数理解析研究所講究録
965 巻 1996 年 87-133

On Absolute CM-periods

京大理学部　吉田敬之 (HIROYUKI YOSHIDA)

本文は準備中の英文草稿を使用した。ここで内容を概観しておく。序文と §1 ～ §5 の Tex 原稿は昨年 10 月に書き上げた。この時点で、筆者は Anderson [A], Colmez [C] の結果を知らなかった。その後、§2 の定理は Anderson の結果と、§3 の予想は実質的に Colmez の予想と一致することがわかった。但し、筆者の定式化のほうがより簡潔な形をしている。新しい内容は、§5, §6 にある。

ここでの主眼は、K を CM 体、F を K の最大実部分体、χ を拡大 K/F に対応する $F_A^\times$ の Hecke character とするとき、$P=\exp(L'_F(0,\chi)/L_F(0,\chi))$ を CM-periods と関係づけた ($P\sim\pi^{[F:\mathbf{Q}]}\prod_{\sigma\in J_K}p_K(\sigma,\sigma)$, Conjecture 3.4) 訳であるが、$P$ が円周率 π と同様に、Galois 共役写像で不変な "Absolute period " として振舞うという点にある。読者は例えば、ψ を even Dichilet character とするとき、関係 $(L(n,\psi)/g(\psi)\pi^n)^\sigma=L(n,\psi^\sigma)/g(\psi^\sigma)\pi^n$, $0<n\in 2\mathbf{Z}$, $\sigma\in \mathrm{Aut}(\mathbf{C})$, $g(\psi)$ は Gauss 和等を想起されたい。§5 の実験結果は、この主張を十分に裏付けていると思う。

§6 は昨年 11 月以降に書いたものでまだ完成していない。Conjecture 6.1 で P の absolute period としての性格を定式化した。character による twisting、L 函数の値をとる点の shift について Motive 理論からわかる結果と、この予想が consistent であることが示せる。虚 2 次体の場合の予想は、ここに書いておいた方法で困難なく証明できる。さて Conjecture 6.1 は §6.1 の記号を用いると

$$\{(\frac{\prod_{i=1}^n L(m/2,\lambda_i)}{\pi^A P^e})\}^\sigma=\zeta(\frac{\prod_{i=1}^n L(m/2,\lambda_i^\sigma)}{\pi^A P^e}),\qquad \zeta \text{ is a root of unity}$$

を主張する。ここで ζ を具体的に定めるのは非常に面白い問題である。虚 2 次体のとき、Prop. 6.2 で示した以上の結果も得ている。虚 2 次体の場合は、本質的には演習問題であると思うが、一般の場合を徹底的に分析していくと、かなり面白く新しい側面も現れる。

以上に補足した §6 についての 3 点は、近い将来に詳しく書きたいと思っている。

ON ABSOLUTE CM-PERIODS

BY HIROYUKI YOSHIDA

Introduction. In this paper, we shall study a new relation between the derivative of Artin's L-function at $s=0$ and periods of abelian varieties with complex multiplication.

For an algebraic number field K, let J_K be the set of all isomorphisms of K into $\mathbf{C}$ and I_K be the free abelian group generated by J_K. Assume that K is a CM-field and let Φ be a CM-type of K. We can find an abelian variety A of type (K,Φ) defined over $\overline{\mathbf{Q}}$. For every $\sigma \in \Phi$, there is a holomorphic differential 1-form $\omega_\sigma(\neq 0)$ on A such that ω_σ is multiplied by a^σ for the action of $a \in K \cap \mathrm{End}(A)$ and that ω_σ is rational over $\overline{\mathbf{Q}}$. Then there exists a constant $p_K(\sigma,\Phi) \in \mathbf{C}^\times$ such that

$$(1) \qquad \int_c \omega_\sigma \sim \pi\, p_K(\sigma,\Phi) \qquad \text{for every} \quad c \in H_1(A,\mathbf{Z}).$$

Here, for a, $b \in \mathbf{C}$, we write $a \sim b$ if $b \neq 0$ and $a/b \in \overline{\mathbf{Q}}$. We know that $p_K(\sigma,\Phi) \mod \overline{\mathbf{Q}}^\times$ does not depend on the choice of A and ω_σ. Shimura showed ([Sh5], [Sh6]) that p_K can be extended (or factorized) to the bilinear form from $I_K \times I_K$ to $\mathbf{C}^\times/\overline{\mathbf{Q}}^\times$, which enjoys several functorial properties (see §1). Thus we have the "CM-period" $p_K(\sigma,\tau) \in \mathbf{C}^\times$, uniquely determined mod $\overline{\mathbf{Q}}^\times$, for every σ, $\tau \in I_K$.

Now let K be a CM-field which is normal over $\mathbf{Q}$ and put $G = \mathrm{Gal}(K/\mathbf{Q})$. Let $\rho \in G$ be the complex conjugation. As is well known, ρ belongs to the center of G. The central theme of this paper is the following

Main Conjecture. *Let ψ be a representation of G and let χ_ψ be the character of ψ. We assume that $\psi(\rho) = -id.$ and that χ_ψ is $\mathbf{Q}$-valued. Let $L(s,\psi)$ be the Artin L-function attached to ψ. Then*

$$(2) \qquad \exp\Big(\frac{L'(0,\psi)}{L(0,\psi)}\Big) \sim \pi^{\dim\psi} \prod_{\sigma \in G} p_K(id,\sigma)^{\chi_\psi(\sigma)}.$$

We note that $L(0,\psi) \in \mathbf{Q}^\times$. Let F be the maximal real subfield of K and χ be the Hecke character which corresponds to the quadratic extension K/F. Let $L_F(s,\chi)$ denote the Hecke L-function attached to χ. Then (2) implies (cf. Proposition 3.5)

$$(3) \qquad \exp\Big(\frac{L_F'(0,\chi)}{L_F(0,\chi)}\Big) \sim (\pi^{1/2} p_K(\mathrm{id.},\mathrm{id.}))^{[K:\mathbf{Q}]}.$$

In §2, we shall prove (2) when K is abelian over $\mathbf{Q}$ (Theorem 2.7). When K is an imaginary quadratic field of discriminant $-d$, the Chowla-Selberg formula ([SC], §12) states

$$\varpi \sim \sqrt{\pi} \prod_{a=1}^{d-1} \Gamma(\frac{a}{d})^{w\chi(a)/4h}, \tag{4}$$

where ϖ is a period of integral of a holomorphic differential form on an elliptic curve with complex multiplication by K, h is the class number of K, w is the number of roots of unity contained in K and χ is the Dirichlet character corresponding to the quadratic extension $K/\mathbf{Q}$. By (1), we have $\varpi \sim \pi p_K(\mathrm{id.}, \mathrm{id.})$. We also have (cf. (2.11) and (4.2))

$$L_{\mathbf{Q}}(0, \chi) = \frac{2h}{w}, \qquad \exp(\frac{L'_{\mathbf{Q}}(0, \chi)}{L_{\mathbf{Q}}(0, \chi)}) = \frac{1}{d} \prod_{a=1}^{d-1} \Gamma(\frac{a}{d})^{w\chi(a)/2h}.$$

Now (3) tells that

$$\exp(\frac{L'_{\mathbf{Q}}(0, \chi)}{L_{\mathbf{Q}}(0, \chi)}) \sim \pi p_K(\mathrm{id.}, \mathrm{id.})^2.$$

Therefore (3) gives a generalization of the Chowla-Selberg formula.

Gross [G] obtained an algebro-geometric proof of (4) based on the calculation of periods of Fermat curves due to Rohrlich and on considerations of periods for families of abelian varieties with complex multiplication by K. Our proof of (2) for an abelian number field K uses Rohrlich's calculation of periods and Shimura's factorization theorem of CM-periods. Taking sufficiently many CM-types of K and using Shimura's theorem, we can give an explicit formula for $p_K(\mathrm{id.}, \sigma)$, $\sigma \in G$ (Theorem 2.6). We note that Shimura predicted that his theorem on CM-periods would give a generalization of the Chowla-Selberg formula for abelian case (cf. [Sh5], p. 571). However, as far as the author knows, explicit relations with the derivatives of L-functions were hitherto unnoticed.

In §3, we shall discuss functorial properties of our Main Conjecture. We shall also formulate a stronger conjecture than (2) (Conjecture 3.2). In §4, we shall collect several general facts on CM-fields and shall recall another important theorem of Shimura which expresses critical values of Hecke's L-function with a Grössencharacter of A_0-type by CM-periods. After these preparations, in §5 we shall submit Main Conjecture to numerical tests. We shall treat the case where $\mathrm{Gal}(K/\mathbf{Q})$ is the dihedral group of order 8, i.e., the case which goes back to Hecke. We shall discuss three numerical examples in detail. These examples will give us strong confidence in the conjecture.

It would be worth to point out three implications of our conjecture. First the conjecture predicts certain arithmetic property at CM-points of non-holomorphic automorphic forms which appear in a limit formula of Kronecker's type (cf. Asai [As]). It would be interesting to investigate the conjecture in this connection. Secondly our conjecture is in some sense "complementary" to the Stark-Shintani conjecture ([St], [Shi4]). In fact, Stark's conjecture (in crude form) predicts, for Artin's L-function $L(s, \psi)$, that $L'(0, \psi)$ (or the leading coefficient of the Taylor expansion of $L(s, \psi)$ at $s = 0$) can be expressed using logarithms of units of algebraic number fields when $L(0, \psi) = 0$. In our conjecture, $L(0, \psi) \neq 0$ and

$\exp(L'(0,\psi))$ is of highly transcendental nature. Thirdly the most general conjecture in our present knowledge which predicts special values of motivic L-functions is Beilinson's: it gives (the transcendental part of) the leading coefficient of the Taylor expansion of the L-function at integral points (cf. [RSS] and several articles in [JKS] on this topic). As noted above, our conjecture gives the next coefficient to the leading one of the Taylor expansion, which does not seem to be immediately predictable by Beilinson's conjecture. It would be very interesting to investigate whether such coefficients can be predicted in the framework of the theory of motives.

