

## A PROPERTY OF FUKUI'S EXTREMAL FUNCTION

須川敏幸 (京大・理)  
TOSHIYUKI SUGAWA

### 1. INTRODUCTION.

Let  $p$  be a positive integer and  $\mathcal{A}_p$  denote the class of analytic functions  $f$  in the unit disk  $\Delta$  with Taylor expansions of the form:  $f(z) = z^p + a_{p+1}z^{p+1} + \dots$ . For a constant  $\alpha \in [0, 1]$ , a function  $f$  in  $\mathcal{A}_p$  is called  $p$ -valently starlike of order  $\alpha$  (respectively,  $p$ -valently convex of order  $\alpha$ ), if  $f$  satisfies the condition  $\operatorname{Re} \frac{zf'}{f} \geq \alpha$  in  $\Delta$  (respectively,  $\operatorname{Re} (1 + \frac{zf''}{f'}) \geq \alpha$  in  $\Delta$ ). We denote by  $S_p^*(\alpha)$  and  $K_p(\alpha)$  the class of  $p$ -valently starlike and convex functions of order  $\alpha$ , respectively. The Marx-Strokhäcker theorem asserts that  $K_1(0) \subset S_1^*(\frac{1}{2})$ . Later, several authors have made efforts toward the generalization of this to the case for general  $p$ . At least, the following result has been proved by S. Fukui and M. Nunokawa.

**Theorem 1.1.** *For any  $p \geq 2$ , it holds that  $K_p(0) \subset S_p^*(0)$ .*

Recently, S. Fukui proved that there exists no positive constant  $\alpha > 0$  such that  $K_p(0) \subset S_p^*(\alpha)$ . In fact, he introduced a function  $f \in K_p(0)$  which is extremal in some sense, and satisfies that  $\inf_{z \in \Delta} \operatorname{Re} \frac{zf'(z)}{f(z)} = 0$  (for a precise definition of  $f$ , see Section 2). Further, he exhibited that his function  $f$  can be written as  $f(z) = \frac{z^p}{(z-1)^{2p}} h(z)$  with a polynomial  $h$  of degree  $p$ , and he showed a remarkable property that the real part of  $h(e^{i\theta})$  is a constant multiple of  $(1 - \cos \theta)^p$  at least in case  $p = 2, 3, 4, 5$ .

In this note, we review Fukui's extremal function and give a complete (self-contained) proof for the next theorem, which is essentially equivalent to a classical result (Lemma 3.1) concerned with trigonometric series.

**Theorem 1.2.** *For any integer  $p \geq 2$ , Fukui's extremal function  $f(z) = \frac{z^p}{(z-1)^{2p}} h(z)$  satisfies that*

$$\operatorname{Re} h(e^{i\theta}) = C_p (1 - \cos \theta)^p,$$

$$\text{where } C_p = 2^p \frac{(p!)^2}{(2p)!} = \frac{p(p-1)\cdots 2 \cdot 1}{(2p-1)(2p-3)\cdots 3 \cdot 1}.$$

We should note that the above theorem was proved also by Yamakawa and Nunokawa, independently.

## 2. AN EXTREMALITY OF FUKUI'S FUNCTION.

For an integer  $p \geq 2$ , S. Fukui considered a function  $f$  in  $\mathcal{A}_p$  which satisfies the differential equation:

$$f'(z) = \frac{pz^{p-1}}{(z-1)^{2p}}.$$

As is easily seen, the function  $f$  enjoys the property:

$$1 + \frac{zf''(z)}{f'(z)} = p \cdot \frac{1+z}{1-z}$$

in  $\Delta$ , thus it follows that  $f \in K_p(0)$ . Here, we set  $h(z) = \frac{(z-1)^{2p}}{z^p} f(z)$ , then we have

$$\begin{aligned} h'(z) &= \frac{(z-1)^{2p}}{z^p} f'(z) + \{2pz - p(z-1)\} \frac{(z-1)^{2p-1}}{z^{p+1}} f(z) \\ &= \frac{p}{z} + \frac{p(z+1)}{z(z-1)} h(z). \end{aligned}$$

