions $f(z)$ analytic in the open unit disk $U=\{z:|z|<1\}$ and normalized so that $f(0)=f'(0)-1=0$.

A function $f(z)\in A$ is called starlike with respect to the origin if

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>0 \qquad in \quad U.$$

It is well known that every starlike function is univalent in U.

Ozaki [2] proved that if $f(z)\in A$ and

$$(1) \qquad 1+\mathrm{Re}\frac{zf''(z)}{f'(z)}<\frac{3}{2} \qquad in \quad U,$$

then $f(z)$ is univalent in U.

R. Singh and S. Singh [4, Theorem 6] proved that if $f(z)\in A$ and satisfies the condition (1), then $f(z)$ is starlike in U.

In this paper, we need the following lemma.

Lemma 1. Let $f(z) \in A$ and starlike with respect to the origin in U.

Let $C(r,\theta) = \{f(te^{i\theta}) : 0 \le t \le r\}$ and let $T(r,\theta)$ be the total variation of $arg f(te^{i\theta})$ on $C(r,\theta)$, so that

$$T(r,\theta) = \int_0^r |\frac{\partial}{\partial t} arg f(te^{i\theta})| dt.$$

Then we have

$$T(r,\theta) < \pi.$$

We owe this lemma to Sheil-Small [5, Theorem 1].

2 Main result.

Main Theorem. Let $f(z) \in A$ and

(2) $$1 + \mathrm{Re}\frac{zf''(z)}{f'(z)} < 1 + \frac{\alpha}{2} \qquad in \quad U,$$

where $0 < \alpha \le 1$.

Then we have

$$|arg\frac{zf'(z)}{f(z)}| < \frac{\pi}{2}\alpha \qquad in \quad U$$

or $f(z)$ is starlike in U.

Proof. Let us put

(3) $$\frac{2}{\alpha}(1 + \frac{\alpha}{2} - 1 - \frac{zf''(z)}{f'(z)}) = \frac{zg'(z)}{g(z)}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

From the assumption (2), we have that

$$\mathrm{Re}\frac{zg'(z)}{g(z)} > 0 \qquad in \quad U.$$

This shows that $g(z)$ is starlike and univalent in U.

From (3) and by an easy calculation (see e.g. [1]), we have

$$f'(z) = (\frac{g(z)}{z})^{-\alpha/2}.$$

Since $g(z)$ is univalent in U, we have that

$$f'(z) \ne 0 \qquad in \quad U.$$

Therefore, we have

$$\text{(4)} \qquad \begin{aligned} \frac{f(z)}{zf'(z)} &= \int_0^1 \frac{f'(tz)}{f'(z)}dt \\ &= \int_0^1 t^{\alpha/2}\left(\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right)^{-\alpha/2}dt \end{aligned}$$

where $z = re^{i\theta}$, $0 \leq \theta < 2\pi$ and $0 < r < 1$.

Since $g(z)$ is starlike in U, from Lemma 1, we have

$$\text{(5)} \qquad -\pi < arg g(tre^{i\theta}) - arg g(re^{i\theta}) < \pi$$

for $0 < t \leq r$.

Putting

$$s = t^{\alpha/2}\left(\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right)^{-\alpha/2},$$

then we have

$$\text{(6)} \qquad arg s = -\frac{\alpha}{2} arg\left(\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right).$$

From (5) and (6), s lies in the convex sector

$$|arg s| \leq \frac{\pi}{2}\alpha$$

and the same is true of its integral mean of (4), (see e.g. [3, Lemma 1]).

Therefore we have

$$|arg \frac{f(z)}{zf'(z)}| < \frac{\pi}{2}\alpha \qquad in \quad U$$

or

$$|arg \frac{zf'(z)}{f(z)}| < \frac{\pi}{2}\alpha \qquad in \quad U.$$

This shows that

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0 \qquad in \quad U.$$

This completes our proof and this is an another proof of [4, Theorem 6].

References

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