Notation and Terminology. Throughout the paper, we fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ in $\mathbf{C}$. By an algebraic number field, we understand an algebraic extension of $\mathbf{Q}$ of finite degree contained in $\overline{\mathbf{Q}}$. We denote by ρ the complex conjugation. For an algebraic number field K, J_K denotes the set of all isomorphisms of K into $\mathbf{C}$ and I_K denotes the free abelian group generated by J_K. The ring of integers of K is denoted by $\mathfrak{O}_K$. We denote by $K_A^\times$ the idele group of K. For $a \in K$, $a \gg 0$ means that a is totally positive. We abbreviate $\rho|K$ to ρ if no confusion is likely. For an extension L of K of finite degree, $\mathrm{Res}_{L/K}$ denotes the restriction homomorphism from I_L to I_K; $\mathrm{Inf}_{L/K}$ denotes the homomorphism from I_K to I_L such that, for $\sigma \in J_K$, $\mathrm{Inf}_{L/K}(\sigma)$ is the sum of all elements of J_L whose restrictions to K coincide with σ. The norm map from L to K is denoted by $N_{L/K}$. By a CM-field, we understand a totally imaginary quadratic extension of a totally real algebraic number field. For a CM-field K, $\Phi \in I_K$ is called a CM-type if $\Phi + \Phi\rho$ is the sum of all elements in J_K. If $\Phi = \sum_{i=1}^n \sigma_i$, we often identify Φ with the set of isomorphisms $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ or with the representation $\oplus_{i=1}^n \sigma_i$ of K by $n \times n$ complex matrices. For a finite group G, a subgroup H of G and a representation ψ of H, the induced representation from ψ is denoted by $\mathrm{Ind}_H^G \psi$ or $\mathrm{Ind}(\psi; H \longrightarrow G)$. For $m_1, \cdots, m_r \in \mathbf{Z}$, $(m_1, \cdots, m_r)$ denotes the greatest common divisor of $m_1, \cdots, m_r$ if one of them is non-zero. For $a, b \in \mathbf{C}$, we write $a \sim b$ if $b \neq 0$ and $a/b \in \overline{\mathbf{Q}}$.

§1. CM-periods

In this section, we shall review basic properties of CM-periods which are essential for succeeding sections.

Let K be a CM-field of degree $2n$ and let Φ be a CM-type of K. We can find an abelian variety A defined over $\overline{\mathbf{Q}}$ such that A is of type (K, Φ). By this word, we understand that

(i) $\dim A = n$ and $K \subseteq \mathrm{End}(A) \otimes \mathbf{Q}$.

(ii) The representation of K on the space of holomorphic differential 1-forms on A is equivalent to Φ.

By definition, for every $\sigma \in \Phi$, there exists a non-zero holomorphic differential 1-form ω_σ rational over $\overline{\mathbf{Q}}$ such that

$$a_*\omega_\sigma = a^\sigma \omega_\sigma \qquad \text{for every} \quad a \in K \cap \mathrm{End}(A).$$

Here a_* denotes the action of a on differential 1-forms. It can be shown that there exists $p_K(\sigma, \Phi) \in \mathbf{C}^\times$ such that

$$\int_c \omega_\sigma \sim \pi\, p_K(\sigma, \Phi) \qquad \text{for every} \quad c \in H_1(A, \mathbf{Z}). \tag{1.1}$$

$p_K(\sigma, \Phi) \mod \overline{\mathbf{Q}}^\times$ does not depend on the choice of A and ω_σ. The following factorization theorem is proved by Shimura [Sh6], Theorem 1.1, [Sh7], Theorem 32.5.

Theorem S1. *For every CM-field K, there exists a map $p_K : I_K \times I_K \longrightarrow \mathbf{C}^\times$ with the following properties.*
(1) $p_K(\sigma, \Phi)$ is defined by (1.1) if Φ is a CM-type of K.
(2) $p_K(\sigma_1 + \sigma_2, \tau) \sim p_K(\sigma_1, \tau)p_K(\sigma_2, \tau)$, $p_K(\sigma, \tau_1 + \tau_2) \sim p_K(\sigma, \tau_1)p_K(\sigma, \tau_2)$ for every σ, σ_1, σ_2, τ, τ_1, $\tau_2 \in I_K$.
(3) $p_K(\xi\rho, \eta) \sim p_K(\xi, \eta\rho) \sim p_K(\xi, \eta)^{-1}$ for every ξ, $\eta \in I_K$.
(4) $p_K(\xi, \mathrm{Res}_{L/K}(\zeta)) \sim p_L(\mathrm{Inf}_{L/K}(\xi), \zeta)$ if $\xi \in I_K$, $\zeta \in I_L$ and $K \subset L$, L is a CM-field.
(5) $p_K(\mathrm{Res}_{L/K}(\zeta), \xi) \sim p_L(\zeta, \mathrm{Inf}_{L/K}(\xi))$ if $\xi \in I_K$, $\zeta \in I_L$ and $K \subset L$, L is a CM-field.
(6) $p_{K'}(\gamma\xi, \gamma\eta) \sim p_K(\xi, \eta)$ if γ is an isomorphism of K' onto K.

Remark 1.1. (1) $p_K(\sigma, \tau) \mod \overline{\mathbf{Q}}^\times$ is uniquely determined.
(2) We can take $p_K(\sigma, \tau)$ from $\mathbf{R}^\times$ for every σ, $\tau \in I_K$. This can be seen using Shimura [Sh5], Proposition 1.6 and following the proof of Theorem 1.1 of [Sh6].
(3) If we consider periods of differential 1-forms of the second kind, we can interpret (3) as a generalized Legendre's relation.

§2. The case of abelian fields

In this section, we shall give a proof of the Main Conjecture in the case of abelian fields.

Let $n \geq 3$ be an integer. We set $\zeta_n = e^{2\pi i/n}$ and put $K = \mathbf{Q}(\zeta_n)$. This notation will be retained until the end of the proof of Theorem 2.5. For $a \in (\mathbf{Z}/n\mathbf{Z})^\times$, let $\sigma(a) \in \mathrm{Gal}(K/\mathbf{Q})$ denote the automorphism given by $\zeta_n^{\sigma(a)} = \zeta_n^a$.

We consider the Fermat curve

$$F_n : x^n + y^n = 1.$$

The genus of F_n is $(n-1)(n-2)/2$. For a triplet of integers r, s, t such that

$$(2.1) \qquad 0 < r, s, t < n, \qquad r + s + t \equiv 0 \mod n,$$

we consider a differential form

$$\eta_{r,s,t} = x^{r-1}y^{s-n}dx$$

on F_n. Then $\eta_{r,s,t}$, $r + s + t = n$ make a basis of the space of differential forms of the first kind on F_n. Rohrlich showed that (cf. the Theorem of the appendix of [G])

$$(2.2) \qquad \int_\gamma \eta_{r,s,t} \sim B(\frac{r}{n}, \frac{s}{n}) \qquad \text{for every} \quad \gamma \in H_1(F_n, \mathbf{Z}),$$

where B denotes the beta function. For $a \in \mathbf{Z}$, let $\langle\langle a \rangle\rangle$ denote the integer such that

$$0 \leq \langle\langle a \rangle\rangle < n, \qquad \langle\langle a \rangle\rangle \equiv a \mod n.$$

For $a \in \mathbf{Z}/n\mathbf{Z}$, by abuse of notation, we set $\langle\langle a\rangle\rangle = \langle\langle \tilde{a}\rangle\rangle$ taking $\tilde{a} \in \mathbf{Z}$ such that $\tilde{a} \mod n = a$. For integers r, s, t satisfying (2.1), we set

$$H_{r,s,t} = \{a \in (\mathbf{Z}/n\mathbf{Z})^\times \mid \langle\langle ar\rangle\rangle + \langle\langle as\rangle\rangle + \langle\langle at\rangle\rangle = n\}.$$

Then we have $|H_{r,s,t}| = \varphi(n)/2$. Assume that $(r,s,t,n) = 1$. Then there exists an abelian variety $A_{r,s,t}$ defined over $\mathbf{Q}$ which is a factor of the Jacobian variety J_n of F_n such that $\dim A_{r,s,t} = \varphi(n)/2$ and that $\mathbf{Z}[\zeta_n] \subseteq \mathrm{End}(A_{r,s,t})$. The differential forms $\eta_{\langle\langle ar\rangle\rangle,\langle\langle as\rangle\rangle,\langle\langle at\rangle\rangle}$, $a \in H_{r,s,t}$ correspond to the basis of $\overline{\mathbf{Q}}$-rational holomorphic differential 1-forms on $A_{r,s,t}$. Hence the CM-type of K determined by $A_{r,s,t}$ is

$$\Phi_{r,s,t} = \{\sigma(a) \mid a \in H_{r,s,t}\}.$$

Now by (2.2) and Theorem S1, (1), (2), we obtain

$$\begin{aligned}\pi p_K(\sigma(a), \Phi_{r,s,t}) &\sim \pi \prod_{u \in \Phi_{r,s,t}} p_K(\sigma(a), \sigma(u)) \\ &\sim B\Big(\frac{\langle\langle ar\rangle\rangle}{n}, \frac{\langle\langle as\rangle\rangle}{n}\Big) \qquad \text{for every} \quad a \in H_{r,s,t}.\end{aligned} \tag{2.3}$$

We are going to write (2.3) in more convenient form for calculation. For $1 \le a \le n-2$, set $r = 1$, $s = a$, $t = n-(a+1)$. For $x \in \mathbf{R}$, let $\langle x\rangle$ denote the fractional part of x, i.e., $0 \le \langle x\rangle < 1$, $x - \langle x\rangle \in \mathbf{Z}$ and let $[x] = x - \langle x\rangle$ denote the integer part of x. We can show easily that, for $u \in (\mathbf{Z}/n\mathbf{Z})^\times$, $\langle \frac{au}{n}\rangle + \langle \frac{u}{n}\rangle < 1$ if and only if $\langle\langle au\rangle\rangle + \langle\langle u\rangle\rangle + \langle\langle n-(a+1)u\rangle\rangle = n$. Here we set $\langle \frac{u}{n}\rangle = \langle \frac{\tilde{u}}{n}\rangle$ taking $\tilde{u} \in \mathbf{Z}$ so that $\tilde{u} \mod n = u$. Put

$$T_a = \{t \in (\mathbf{Z}/n\mathbf{Z})^\times \mid \langle \frac{at}{n}\rangle + \langle \frac{t}{n}\rangle < 1\}, \qquad \Phi_a = \{\sigma(t) \mid t \in T_a\}.$$

We have

$$H_{1,a,n-(a+1)} = T_a, \qquad \Phi_{1,a,n-(a+1)} = \Phi_a.$$

Hence (2.3) can be written as[1]

$$\prod_{t \in T_a} p_K(\sigma(1), \sigma(t)) \sim \pi^{-1} B\Big(\frac{a}{n}, \frac{1}{n}\Big). \tag{2.4}$$

For $t \in (\mathbf{Z}/n\mathbf{Z})^\times$, put

$$\epsilon_{at} = \begin{cases} 1 & \text{if} \quad t \in T_a, \\ -1 & \text{if} \quad t \notin T_a. \end{cases}$$

Since $p_K(\sigma(1), \sigma(-t)) \sim p_K(\sigma(1), \sigma(t))^{-1}$ by Theorem S1, (3), we obtain

$$\prod_{t=1,(t,n)=1}^{[(n-1)/2]} p_K(\sigma(1), \sigma(t))^{\epsilon_{at}} \sim \pi^{-1} \frac{\Gamma(\frac{a}{n})\Gamma(\frac{1}{n})}{\Gamma(\frac{a+1}{n})} \tag{2.5}$$

[1] When n is an odd prime, (2.4) follows immediately from a formula of Weil [W2], p. 815.

by (2.4). We note that (2.5) holds for every a, $1 \le a \le n-2$.