Thus,  $h$  is an analytic solution of the differential equation

$$(2.1) \quad zh'(z) + p \cdot \frac{1+z}{1-z} h(z) = p.$$

Let  $h(z) = \sum_{n=0}^{\infty} A_n z^n$  be the power series expansion, then from (2.1) it follows that

$$\begin{aligned} &\sum_{n=0}^{\infty} n A_n z^n + p \left( 1 + 2 \sum_{n=1}^{\infty} z^n \right) \sum_{n=0}^{\infty} A_n z^n \\ &= \sum_{n=0}^{\infty} (n+p) A_n z^n + 2p \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n-1} A_k \right) z^n = p, \end{aligned}$$

thus the coefficients should satisfy  $A_0 = 1$  and

$$(2.2) \quad (n+p)A_n + 2p \sum_{k=0}^{n-1} A_k = 0 \quad (n \geq 1).$$

From (2.2) with  $n = 1$ , we see that  $A_1 = -\frac{2p}{p+1}$ . If  $n \geq 2$ , subtracting  $(n+p-1)A_{n-1} + 2p \sum_{k=0}^{n-2} A_k = 0$  from (2.2), we have

$$(n+p)A_n + (p-n+1)A_{n-1} = 0.$$

Therefore, we obtain that  $A_n = 0$  for  $n > p$  and

$$\begin{aligned}
 A_n &= \left(-\frac{p-n+1}{p+n}\right) \left(-\frac{p-n+2}{p+n-1}\right) \cdots \left(-\frac{p-1}{p+2}\right) A_1 \\
 &= 2(-1)^n \frac{p(p-1) \cdots (p-n+1)}{(p+n) \cdots (p+1)} \\
 (2.3) \quad &= (-1)^n \frac{2(p!)^2}{(p+n)!(p-n)!}
 \end{aligned}$$

for  $1 \leq n \leq p$ . In particular,  $h(z)$  is a polynomial of degree  $p$ . First note that  $h(1) = 0$  by an equivalent relation of (2.1):

$$(2.4) \quad z(1-z)h'(z) + p(1+z)h(z) + p(z-1) = 0.$$

Differentiation of (2.4) yields that

$$(2.5) \quad z(1-z)h''(z) + (p+1+(p-2)z)h'(z) + ph(z) + p = 0,$$

in particular,  $h'(1) = -\frac{p}{2p-1}$ . Further differentiating (2.5), we obtain

$$z(1-z)h'''(z) + (p+2+(p-4)z)h''(z) + (2p-2)h'(z) = 0,$$

and  $h''(1) + h'(1) = 0$ , especially. Now we sum up the above result as

$$h(1) = 0, \quad h'(1) = -h''(1) = -\frac{p}{2p-1} (\neq 0).$$

Here we note that

$$\frac{zf'(z)}{f(z)} = \frac{z \cdot pz^{p-1}}{(z-1)^{2p}f(z)} = \frac{p}{h(z)},$$

so Theorem 1.1 implies that  $\operatorname{Re} \frac{p}{h(z)} \geq 0$ . Hence, we obtain that  $\inf_{z \in \Delta} \operatorname{Re} \frac{p}{h(z)} = 0$  in fact by showing the following elementary

**Lemma 2.1.** *Suppose that  $h(z)$  is analytic near  $z = 1$  and that*

$$h(1) = 0, \quad h'(1) \neq 0.$$

*Then we have*

$$\liminf_{\Delta \ni z \rightarrow 1} \operatorname{Re} \frac{1}{h(z)} \leq \frac{-\operatorname{Re}[h'(1) + h''(1)]}{2|h'(1)|^2}.$$

*Proof.* We write  $h(e^{i\theta}) = u(\theta) + iv(\theta)$ . Then we have

$$\begin{aligned}
 u'(\theta) + iv'(\theta) &= ie^{i\theta}h'(e^{i\theta}), \text{ and} \\
 u''(\theta) + iv''(\theta) &= -e^{i\theta}h'(e^{i\theta}) - e^{2i\theta}h''(e^{i\theta}).
 \end{aligned}$$

Letting  $\theta = 0$ , we have  $u(0) = v(0) = 0$ ,

$$\begin{aligned} u'(0)^2 + v'(0)^2 &= |h'(1)|^2, \quad \text{and} \\ u''(0) &= -\operatorname{Re}[h'(1) + h''(1)]. \end{aligned}$$

Thus, by de l'Hospital's theorem, we see that

$$\begin{aligned} \liminf_{\Delta \ni z \rightarrow 1} \operatorname{Re} \frac{1}{h(z)} &= \liminf_{\Delta \ni z \rightarrow 1} \frac{\operatorname{Re} h(z)}{|h(z)|^2} \\ &\leq \lim_{\theta \rightarrow 0} \frac{\operatorname{Re} h(e^{i\theta})}{|h(e^{i\theta})|^2} = \lim_{\theta \rightarrow 0} \frac{u(\theta)}{u(\theta)^2 + v(\theta)^2} = \lim_{\theta \rightarrow 0} \frac{u'(\theta)}{2(u'(\theta)u(\theta) + v'(\theta)v(\theta))} \\ &= \frac{1}{2} \lim_{\theta \rightarrow 0} \frac{u''(\theta)}{u''(\theta)u(\theta) + u'(\theta)^2 + v''(\theta)v(\theta) + v'(\theta)^2} \\ &= \frac{u''(0)}{2(u'(0)^2 + v'(0)^2)} = \frac{-\operatorname{Re}[h'(1) + h''(1)]}{2|h'(1)|^2}. \end{aligned}$$