To relate CM-periods to derivatives of Dirichlet's L-functions at $s=0$, we first consider Hurwitz-Lerch's zeta function. For $0 < a \le 1$, set

$$\zeta(s,a) = \sum_{m=0}^{\infty}(a+m)^{-s}, \qquad \Re(s) > 1.$$

Then $\zeta(s,a)$ can be meromorphically continued to the whole s-plane, holomorphic except for a simple pole at $s=1$. We have

$$\zeta(0,a) = \frac{1}{2} - a. \tag{2.6}$$

$$\zeta'(0,a) = \log\Gamma(a) - \frac{1}{2}\log(2\pi). \tag{2.7}$$

(cf. Whittaker-Watson [WW], p. 271.) For $c \in \mathbf{Z}$, $1 \le c \le n-1$, we set

$$\zeta_{\mathbf{Q}}(s,c) = \sum_{m=1, m\equiv c \bmod n}^{\infty} m^{-s}.$$

Then we have

$$\zeta_{\mathbf{Q}}(s,c) = n^{-s}\zeta(s,\frac{c}{n}).$$

By (2.6) and (2.7), we get

$$\zeta_{\mathbf{Q}}(0,c) = \frac{1}{2} - \frac{c}{n}, \tag{2.8}$$

$$\zeta'_{\mathbf{Q}}(0,c) = \log\Gamma(\frac{c}{n}) - \frac{1}{2}\log 2\pi - \log n \cdot \zeta_{\mathbf{Q}}(0,c). \tag{2.9}$$

Let η be a Dirichlet character of conductor n which is not necessarily primitive. Since $L(s,\eta) = \sum_{c=1}^{n-1}\eta(c)\zeta_{\mathbf{Q}}(s,c)$, we obtain

$$L(0,\eta) = -\frac{1}{n}\sum_{c=1}^{n-1}\eta(c)c, \tag{2.10}$$

$$L'(0,\eta) = \sum_{c=1}^{n-1}\eta(c)\log\Gamma(\frac{c}{n}) - \log n \cdot L(0,\eta) \tag{2.11}$$

if η is not trivial.[2]

[2] We followed Shintani [Shi5] to derive these formulas.

Now let η be a primitive Dirichlet character of conductor f. We assume that f divides n. Let η_* be the Dirichlet character of conductor n obtained from η. Let

$$L(s,\eta_*)=\sum_{m=1}^{\infty}\eta_*(m)m^{-s}$$

be the L-function attached to η_*. Since

$$\eta_*(m)=\begin{cases}\eta(m) & \text{if } (m,n)=1,\\ 0 & \text{if } (m,n)>1,\end{cases}$$

we have

$$L(s,\eta_*)=\prod_{p\mid\frac{n}{f}}(1-\eta(p)p^{-s})\times L(s,\eta). \tag{2.12}$$

It is well known that $L(0,\eta)\neq 0$ if η is odd and primitive.

Let $\widehat{G}$ denote the set of all primitive Dirichlet characters whose conductors divide n. Let $\widehat{G}_-$ (resp. $\widehat{G}_+$) denote the subset of $\widehat{G}$ consisting of all odd (resp. even) Dirichlet characters in $\widehat{G}$.

Lemma 2.1. *For $1\le a\le n-2$ and $t\in(\mathbf{Z}/n\mathbf{Z})^\times$, we have*

$$\epsilon_{at}=-(2\langle\frac{t}{n}\rangle-1)-(2\langle\frac{at}{n}\rangle-1)+(2\langle\frac{(a+1)t}{n}\rangle-1).$$

Proof. Put $\mu_{at}=\langle\frac{t}{n}\rangle+\langle\frac{at}{n}\rangle-\langle\frac{(a+1)t}{n}\rangle$. It suffices to show $\epsilon_{at}=1-2\mu_{at}$. Since $\mu_{at}\in\mathbf{Z}$, $0\le\langle\frac{t}{n}\rangle,\langle\frac{at}{n}\rangle,\langle\frac{(a+1)t}{n}\rangle<1$, we have $\mu_{at}=0$ or 1. If $\langle\frac{t}{n}\rangle+\langle\frac{at}{n}\rangle<1$, then $\mu_{at}<1$; hence $\mu_{at}=0$. If $\langle\frac{t}{n}\rangle+\langle\frac{at}{n}\rangle>1$, then $\mu_{at}>0$; hence $\mu_{at}=1$. This proves our Lemma.

Lemma 2.2. *Let $\eta\in\widehat{G}_-$ and let f be the conductor of η. For $1\le a\le n-1$, put*

$$\frac{a}{n}=\frac{b}{m},\qquad (m,b)=1,\qquad 1\le b\le m-1.$$

Then we have

$$\sum_{t=1,(t,n)=1}^{[(n-1)/2]}(2\langle\frac{at}{n}\rangle-1)\eta(t)$$
$$=\begin{cases}-\frac{\varphi(n)}{\varphi(m)}\eta(b)^{-1}\prod_{p\mid\frac{m}{f}}(1-\eta(p))L(0,\eta) & \text{if } f\mid m,\\ 0 & \text{if } f\nmid m.\end{cases}$$

Proof. Let S denote the sum in question. Since η is odd, we have

$$S = \sum_{t=1,(t,n)=1}^{n-1} \left(\left\langle\frac{at}{n}\right\rangle - \frac{1}{2}\right)\eta(t) = \sum_{t=1,(t,n)=1}^{n-1} \left\langle\frac{at}{n}\right\rangle\eta(t) = \sum_{t=1,(t,n)=1}^{n-1} \left\langle\frac{bt}{m}\right\rangle\eta(t).$$

Put $\langle\langle bt\rangle\rangle_m = m\langle\frac{bt}{m}\rangle$, $n = md$. Then we have

$$S = \frac{1}{m}\sum_{t=1,(t,n)=1}^{n-1} \langle\langle bt\rangle\rangle_m \eta_*(t) = \frac{1}{m}\sum_{k=0}^{d-1}\sum_{u=1}^{m-1}\langle\langle bu\rangle\rangle_m \eta_*(mk+u).$$

We have $\eta_*(mk+u) = 0$ if $(u,m) > 1$. Assume $(u,m) = 1$. Take $u' \in (\mathbf{Z}/n\mathbf{Z})^\times$ so that $u' \equiv u \mod m$. Then we have

$$\sum_{k=0}^{d-1}\eta_*(mk+u) = \sum_{v\in(\mathbf{Z}/n\mathbf{Z})^\times, v\equiv u \mod m} \eta_*(v) = \sum_{v\in(\mathbf{Z}/n\mathbf{Z})^\times, v\equiv u' \mod m} \eta_*(u')\eta_*(u'^{-1}v)$$
$$= \eta_*(u') \sum_{v\in(\mathbf{Z}/n\mathbf{Z})^\times, v\equiv 1 \mod m} \eta_*(v).$$

Set $Y_m = \{v \in (\mathbf{Z}/n\mathbf{Z})^\times \mid v \equiv 1 \mod m\}$. Then η_* is trivial on Y_m if and only if f divides m. If $f \mid m$, then $\eta_*(u') = \eta(u)$. Therefore we obtain

$$\sum_{k=0}^{d-1}\eta_*(mk+u) = \begin{cases} \frac{\varphi(n)}{\varphi(m)}\eta(u) & \text{if } f \mid m, \\ 0 & \text{if } f \nmid m. \end{cases}$$

We have shown that $S = 0$ if $f \nmid m$. Assume $f \mid m$. Then we have

$$S = \frac{1}{m}\sum_{u=1,(u,m)=1}^{m-1} \langle\langle bu\rangle\rangle_m \cdot \frac{\varphi(n)}{\varphi(m)}\eta(u) = \frac{\varphi(n)}{\varphi(m)} \cdot \frac{1}{m}\sum_{v=1,(v,m)=1}^{m-1} v\eta(b)^{-1}\eta(v)$$
$$= \frac{\varphi(n)}{\varphi(m)} \cdot \eta(b)^{-1} \cdot \frac{1}{m}\sum_{v=1,(v,m)=1}^{m-1} \eta(v)v.$$

By (2.10) and (2.12), we have

$$\frac{1}{m}\sum_{v=1,(v,m)=1}^{m-1} \eta(v)v = -\prod_{p\mid\frac{m}{f}}(1-\eta(p))L(0,\eta).$$

This completes the proof.

Lemma 2.3. *For $1 \le a \le n-2$, regard ϵ_{at} as a function of t, $1 \le t \le [(n-1)/2]$, $(t,n)=1$. Then the dimension of the vector space V spanned by ϵ_{at} over $\mathbf{C}$ is equal to $\varphi(n)/2$, where $\varphi(n) = |(\mathbf{Z}/n\mathbf{Z})^\times|$. In other words, the rank of $n-2 \times (\varphi(n)/2)$-matrix (ϵ_{at}) is $\varphi(n)/2$.*

Proof. For $1 \le a \le n-1$, we set

$$f_a(t) = 2\langle \frac{at}{n} \rangle - 1, \qquad 1 \le t \le [(n-1)/2], \quad (t,n)=1.$$

We see easily that

$$f_a + f_{n-a} = 0, \qquad 1 \le a \le n-1.$$

By Lemma 2.1, we have

$$\epsilon_{at} = -f_1(t) - f_a(t) + f_{a+1}(t), \qquad 1 \le a \le n-2.$$

Hence we have

$$V \ni 2f_1 - f_2, f_1 + f_2 - f_3, f_1 + f_3 - f_4, \cdots, f_1 + f_{n-2} - f_{n-1}.$$

Adding successive terms, we get

$$V \ni 2f_1 - f_2, 3f_1 - f_3, 4f_1 - f_4, \cdots, (n-1)f_1 - f_{n-1}.$$

Since $f_{n-1} = -f_1$, we get $V \ni f_1$. Therefore we have $V \ni f_1, f_2, f_3, \cdots, f_{n-1}$. It suffices to show that $\dim\langle f_1, f_2, \cdots, f_{n-1}\rangle_{\mathbf{C}} = \varphi(n)/2$.

Now we enlarge the domain of definition of f_a: we set

$$f_a(t) = 2\langle \frac{at}{n} \rangle - 1, \qquad 1 \le t \le n-1, \quad (t,n)=1$$

and we regard f_a as a function on $(\mathbf{Z}/n\mathbf{Z})^\times$. Since $f_a(n-t) = -f_a(t)$, f_a is an odd function. Therefore it suffices to show that the space of odd functions on $(\mathbf{Z}/n\mathbf{Z})^\times$ is spanned by f_a, $1 \le a \le n-1$. For this purpose, set

$$\begin{aligned} W = \{ g \mid\ & g \text{ is an odd function on } (\mathbf{Z}/n\mathbf{Z})^\times \text{ such that} \\ & \sum_{t \in (\mathbf{Z}/n\mathbf{Z})^\times} g(t) f_a(t) = 0 \qquad \text{for every} \quad f_a, 1 \le a \le n-1 \}. \end{aligned}$$

It is sufficient to show that $W = \{0\}$. Since $\sum_{t \in (\mathbf{Z}/n\mathbf{Z})^\times} g(t) = 0$ if g is odd, we have

$$\begin{aligned} W = \{ g \mid\ & g \text{ is an odd function on } (\mathbf{Z}/n\mathbf{Z})^\times \text{ such that} \\ & \sum_{t \in (\mathbf{Z}/n\mathbf{Z})^\times} g(t) \langle \frac{at}{n} \rangle = 0 \qquad \text{for every} \quad a, 1 \le a \le n-1 \}. \end{aligned}$$

If $g \in W$, $c \in (\mathbf{Z}/n\mathbf{Z})^\times$, we see easily that $g(ct) \in W$. Therefore $(\mathbf{Z}/n\mathbf{Z})^\times$ acts on W. Assume $W \neq \{0\}$. Then we have a representation of $(\mathbf{Z}/n\mathbf{Z})^\times$ on W which splits into a direct sum of one dimensional representations. Let ψ be a one dimensional constituent and let $g \in W$ be a non-zero function which transforms according to ψ. Since

$$g(ct) = \psi(c)g(t), \qquad c, t \in (\mathbf{Z}/n\mathbf{Z})^\times,$$

g is a constant multiple of ψ. Hence we may assume that $\psi \in W$. Since ψ must be odd, we have $\psi = \eta_*$ with some $\eta \in \widehat{G}_-$. Let f be the conductor of η and put $a = n/f$. We have $1 \leq a < n$, $a/n = 1/f$. By Lemma 2.2, we have

$$\sum_{t \in (\mathbf{Z}/n\mathbf{Z})^\times} f_a(t)\eta_*(t) = 2 \sum_{t=1,(t,n)=1}^{[(n-1)/2]} (2\langle \frac{at}{n} \rangle - 1)\eta(t) = -\frac{2\varphi(n)}{\varphi(f)} \cdot L(0,\eta) \neq 0.$$

This is a contradiction and completes the proof.

For $\eta \in \widehat{G}$, let M_η be the field generated over $\mathbf{Q}$ by $\eta(m)$, $m \in \mathbf{Z}$. We see easily that $M_\eta = \mathbf{Q}(\eta_*(m) \mid m \in \mathbf{Z})$. Let J_η be the set of all isomorphisms of M_η into $\mathbf{C}$. Then $\{\eta^\sigma \mid \sigma \in J_\eta\}$ is the set of all conjugates of η over $\mathbf{Q}$.

Lemma 2.4. *Let $\eta \in \widehat{G}_-$ and f be the conductor of η. For every $a \in M_\eta$, we have*

$$\prod_{c=1}^{f-1} \Gamma(\frac{c}{f})^{\sum_\sigma a^\sigma \prod_p (1-\eta^\sigma(p))\eta^\sigma(c)} \sim \prod_{c=1,(c,n)=1}^{n-1} \Gamma(\frac{c}{n})^{\sum_\sigma a^\sigma \eta^\sigma(c)},$$

where σ extends over J_η and p runs over all prime divisors of n/f.

Proof. Take any $a \in M_\eta$. By (2.11), we have

$$\prod_\sigma \exp(a^\sigma L'(0,\eta_*^\sigma)) = n^{-\sum_\sigma a^\sigma L(0,\eta_*^\sigma)} \prod_{c=1}^{n-1} \Gamma(\frac{c}{n})^{\sum_\sigma a^\sigma \eta_*^\sigma(c)},$$

where σ extends over J_η. Since

$$\sum_{\sigma \in J_\eta} a^\sigma L(0,\eta_*^\sigma) = \sum_{\sigma \in J_\eta} (aL(0,\eta_*))^\sigma \in \mathbf{Q},$$

we obtain

$$\prod_{\sigma \in J_\eta} \exp(a^\sigma L'(0,\eta_*^\sigma)) \sim \prod_{c=1}^{n-1} \Gamma(\frac{c}{n})^{\sum_{\sigma \in J_\eta} a^\sigma \eta_*^\sigma(c)}. \tag{2.13}$$

By (2.11), we similarly obtain

$$\prod_{\sigma \in J_\eta} \exp(b^\sigma L'(0,\eta^\sigma)) \sim \prod_{c=1}^{f-1} \Gamma(\frac{c}{f})^{\sum_{\sigma \in J_\eta} b^\sigma \eta^\sigma(c)} \qquad \text{for every} \quad b \in M_\eta. \tag{2.14}$$

By (2.12), we have

$$\prod_{\sigma\in J_\eta}\exp(a^\sigma L'(0,\eta_*^\sigma))\sim\prod_{\sigma\in J_\eta}\exp(a^\sigma\prod_{p|\frac{n}{f}}(1-\eta^\sigma(p))L'(0,\eta^\sigma)). \tag{2.15}$$

Now the assertion follows immediately from (2.13) $\sim$ (2.15), taking $b=a\prod_{p|\frac{n}{f}}(1-\eta(p))$ in (2.14).

Theorem 2.5. *We have*

$$p_K(\sigma(1),\sigma(t))\sim\pi^{-\delta_{1t}/2}\prod_{\eta\in\widehat{G}_-}\prod_{c=1}^{n-1}\Gamma(\frac{c}{n})^{\eta(tc)/(L(0,\eta)\varphi(n))} \tag{2.16}$$

for $1\le t\le[(n-1)/2]$, $(t,n)=1$. *Here* δ *denotes Kronecker's delta, i.e.,* $\delta_{11}=1$, $\delta_{1t}=0$ *if* $t\neq 1$.

Proof. Set

$$p'_K(\sigma(1),\sigma(t))=\pi^{-\delta_{1t}/2}\prod_{\eta\in\widehat{G}_-}\prod_{c=1}^{n-1}\Gamma(\frac{c}{n})^{\eta(tc)/(L(0,\eta)\varphi(n))}.$$

We shall first show that $p'_K(\sigma(1),\sigma(t))$ satisfies (2.5), i.e.,

$$\prod_{t=1,(t,n)=1}^{[(n-1)/2]}p'_K(\sigma(1),\sigma(t))^{\epsilon_{at}}\sim\pi^{-1}\frac{\Gamma(\frac{a}{n})\Gamma(\frac{1}{n})}{\Gamma(\frac{a+1}{n})} \tag{2.17}$$

for $1\le a\le n-2$. Take any $a\in\mathbf{Z}$, $1\le a\le n-1$ and put

$$\frac{a}{n}=\frac{b}{m},\qquad (b,m)=1,\qquad 1\le b\le m-1.$$

For $\eta\in\widehat{G}$, let f_η denote the conductor of η. By (2.9), if $\eta\neq 1$, we have

$$L'(0,\eta)=\sum_{c=1}^{n-1}\eta(c)\log\Gamma(\frac{c}{n})-\log n\cdot L(0,\eta)=\sum_{c=1}^{f_\eta-1}\eta(c)\log\Gamma(\frac{c}{f_\eta})-\log f_\eta\cdot L(0,\eta).$$

Hence we have

$$\prod_{c=1}^{n-1}\Gamma(\frac{c}{n})^{\sum_\sigma\eta^\sigma(c)/(L(0,\eta^\sigma)\varphi(n))}\sim\prod_{c=1}^{f_\eta-1}\Gamma(\frac{c}{f_\eta})^{\sum_\sigma\eta^\sigma(c)/(L(0,\eta^\sigma)\varphi(n))}\qquad\text{if}\quad\eta\neq 1,$$

where σ extends over J_η. Therefore we have

$$p'_K(\sigma(1),\sigma(t))\sim\pi^{-\delta_{1t}/2}\prod_{\eta\in\widehat{G}_-,f_\eta\nmid m}\prod_{c=1}^{n-1}\Gamma(\frac{c}{n})^{\eta(tc)/(L(0,\eta)\varphi(n))}$$

$$\times\prod_{\eta\in\widehat{G}_-,f_\eta|m}\prod_{c=1}^{f_\eta-1}\Gamma(\frac{c}{f_\eta})^{\eta(tc)/(L(0,\eta)\varphi(n))}.$$

By Lemma 2.2, we obtain

$$\prod_{t=1,(t,n)=1}^{[(n-1)/2]} p'_K(\sigma(1),\sigma(t))^{2\langle at/n\rangle-1}$$

$$\sim \pi^{(1-2a/n)/2} \prod_{\eta\in\widehat{G}_-, f_\eta|m} \prod_{c=1}^{f_\eta-1} \Gamma(\frac{c}{f_\eta})^{-\eta(b)^{-1}\eta(c)\prod_p(1-\eta(p))/\varphi(m)},$$

where p extends over all prime divisors of m/f_η. By Lemma 2.4, for $\eta\in\widehat{G}_-$, we have

$$\prod_{\sigma\in J_\eta}\prod_{c=1}^{f-1}\Gamma(\frac{c}{f})^{-\eta^\sigma(b)^{-1}\eta^\sigma(c)\prod_p(1-\eta^\sigma(p))/\varphi(m)} \sim \prod_{\sigma\in J_\eta}\prod_{c=1,(c,m)=1}^{m-1}\Gamma(\frac{c}{m})^{-\eta^\sigma(b)^{-1}\eta^\sigma(c)/\varphi(m)},$$

where f is the conductor of η and $f \mid m$ is assumed; p extends over all prime divisors of m/f. Therefore we obtain

$$\prod_{t=1,(t,n)=1}^{[(n-1)/2]} p'_K(\sigma(1),\sigma(t))^{2\langle at/n\rangle-1} \sim \pi^{(1-2a/n)/2}\prod_{c=1,(c,m)=1}^{m-1}\Gamma(\frac{c}{m})^{-\sum_\eta \eta(b)^{-1}\eta(c)/\varphi(m)},$$

where η extends over Dirichlet characters in $\widehat{G}_-$ such that $f_\eta \mid m$. Since

$$\sum_{\eta\in\widehat{G}_-, f_\eta|m}\eta(b)^{-1}\eta(c) = \begin{cases} \varphi(m)/2 & \text{if } c\equiv b \mod m, \\ -\varphi(m)/2 & \text{if } c\equiv -b \mod m, \\ 0 & \text{otherwise,}\end{cases}$$

we have

$$\prod_{t=1,(t,n)=1}^{[(n-1)/2]} p'_K(\sigma(1),\sigma(t))^{2\langle at/n\rangle-1} \sim \pi^{(1-2a/n)/2}\Gamma(\frac{b}{m})^{-1/2}\Gamma(\frac{m-b}{m})^{1/2}$$

$$\sim \pi^{(1-2a/n)/2}\pi^{1/2}\Gamma(\frac{b}{m})^{-1} = \pi^{1-a/n}\Gamma(\frac{a}{n})^{-1}.$$