□

### 3. PROOF OF THEOREM 1.2.

This section devoted to a proof of Theorem 1.2. We remark that this theorem produces a more direct proof of the extremality of Fukui's function:  $\inf_{z \in \Delta} \operatorname{Re} \frac{p}{h(z)} = 0$ .

First, we prepare the next

**Lemma 3.1.** *For a positive integer  $p$ ,*

$$(3.1) \quad \sin^{2p} \frac{\theta}{2} = \sum_{n=0}^p 2^{1-\varepsilon-2p} (-1)^n \binom{2p}{p-n} \cos n\theta,$$

where  $\varepsilon = \delta_{0,n}$ , i.e.,  $\varepsilon = 1$  if  $n = 0$  and  $\varepsilon = 0$  otherwise.

However this is a known result, we include a proof for convenience of the reader.

*Proof.* Since the function  $\sin^{2p} \frac{\theta}{2}$  is even, its Fourier expansion takes a form:  $\sin^{2p} \frac{\theta}{2} = \sum_{n=0}^{\infty} B_n \cos n\theta$ , here

$$B_n = \frac{1}{2^\varepsilon \pi} \int_0^{2\pi} \sin^{2p} \frac{\theta}{2} \cos n\theta d\theta$$

for  $n = 0, 1, 2, \dots$ . Now we introduce an auxiliary double sequence

$$\alpha_{k,n} = \int_0^{2\pi} \sin^{2k} \frac{\theta}{2} \cos n\theta d\theta$$

for  $k, n = 0, 1, 2, \dots$ . Then, in order to prove the assertion, it suffices to show that

$$\alpha_{k,n} = \begin{cases} \pi(-1)^n 2^{1-2k} \binom{2k}{k-n} & \text{if } 0 \leq n \leq k, \\ 0 & \text{if } k < n. \end{cases}$$

Using a descending formula:

$$\begin{aligned} \alpha_{k,n} &= \int_0^{2\pi} \sin^{2k-1} \frac{\theta}{2} \left( \sin \frac{\theta}{2} \cos n\theta \right) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \sin^{2k-1} \frac{\theta}{2} \left( \sin\left(n + \frac{1}{2}\right)\theta - \sin\left(n - \frac{1}{2}\right)\theta \right) d\theta \\ &= \frac{1}{4} \int_0^{2\pi} \sin^{2k-2} \frac{\theta}{2} \left[ (\cos n\theta - \cos(n+1)\theta) - (\cos(n-1)\theta - \cos n\theta) \right] d\theta \\ &= \frac{1}{4} (2\alpha_{k-1,n} - \alpha_{k-1,n+1} - \alpha_{k-1,n-1}), \end{aligned}$$

we can show the above equation by induction with respect to  $k$ .  $\square$

Now we return to Fukui's extremal function  $f$ . By (2.3), we have

$$\begin{aligned} h(z) &= \sum_{n=0}^p A_n z^n = 1 + \sum_{n=1}^p (-1)^n \frac{2(p!)^2}{(p+n)!(p-n)!} z^n \\ &= \sum_{n=0}^p (-1)^n 2^{1-\varepsilon} \frac{(p!)^2}{(p+n)!(p-n)!} z^n. \end{aligned}$$

In particular, we obtain

$$(3.2) \quad \operatorname{Re} h(e^{i\theta}) = \sum_{n=0}^p (-1)^n 2^{1-\varepsilon} \frac{(p!)^2}{(p+n)!(p-n)!} \cos n\theta.$$

On the other hand, we have

$$(1 - \cos \theta)^p = \left( 2 \sin^2 \frac{\theta}{2} \right)^p = \sum_{n=0}^p (-1)^n 2^{1-\varepsilon-p} \binom{2p}{p-n} \cos n\theta$$

by (3.1), and

$$C_p 2^{1-\varepsilon-p} \binom{2p}{p-n} = 2^{1-\varepsilon} \frac{(p!)^2}{(2p)!} \frac{(2p)!}{(p+n)!(p-n)!} = 2^{1-\varepsilon} \frac{(p!)^2}{(p+n)!(p-n)!},$$

hence now Theorem 1.2 is proved.