By Lemma 2.1, we have

$$\prod_{t=1,(t,n)=1}^{[(n-1)/2]} p'_K(\sigma(1),\sigma(t))^{\epsilon_{at}} \sim \pi^{1/n-1}\Gamma(\frac{1}{n})\pi^{a/n-1}\Gamma(\frac{a}{n})\pi^{1-(a+1)/n}\Gamma(\frac{a+1}{n})^{-1}$$

$$= \pi^{-1}\frac{\Gamma(\frac{1}{n})\Gamma(\frac{a}{n})}{\Gamma(\frac{a+1}{n})}$$

Thus we have established (2.17). By (2.5), we get

$$(2.18)\qquad \prod_{t=1,(t,n)=1}^{[(n-1)/2]} p_K(\sigma(1),\sigma(t))^{\epsilon_{at}} \sim \prod_{t=1,(t,n)=1}^{[(n-1)/2]} p'_K(\sigma(1),\sigma(t))^{\epsilon_{at}}, \qquad 1 \le a \le n-2.$$

By Lemma 2.4, for every t_0, $1 \le t_0 \le [(n-1)/2]$, $(t_0, n) = 1$, there exist integers u_a, $1 \le a \le n-2$ and a positive integer m such that $\sum_{a=1}^{n-2} \epsilon_{at} u_a = m\delta_{t_0 t}$. Taking u_a-th power of (2.18) and making a product over a, we obtain $p'_K(\sigma(1),\sigma(t_0))^m \sim p_K(\sigma(1),\sigma(t_0))^m$. This completes the proof.

Let K be an algebraic number field which is abelian over $\mathbf{Q}$. Then K is a CM-field if K is totally imaginary. Set $G = \mathrm{Gal}(K/\mathbf{Q})$ and let $\widehat{G}$ be the set of all irreducible characters of G. Let $\widehat{G}_+$ (resp. $\widehat{G}_-$) be the subset of $\widehat{G}_+$ consisting of characters η such that $\eta(\rho) = 1$ (resp. $\eta(\rho) = -1$). When $K = \mathbf{Q}(\zeta_n)$, this notation is consistent with the previous one if we identify $\eta \in \widehat{G}$ with the corresponding primitive Dirichlet character.

Theorem 2.6. *Let K be a totally imaginary algebraic number field which is abelian over $\mathbf{Q}$. Then we have*

$$(2.19)\qquad p_K(id,\sigma) \sim \pi^{-\mu(\sigma)/2} \prod_{\eta \in \widehat{G}_-} \exp\Big(\frac{\eta(\sigma)}{[K:\mathbf{Q}]}\frac{L'(0,\eta)}{L(0,\eta)}\Big), \qquad \sigma \in G$$

where

$$\mu(\sigma) = \begin{cases} 1 & \text{if } \sigma = 1, \\ -1 & \text{if } \sigma = \rho, \\ 0 & \text{if } \sigma \neq 1, \rho. \end{cases}$$

Proof. If $K = \mathbf{Q}(\zeta_n)$, the assertion follows from (2.11), Theorem 2.5 and Theorem S1, (3). Now let K be a totally imaginary subfield of $\mathbf{Q}(\zeta_n)$. We set

$$L = \mathbf{Q}(\zeta_n), \qquad \widetilde{G} = \mathrm{Gal}(L/\mathbf{Q}), \qquad H = \mathrm{Gal}(L/K).$$

By Theorem S1, (5), we have

$$p_K(\mathrm{id},\sigma) \sim \prod_{\tau \in H} p_L(\mathrm{id},\tau\widetilde{\sigma})$$

where $\widetilde{\sigma} \in \widetilde{G}$ denotes an element such that $\widetilde{\sigma}|K = \sigma$. Since (2.19) holds for p_L, we obtain

$$\begin{aligned} p_K(\mathrm{id},\sigma) &\sim \prod_{\tau \in H} \pi^{-\mu(\tau\widetilde{\sigma})/2} \prod_{\eta \in \widehat{\widetilde{G}}_-} \exp\Big(\frac{\eta(\tau\widetilde{\sigma})}{[L:\mathbf{Q}]}\cdot\frac{L'(0,\eta)}{L(0,\eta)}\Big) \\ &= \pi^{-\sum_{\tau\in H}\mu(\tau\widetilde{\sigma})/2} \prod_{\eta \in \widehat{\widetilde{G}}_-} \exp\Big(\frac{\sum_{\tau\in H}\eta(\tau\widetilde{\sigma})}{[L:\mathbf{Q}]}\cdot\frac{L'(0,\eta)}{L(0,\eta)}\Big). \end{aligned}$$

We see easily that $\sum_{\tau\in H}\mu(\tau\widetilde{\sigma}) = \mu(\sigma)$. It is clear that $\sum_{\tau\in H}\eta(\tau\widetilde{\sigma}) = 0$ if $\eta|H$ is non-trivial; if $\eta|H$ is trivial, then η can be identified with an element of $\widehat{G}_-$ and we have $\sum_{\tau\in H}\eta(\tau\widetilde{\sigma}) = |H|\,\eta(\sigma)$. Hence the assertion follows.

Theorem 2.7. *Let the assumption be the same as in Theorem 2.6. Let ψ be a virtual character of G which is a $\mathbf{Z}$-linear combination of characters in $\widehat{G}_-$. We assume that ψ is $\mathbf{Q}$-valued and $\psi \neq 0$. Then we have*

$$\exp(\frac{L'(0,\psi)}{L(0,\psi)}) \sim \pi^{\dim\psi} \prod_{\sigma\in G} p_K(id,\sigma)^{\psi(\sigma^{-1})}. \tag{2.20}$$

Proof. It suffices to prove the theorem assuming $\psi = \sum_{\tau\in J_\omega} \omega^\tau$ with $\omega \in \widehat{G}_-$. By Theorem 2.6, we obtain

$$\begin{aligned}\prod_{\sigma\in G} p_K(\mathrm{id},\sigma)^{\psi(\sigma^{-1})} &\sim \pi^{-\psi(1)/2+\psi(\rho)/2} \prod_{\sigma\in G}\prod_{\eta\in\widehat{G}_-} \exp(\frac{\eta(\sigma)}{[K:\mathbf{Q}]}\frac{L'(0,\eta)}{L(0,\eta)})^{\psi(\sigma^{-1})} \\ &= \pi^{-\dim\psi} \prod_{\eta\in\widehat{G}_-} \exp(\frac{1}{[K:\mathbf{Q}]}\sum_{\sigma\in G}\eta(\sigma)\psi(\sigma^{-1})\cdot\frac{L'(0,\eta)}{L(0,\eta)}).\end{aligned}$$

By the orthogonality relations for characters, we have

$$\sum_{\sigma\in G}\eta(\sigma)\psi(\sigma^{-1}) = \begin{cases} |G| & \text{if} \quad \eta\cong\omega^\tau \quad \text{for some} \quad \tau\in J_\omega, \\ 0, & \text{otherwise.}\end{cases}$$

Hence we get

$$\prod_{\sigma\in G} p_K(\mathrm{id},\sigma)^{\psi(\sigma^{-1})} \sim \pi^{-\dim\psi}\prod_{\tau\in J_\omega}\exp(\frac{L'(0,\omega^\tau)}{L(0,\omega^\tau)}).$$

Since

$$L(s,\psi) = \prod_{\tau\in J_\omega} L(s,\omega^\tau), \qquad \frac{L'(0,\psi)}{L(0,\psi)} = \sum_{\tau\in J_\omega}\frac{L'(0,\omega^\tau)}{L(0,\omega^\tau)},$$

the assertion follows.

Corollary 2.8. *Let K be an abelian CM-field and F be the maximal real subfield of K. Let χ be the Hecke character of $F_A^\times$ which corresponds to the quadratic extension K/F. Let $L_F(s,\chi)$ denote the Hecke L-function attached to χ. Then we have*

$$\exp(L'_F(0,\chi)) \sim (\pi^{1/2}p_K(id,id))^{[K:\mathbf{Q}]L_F(0,\chi)}.$$

For a proof in more general context, see Proposition 3.5 in the next section.

§3. Conjectures

We expect that essential parts of the results in §2 will generalize to an arbitrary CM-field. In this section, we shall give a precise formulation of conjectures.

Let K be a CM-field. We assume that K is normal over $\mathbf{Q}$ and set $G = \mathrm{Gal}(K/\mathbf{Q})$. Let $\rho \in G$ be the complex conjugation. It is well known that ρ belongs to the center of G. Let ψ be a representation of G. We call ψ *odd* (resp. *even*) if $\psi(\rho) = -\mathrm{id}$ (resp. $\psi(\rho) = \mathrm{id}$). Let $\widehat{G}$ be the set of all equivalence classes of irreducible representations of G. Let $\widehat{G}_-$ (resp. $\widehat{G}_+$) be the subset of $\widehat{G}$ which consists of all equivalence classes of irreducible odd (resp. even) representations. We have $\widehat{G} = \widehat{G}_+ \cup \widehat{G}_-$ (disjoint union). If $\eta \in \widehat{G}_-$, then $L(1,\eta) \neq 0$ and the Gamma factor to go with $L(s,\eta)$ is $\Gamma((s+1)/2)^{\dim \eta}$. Hence $L(0,\eta) \neq 0$ for $\eta \in \widehat{G}_-$.

Conjecture 3.1. *Let ψ be a virtual representation of G which is a $\mathbf{Z}$-linear combination of representations in $\widehat{G}_-$. We assume that the character χ_ψ of ψ is $\mathbf{Q}$-valued and that $\psi \neq 0$. Then we have*

$$\exp\Big(\frac{L'(0,\psi)}{L(0,\psi)}\Big) \sim \pi^{\dim \psi} \prod_{\sigma\in G} p_K(id,\sigma)^{\chi_\psi(\sigma^{-1})}. \tag{3.1}$$

This Conjecture generalizes Theorem 2.7. We note that $\chi_\psi(\sigma) = \chi_\psi(\sigma^{-1})$. Conjecture 3.1 expresses $\exp(L'(0,\psi))$ in terms of p_K. It seems impossible to generalize Theorem 2.6, i.e., the expression of p_K in terms of $\exp(L'(0,\psi))$. This can be explained as follows. It may be conjectured that $p_K(\mathrm{id},\sigma_i)$, $1 \leq i \leq [K:\mathbf{Q}]/2$ are algebraically independent over $\mathbf{Q}$ if $\sum_{i=1}^{[K:\mathbf{Q}]/2} \sigma_i$ is a CM-type of K (cf. [Sh6], p. 319). If G is not abelian, we can easily show that $|\widehat{G}_-| \leq \frac{[K:\mathbf{Q}]}{2} - 3$. Hence the expression of p_K in terms of $\exp(L'(0,\psi))$, $\psi \in \widehat{G}_-$ would be impossible.

We can see that Conjecture 3.1 is compatible with field extensions. In fact, let L be an extension of K. We assume that L is a CM-field normal over $\mathbf{Q}$. Put

$$\widetilde{G} = \mathrm{Gal}(L/\mathbf{Q}), \qquad H = \mathrm{Gal}(L/K).$$

We assume Conjecture 3.1 for L and shall show (3.1) for K. Regarding ψ as a virtual character of $\widetilde{G}$, we have

$$\begin{aligned}\exp\Big(\frac{L'(0,\psi)}{L(0,\psi)}\Big) &\sim \pi^{\dim \psi} \prod_{\widetilde{\sigma}\in\widetilde{G}} p_L(\mathrm{id},\widetilde{\sigma})^{\chi_\psi(\widetilde{\sigma}^{-1})} \\ &\sim \pi^{\dim \psi} \prod_{\mu\in G}\prod_{\tau\in H} p_L(\mathrm{id},\tau\widetilde{\mu})^{\chi_\psi((\tau\widetilde{\mu})^{-1}))} \sim \pi^{\dim\psi} \prod_{\mu\in G}\Big(\prod_{\tau\in H} p_L(\mathrm{id},\tau\widetilde{\mu})\Big)^{\chi_\psi(\mu^{-1})},\end{aligned}$$

where $\widetilde{\mu} \in \widetilde{G}$ is an extension of $\mu \in G$. Now (3.1) for K follows from Theorem S1, (5).

It may be the case that (3.1) remains true without assuming χ_ψ is $\mathbf{Q}$-valued, if we could define p_K more precisely so that $\prod_{\sigma\in G} p_K(\mathrm{id},\sigma)^{\chi_\psi(\sigma^{-1})}$ is well defined. A conjecture of the same strength can be formulated as follows.

Conjecture 3.2. *Let c be a conjugacy class in G. Then*

$$\prod_{\sigma\in c} p_K(id,\sigma) \sim \pi^{-\mu(c)/2} \prod_{\eta\in\widehat{G}_-} \exp\Big(\frac{|c|\chi_\eta(c)}{[K:\mathbf{Q}]}\frac{L'(0,\eta)}{L(0,\eta)}\Big), \tag{3.2}$$

where

$$\mu(c) = \begin{cases} 1 & \text{if } c = \{1\}, \\ -1 & \text{if } c = \{\rho\}, \\ 0 & \text{if } c \neq \{1\}, \{\rho\}. \end{cases}$$

This conjecture generalizes Theorem 2.6 and stronger than Conjecture 3.1 as shown below. We note that $2|\widehat{G}_-|$ is equal to the number of conjugacy classes c of G such that $c \neq c\rho$. For a conjugacy class c of G, the order of $g \in c$ is called the order of c; for $a \in \mathbf{Z}$, the conjugacy class of g^a will be denoted by c^a.

Proposition 3.3. *The following two assertions are equivalent.*
(1) For every conjugacy class c of G, we have

$$(3.3) \qquad \prod_{a=1,(a,n)=1}^{n} \prod_{\sigma \in c^a} p_K(id, \sigma) \sim \prod_{a=1,(a,n)=1}^{n} \pi^{-\mu(c^a)/2} \prod_{\eta \in \widehat{G}_-} \exp\left(\frac{|c^a|\chi_\eta(c^a)}{[K:\mathbf{Q}]} \cdot \frac{L'(0,\eta)}{L(0,\eta)}\right),$$

where n is the order of c.
(2) Conjecture 3.1 holds.

Proof. For $\omega \in \widehat{G}$, let M_ω be the field generated over $\mathbf{Q}$ by the values of χ_ω and put $J_\omega = J_{M_\omega}$. We divide $\widehat{G}$ into a disjoint union of orbits under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$: ω_1 and ω_2 belongs to the same orbit if and only if $\chi_{\omega_1^\tau} = \chi_{\omega_2}$ for some $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Each orbit is contained in $\widehat{G}_+$ or $\widehat{G}_-$. From each orbit we choose a representation ω and put $\psi = \oplus_{\tau \in J_\omega} \omega^\tau$, $J_\psi = J_\omega$. Then the character of ψ is $\mathbf{Q}$-valued. Let R be the set of equivalence classes of representations obtained in this manner. Let R_- (resp. R_+) be the subset of R whch consists of equivalence classes of odd (resp. even) representations. Clearly we see that every $\mathbf{Q}$-valued virtual character of G is a $\mathbf{Z}$-linear combination of characters of representations in R.

For a conjugacy class c, let $n(c)$ denote the order of c. Let $\omega \in \widehat{G}$ and set $\psi = \oplus_{\tau \in J_\omega} \omega^\tau$. Since $\sum_{a=1,(a,n(c))=1}^{n(c)} \chi_\omega(c^a)$ is $\mathbf{Q}$-valued, we have

$$\begin{aligned} |J_\omega| \sum_{a=1,(a,n(c))=1}^{n(c)} \chi_\omega(c^a) &= \sum_{\tau \in J_\omega} \sum_{a=1,(a,n(c))=1}^{n(c)} \chi_{\omega^\tau}(c^a) \\ &= \sum_{a=1,(a,n(c))=1}^{n(c)} \chi_\psi(c^a) = \varphi(n(c))\chi_\psi(c). \end{aligned}$$

Therefore we obtain

$$(3.4) \qquad \sum_{a=1,(a,n(c))=1}^{n(c)} \chi_\omega(c^a) = \frac{\varphi(n(c))}{|J_\psi|}\chi_\psi(c).$$

By this formula and noting $\dfrac{L'(0,\psi)}{L(0,\psi)} = \sum_{\omega\in J_\omega} \dfrac{L'(0,\omega^\tau)}{L(0,\omega^\tau)}$ if $\omega \in \widehat{G}_-$, we see that (3.3) is equivalent to

$$\prod_{a=1,(a,n(c))=1}^{n(c)} \prod_{\sigma\in c^a} p_K(\mathrm{id},\sigma) \sim \pi^{-\mu(c)/2} \prod_{\psi\in R_-} \exp\left(\frac{|c|\varphi(n(c))}{[K:\mathbf{Q}]}\frac{\chi_\psi(c)}{|J_\psi|}\frac{L'(0,\psi)}{L(0,\psi)}\right), \tag{3.5}$$

Now assume (1). We have

$$\begin{aligned}\prod_{\sigma\in G} p_K(\mathrm{id},\sigma)^{\chi_\psi(\sigma^{-1})} &= \prod_c \prod_{a=1,(a,n(c))=1}^{n(c)} \prod_{\sigma\in c^a} p_K(\mathrm{id},\sigma)^{\chi_\psi(\sigma^{-1})} \\ &\sim \pi^{-(\chi_\psi(1)-\chi_\psi(\rho))/2} \prod_c \prod_{\eta\in R_-} \exp\left(\frac{|c|\varphi(n(c))}{[K:\mathbf{Q}]}\frac{\chi_\eta(c)}{|J_\eta|}\chi_\psi(c^{-1})\frac{L'(0,\eta)}{L(0,\eta)}\right) \\ &= \pi^{-\dim\psi} \prod_{\eta\in R_-} \exp\left(\frac{1}{[K:\mathbf{Q}]|J_\psi|}\sum_{\sigma\in G}\chi_\eta(\sigma)\chi_\psi(\sigma^{-1})\frac{L'(0,\eta)}{L(0,\eta)}\right).\end{aligned}$$

A subset of the form $\cup_{a=1,(a,n(c))=1}^{n(c)} c^a$ of G is called an "Abteilung" in the old terminology which goes back to Frobenius ([F]). The product $\prod_c$ extends over Abteilungen choosing one conjugacy class c from each Abteilung. By the orthogonality relations, we have $\sum_{\sigma\in G}\chi_\eta(\sigma)\chi_\psi(\sigma^{-1})$ is equal to $|G||J_\psi|$ if $\psi\cong\eta$ and 0 if ψ is not equivalent to η. Thus we obtain (2).

Next we assume (2). Set

$$P = \prod_{a=1,(a,n(c))=1}^{n(c)} \prod_{\sigma\in c^a} p_K(\mathrm{id},\sigma), \qquad \tilde{c} = \cup_{a=1,(a,n(c))=1}^{n(c)} c^a.$$

By the orthogonality relation, we have

$$\sum_{\omega\in\widehat{G}} \sum_{a=1,(a,n(c))=1}^{n(c)} \chi_\omega(\sigma)\chi_\omega((c^a)^{-1})|c|/|G| = \begin{cases} 1 & \text{if } \sigma\in\tilde{c}, \\ 0 & \text{otherwise.}\end{cases}$$

Using (3.4), we have

$$\sum_{\psi\in R} \chi_\psi(\sigma)\chi_\psi(c^{-1})|c|\varphi(n(c))/|J_\psi||G| = \begin{cases} 1 & \text{if } \sigma\in\tilde{c}, \\ 0 & \text{otherwise.}\end{cases}$$

Hence we obtain

$$P = \prod_{\psi\in R}\prod_{\sigma\in G} p_K(\mathrm{id},\sigma)^{\chi_\psi(\sigma)\chi_\psi(c^{-1})|c|\varphi(n(c))/|J_\psi||G|}.$$

Using Theorem S1, (3) and the assumption, we have

$$P \sim \prod_{\psi \in R_-} (\pi^{-\dim \psi} \exp(\frac{L'(0,\psi)}{L(0,\psi)}))^{\chi_\psi(c^{-1})|c|\varphi(n(c))/|J_\psi||G|}.$$

Considering regular representations of G and $G/\langle \rho \rangle$, we have

$$\sum_{\psi \in R_-} \dim \psi \cdot \chi_\psi(\sigma)/|J_\psi| = \begin{cases} |G|/2 & \text{if} \quad \sigma = 1, \\ -|G|/2 & \text{if} \quad \sigma = \rho, \\ 0 & \text{if} \quad \sigma \neq 1, \rho. \end{cases}$$

Hence we obtain (2). This completes the proof.

Conjecture 3.4. *Let K be a CM-field (not necessarily normal over $\mathbf{Q}$) and F be its maximal real subfield. Let χ be the Hecke character of $F_A^\times$ which corresponds to the quadratic extension K/F and $L_F(s,\chi)$ be the Hecke L-function attached to χ. Then*

$$\exp(\frac{L_F'(0,\chi)}{L_F(0,\chi)}) \sim \pi^{[K:\mathbf{Q}]/2} \prod_{\sigma \in J_K} p_K(\sigma,\sigma). \tag{3.6}$$

We note that if K is normal over $\mathbf{Q}$, (3.6) can be written as

$$\exp(\frac{L_F'(0,\chi)}{L_F(0,\chi)}) \sim (\pi^{1/2} p_K(\mathrm{id},\mathrm{id}))^{[K:\mathbf{Q}]}.$$

by Theorem S1, (6). Thus Conjecture 3.4 generalizes Corollary 2.8.

Proposition 3.5. *Let L be a CM-field normal over $\mathbf{Q}$. The following two assertions are equivalent.*
(1) Conjecture 3.1 holds for all odd representations of Gal($L/\mathbf{Q}$) with $\mathbf{Q}$-valued characters.
(2) Conjecture 3.4 holds for all CM-subfields of L.

Proof. Put $G = \mathrm{Gal}(L/\mathbf{Q})$. Let K be a CM-subfield of L and set $H = \mathrm{Gal}(L/K)$, $\widetilde{H} = \mathrm{Gal}(L/F)$. Let χ be the non-trivial character of $\mathrm{Gal}(K/F) \cong \widetilde{H}/H$ and lift χ to the character $\widetilde{\chi}$ of $\widetilde{H}$. Put $\psi = \mathrm{Ind}_{\widetilde{H}}^G \widetilde{\chi}$. We shall show

$$\prod_{\sigma \in J_K} p_K(\sigma,\sigma) \sim \prod_{\sigma \in G} p_L(\mathrm{id},\sigma)^{\chi_\psi(\sigma)}. \tag{3.7}$$

For $g \in G$, set

$$n_+(g) = |\{x \in G \mid xgx^{-1} \in H\}|, \qquad n_-(g) = |\{x \in G \mid xgx^{-1} \in H\rho\}|.$$

By the formula of induced characters, we have

$$\chi_\psi(g) = (n_+(g) - n_-(g))/|\widetilde{H}|.$$

We choose a subset S of G so that $G = S \cup S\rho$ is a disjoint union. By Theorem S1, (3), we have

$$\prod_{\sigma\in G} p_L(\mathrm{id},\sigma)^{\chi_\psi(\sigma)} \sim \prod_{\sigma\in S} p_L(\mathrm{id},\sigma)^{2(n_+(\sigma)-n_-(\sigma))/|\widetilde{H}|}.$$

By Theorem S1, (5), (6), we get

$$\prod_{\sigma\in J_K} p_K(\sigma,\sigma) \sim \prod_{g\in G} p_K(g|K, g|K)^{1/|H|}$$
$$\sim \prod_{g\in G}\prod_{h\in H} p_L(g,hg)^{1/|H|} \sim \prod_{g\in G}\prod_{h\in H} p_L(\mathrm{id}, g^{-1}hg)^{1/|H|}.$$

For $\sigma \in S$, we have

$$|\{g\in G, h\in H \mid g^{-1}hg=\sigma\}| = n_+(\sigma), \qquad |\{g\in G, h\in H \mid g^{-1}hg=\sigma\rho\}| = n_-(\sigma).$$

Hence, by Theorem S1, (3), we obtain

$$\prod_{\sigma\in J_K} p_K(\sigma,\sigma) \sim \prod_{\sigma\in S} p_L(\mathrm{id},\sigma)^{(n_+(\sigma)-n_-(\sigma))/|H|}.$$

We have proved (3.7).

Now assume (1). Identify the Hecke character χ with the non-trivial character of $\mathrm{Gal}(K/F)$ as above. Since $L_F(s,\chi) = L(s,\psi)$, we have

$$\exp\left(\frac{L'_F(0,\chi)}{L_F(0,\chi)}\right) \sim \pi^{\dim\psi} \prod_{\sigma\in G} p_L(\mathrm{id},\sigma)^{\chi_\psi(\sigma)}.$$

By (3.7), we obtain (2).

Next assume (2). Let ψ be a virtual representation of G as in Conjecture 3.1. By a theorem of Artin (cf. [Ar], [T], p. 45), there exist a positive integer n, subgroups H_i of G and integers m_i $(1 \le i \le r)$ such that

$$n\chi_\psi = \sum_{i=1}^r m_i \operatorname{Ind}_{H_i}^G 1_{H_i},$$

where I_{H_i} denotes the trivial character of H_i. If $\rho \in H_i$, then $\operatorname{Ind}_{H_i}^G 1_{H_i}$ is an even representation. Hence we may assume that $\rho \notin H_i$ for every i. Put $\widetilde{H}_i = H_i \cup H_i\rho$. We have

$$n\chi_\psi = \sum_{i=1}^r m_i \operatorname{Ind}_{\widetilde{H}_i}^G (1_{\widetilde{H}_i} \oplus \chi_{\widetilde{H}_i}),$$

where $\chi_{\widetilde{H}_i}$ denotes the non-trivial character of $\widetilde{H}_i$ which is trivial on H_i. Since $\operatorname{Ind}_{\widetilde{H}_i}^G 1_{\widetilde{H}_i}$ is even, we have

$$n\chi_\psi = \sum_{i=1}^r m_i \operatorname{Ind}_{\widetilde{H}_i}^G \chi_{\widetilde{H}_i}.$$

By (3.7) and the assumption, (3.1) holds for all representations $\operatorname{Ind}_{\widetilde{H}_i}^G \chi_{\widetilde{H}_i}$. Therefore (3.1) holds for χ_ψ. This completes the proof.

For a totally real algebraic number field F, a CM-field K such that $[K:F]=2$ is called a CM-*extension* of F.

Proposition 3.6. *Let F be a totally real algebraic number field. Let K_1 and K_2 be CM-extensions of F and K_0 be the composite field of K_1 and K_2. If Conjecture 3.4 holds for K_1 and K_2, then it holds also for K_0.*

Proof. We may assume $K_1 \neq K_2$. Let F_0 be the maximal real subfield of K_0. Let χ_0, χ_1 and χ_2 be the Hecke characters which correspond to the quadratic extensions K_0/F_0, K_1/F and K_2/F respectively. Let α and β be the generators of $\mathrm{Gal}(K_0/K_1)$ and $\mathrm{Gal}(K_0/K_2)$ respectively. Considering an induced representation, we find easily that

$$L_{F_0}(s,\chi_0) = L_F(s,\chi_1)L_F(s,\chi_2).$$

Hence, by the assumption and Theorem S1, (5), we obtain

$$\begin{aligned}\exp\Big(\frac{L'_{F_0}(0,\chi_0)}{L_{F_0}(0,\chi_0)}\Big) &= \exp\Big(\frac{L'_F(0,\chi_1)}{L_F(0,\chi_1)}\Big)\exp\Big(\frac{L'_F(0,\chi_2)}{L_F(0,\chi_2)}\Big)\\ &\sim \pi^{[K_1:\mathbf{Q}]/2}\prod_{\sigma\in J_{K_1}} p_{K_1}(\sigma,\sigma)\ \pi^{[K_2:\mathbf{Q}]/2}\prod_{\sigma\in J_{K_2}} p_{K_2}(\sigma,\sigma)\\ &\sim \pi^{[K_0:\mathbf{Q}]/2}\Big(\prod_{\sigma\in J_{K_0}} p_{K_0}(\sigma,\sigma)p_{K_0}(\sigma,\alpha\sigma)p_{K_0}(\sigma,\sigma)p_{K_0}(\sigma,\beta\sigma)\Big)^{1/2}.\end{aligned}$$

We have $\beta = \alpha\rho$ and $\beta\sigma = \alpha\rho\sigma = \alpha\sigma\rho$ for $\sigma \in J_{K_0}$. Now the assertion follows from Theorem S1, (3).

§4. Preparations on CM-fields

In the next section, we shall render Conjecture 3.1 to numerical tests. For this purpose, we collect several general facts on CM-fields in this section.

For an algebraic number field L, let D_L, h_L, E_L, W_L and R_L denote the discriminant, the class number, the group of units, the group of roots of unity in L and the regulator of L respectively. We put $w_L = |W_L|$. Let $\zeta_L(s)$ denote the Dedekind zeta function of L. The analytic class number formula gives

$$\lim_{s\to 1}(s-1)\zeta_L(s) = \frac{2^{r_1+r_2}\pi^{r_2}h_L R_L}{w_L|D_L|^{1/2}}. \tag{4.1}$$

Here r_1 (resp. r_2) denotes the number of real (resp. complex) archimedean places of L.

Let K be a CM-field and F be the maximal real subfield of K. Put $n = [F : \mathbf{Q}]$. Let χ denote the Hecke character which corresponds to the quadratic extension K/F. By (4.1) and by the functional equations for $\zeta_F(s)$ and $\zeta_K(s)$, we obtain

$$L_F(0,\chi) = \frac{2R_K}{w_K R_F}\cdot\frac{h_K}{h_F}. \tag{4.2}$$

By the definition of the regulator, we have

$$\frac{R_K}{R_F} = \frac{2^{n-1}}{[E_K : W_K E_F]}. \tag{4.3}$$

Hence we get

$$L_F(0,\chi) = \frac{2^n}{w_K[E_K : W_K E_F]} \cdot \frac{h_K}{h_F} = \frac{2^{n-1}}{[E_K : E_F]} \cdot \frac{h_K}{h_F}. \tag{4.4}$$

Let $\mathfrak{d}_{K/F}$ denote the relative different of K over F. The next Lemma is well known. We include its proof for the sake of completeness.

Lemma 4.1. *$[E_K : W_K E_F] = 1$ or 2. If $\mathfrak{d}_{K/F}$ does not divide (2), then $E_K = W_K E_F$. If $[E_K : W_K E_F] = 2$, then $K = F(\sqrt{-\epsilon_0})$ with a totally positive unit $\epsilon_0 \in E_F$.*

Proof. For $\epsilon \in E_K$, we have $|\epsilon^{\rho\sigma}/\epsilon^{\sigma}| = |\epsilon^{\sigma\rho}/\epsilon^{\sigma}| = 1$ for every $\sigma \in J_K$. Hence we have $\epsilon^{\rho}/\epsilon \in W_K$ by Kronecker's theorem. Define a mapping ψ from $E_K/W_K E_F$ into W_K/W_K^2 by

$$\psi(\epsilon \mod W_K E_F) = \epsilon^{\rho}/\epsilon \mod W_K^2, \qquad \epsilon \in E_K.$$

Then it can immediately be verified that ψ is well defined and is a homomorphism. If $\epsilon^{\rho}/\epsilon = \zeta^2$, $\zeta \in W_K$, then $(\zeta\epsilon)^{\rho} = \zeta\epsilon$, i.e. $\epsilon \in W_K E_F$. This shows that ψ is injective. Therefore $[E_K : W_K E_F] = 1$ or 2. Assume $[E_K : W_K E_F] = 2$. Then ψ is surjective. Hence there exists an $\epsilon \in E_K$ such that $\epsilon^{\rho}/\epsilon = -1$. Put $\epsilon_0 = -\epsilon^2$. Then $\epsilon_0 = N_{K/F}(\epsilon)$ is a totally positive unit of E_F. We have $K = F(\epsilon) = F(\sqrt{-\epsilon_0})$. Since the different of ϵ over F is 2ϵ, $\mathfrak{d}_{K/F}$ must divide (2). This completes the proof.

Let $I(K)$ denote the ideal group and let Φ be a CM-type of K. Let ψ be a Grössencharacter of conductor $\mathfrak{f}$ of $I(K)$ such that

$$\psi((\alpha)) = \prod_{\sigma\in\Phi} (\alpha^{\sigma\rho}/|\alpha^{\sigma}|)^{t_\sigma} \qquad \text{if} \quad \alpha \equiv 1 \mod {}^{\times}\mathfrak{f},$$

where t_σ, $\sigma \in \Phi$ are non-negative integers. Let $L_K(s,\psi)$ denote the L-function attached to ψ. We quote a fundamental theorem of Shimura for the use in the next section ([Sh2], Theorem 2 combined with [Sh6], Theorem 1.1; or see [Sh7], Theorem 32.12; cf. also [Sh4], §5).

Theorem S2. *For every integer m such that $m - t_\sigma \in 2\mathbf{Z}$ and $-t_\sigma < m \leq t_\sigma$ for every $\sigma \in \Phi$, we have*

$$L_K(m/2,\psi) \sim \pi^{e/2} p_K(\sum_{\sigma\in\Phi} t_\sigma \cdot \sigma, \Phi),$$

where $e = m[F : \mathbf{Q}] + \sum_{\sigma\in\Phi} t_\sigma$.

To compute $L_K(m/2,\psi)$, we apply Shimura's method [Sh2], which we are going to explain briefly.

Let $k \in \mathbf{Z}$, $k > 0$, $r = (r_1, r_2, \cdots, r_n) \in \mathbf{Z}^n$, $r_i \geq 0$ for $1 \leq i \leq n$. Set $\{r\} = \sum_{i=1}^n r_i$, $\mathbf{1} = (1,1,\cdots,1) \in \mathbf{Z}^n$. For $x = (x_1, x_2, \cdots, x_n) \in \mathbf{C}^n$, $a = (a_1, a_2, \cdots, a_n) \in \mathbf{Z}^n$, put $x^a = \prod_{i=1}^n x_i^{a_i}$. ($x^a$ is defined similarly also for $x \in \mathbf{R}_+^n$, $a \in \mathbf{C}^n$.) Let $\mathfrak{H}$ denote the complex upper half plane. For $z \in \mathfrak{H}^n$ and $s \in \mathbf{C}$, define an Eisenstein series $E_{k,r}(z,s)$ by

$$E_{k,r}(z,s) = (z-\bar{z})^{-r}(2\pi i)^{-kn-\{r\}} \sum_{(c,d)/\sim} \left(\frac{c\bar{z}+d}{cz+d}\right)^{r} (cz+d)^{-k\cdot\mathbf{1}}|cz+d|^{-s\cdot\mathbf{1}}, \tag{4.5}$$

where $\sum_{(c,d)/\sim}$ means that (c,d) runs over $\mathfrak{O}_F \oplus \mathfrak{O}_F \backslash \{(0,0)\}$ under the equivalence relation

$$(c,d) \sim (c_1,d_1) \Longleftrightarrow \exists \epsilon \in E_F, c_1 = \epsilon c, d_1 = \epsilon d.$$

Then $E_{k,r}(z,s)$ converges absolutely when $\Re(s) + k > 2$ and can be continued meromorphically to the whole s-plane. Furthermore $E_{k,r}(z,s)$ is holomorphic at $s=0$. We put

$$E_{k,r}(z) = E_{k,r}(z,0). \tag{4.6}$$

Then we have the transformation formula

$$E_{k,r}(\gamma z) = E_{k,r}(z)(cz+d)^{2r}(cz+d)^{k\cdot\mathbf{1}} \qquad \text{for every} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathfrak{O}_F). \tag{4.7}$$

Let K be a CM-extension of F. Let $\mathfrak{A}$ be a fractional ideal of K; we assume that $\mathfrak{A} = \mathfrak{O}_F\omega \oplus \mathfrak{O}_F$ with $\omega \in K$. Let Φ be a CM-type of K such that $\Im(\omega^\sigma) > 0$ for every $\sigma \in \Phi$. Regard ω as a point of $\mathfrak{H}^n$ by $\omega \longrightarrow (\omega^\sigma)_{\sigma\in\Phi}$. If $\Re(s)$ is sufficiently large, then by straightforward calculation, we get

$$\begin{aligned} E_{k,r}(\omega,s) =& [E_K : E_F](2\pi i)^{-kn-\{r\}} \prod_{\sigma\in\Phi} (2\Im(\omega^\sigma)i)^{-r(\sigma)} \\ &\times \sum_{(\alpha)\subseteq\mathfrak{A}} \prod_{\sigma\in\Phi} (\frac{\alpha^{\sigma\rho}}{|\alpha^\sigma|})^{2r(\sigma)+k} N(\alpha)^{-k/2-s/2}, \end{aligned} \tag{4.8}$$

where (α) extends over all principal ideals contained in $\mathfrak{A}$ and $r(\sigma)$ denotes r_j if ω^σ corresponds to the j-th component in $\mathfrak{H}^n$. By analytic continuation, we can use $E_{k,r}(\omega)$ to evaluate the right hand side at $s=0$.

For $t \in \mathbf{C}$, define a differential operator $D_{j,t}$ by $D_{j,t} = \frac{1}{2\pi i}(\frac{t}{2iy_j} + \frac{\partial}{\partial z_j})$. Then we have

$$D_{j,k+2r_j} E_{k,r}(z) = (k+r_j) E_{k,r^*}(z), \tag{4.9}$$

where $r^* = r + (0,0,\cdots,0,1,0,\cdots,0) \in \mathbf{Z}^n$ (added the vector whose j-th component is 1, other component 0).

If the class number of F is 1 and $r = (0,0,\cdots,0)$, we have (cf. Siegel [Si1], (16) $\sim$ (18)), excluding the case $F = \mathbf{Q}$ and $k=2$,

$$E_{k,0}(z) = (2\pi i)^{-kn}\{\zeta_F(k) + (\frac{(2\pi i)^k}{(k-1)!})^n |D_F|^{1/2-k} \sum_{\mathfrak{d}^{-1}|\nu\gg 0} \sigma_{k-1}(\nu) e^{2\pi i S(\nu z)}\}, \tag{4.10}$$

where $\mathfrak{d}$ denotes the different of F over $\mathbf{Q}$, $S(\nu z) = \sum_{j=1}^n \nu^{(j)} z_j$, $\nu^{(j)}$ being the j-th conjugate of $\nu \in F$,

$$\sigma_{k-1}(\nu) = \sum_{\mathfrak{d}^{-1}|(\alpha)|\nu} \operatorname{sgn}(N(\alpha^k)) N((\alpha)\mathfrak{d})^{k-1}. \tag{4.11}$$

We can use (4.8) $\sim$ (4.11) to compute numerical values of $L(m/2, \psi)$ if the conductor of ψ is (1) and if the class number of F in the narrow sense is 1.[3]

§5. Numerical examples

In this section, we shall examine Conjecture 3.1 numerically for some simple non-abelian CM-fields. The following example is discussed in [ST], p. 74.

Let $F = \mathbf{Q}(\sqrt{d})$, $0 < d \in \mathbf{Q}$ be a real quadratic field. Let $x + y\sqrt{d} \in F$, $x, y \in \mathbf{Q}$ be a totally positive element and set $\xi = \sqrt{x + y\sqrt{d}}\ i$, $\xi' = \sqrt{x - y\sqrt{d}}\ i$, $K = \mathbf{Q}(\xi)$. Then $[K : F] = 2$ and K is a CM-field. We assume that K is not normal over $\mathbf{Q}$. This assumption implies $E_K = E_F$ (cf. [Sh3], Proposition A.7, (iii)). In fact, if $E_K \neq E_F$, we have $K = F(\sqrt{-\epsilon_0})$ with a totally positive unit ϵ_0 of F by Lemma 4.1. Since $\epsilon_0^{\mu} = \epsilon_0^{-1}$ for the generator μ of $\mathrm{Gal}(F/\mathbf{Q})$, we have $K = F(\sqrt{-\epsilon_0^{\mu}})$. This shows that K is normal over $\mathbf{Q}$. Let L be the normal closure of K over $\mathbf{Q}$. Then L is a CM-field and we have $L = \mathbf{Q}(\xi, \xi')$, $[L : \mathbf{Q}] = 8$. Define σ, $\tau \in \mathrm{Gal}(L/\mathbf{Q})$ by

$$\sigma : (\xi, \xi') \longrightarrow (\xi', -\xi), \qquad \tau : (\xi, \xi') \longrightarrow (\xi', \xi).$$

Then $\mathrm{Gal}(L/\mathbf{Q})$ is generated by σ and τ which are subject to the relations

$$\sigma^4 = \tau^2 = 1, \qquad \tau\sigma = \sigma^3\tau, \qquad \sigma^2 = \rho.$$

Thus $\mathrm{Gal}(L/\mathbf{Q})$ is the dihedral group of order 8. Define a CM-type Φ of K by $\Phi = \{\mathrm{id}, \sigma|K\}$. Then the reflex of (K, Φ) is (K', Φ'), where $K' = \mathbf{Q}(\xi + \xi')$, $\Phi' = \{\mathrm{id}, \sigma\tau|K'\}$. Put $d' = x^2 - y^2 d$. Since $(\xi + \xi')^2 = -2(x + \sqrt{d'})$, we have $K' = \mathbf{Q}(\sqrt{2(x + \sqrt{d'}}\ i)$ and $F' := \mathbf{Q}(\sqrt{d'})$ is the maximal real subfield of K'. Let F_0 be the maximal real subfield of L. We have $F_0 = \mathbf{Q}(\sqrt{d}, \sqrt{d'})$. We note that

$$(5.1) \qquad \sqrt{d}^{\,\sigma} = -\sqrt{d}, \qquad \sqrt{d'}^{\,\sigma} = -\sqrt{d'}, \qquad \sqrt{d}^{\,\tau} = -\sqrt{d}, \qquad \sqrt{d'}^{\,\tau} = \sqrt{d'}.$$

Put $G = \mathrm{Gal}(L/\mathbf{Q})$. We have

$$G/\langle\rho\rangle \cong \mathrm{Gal}(F_0/\mathbf{Q}) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}.$$

Hence $\widehat{G}_+$ consists of four one dimensional representations. We see easily that $\widehat{G}_- = \{\eta\}$, where η is the unique irreducible two dimensional representation of G. Let χ be the non-trivial character of $\mathrm{Gal}(K/F)$ and regard χ as a character of $\mathrm{Gal}(L/F)$. Then we find immediately that

$$\eta \cong \mathrm{Ind}(\chi; \mathrm{Gal}(L/F) \longrightarrow \mathrm{Gal}(L/\mathbf{Q})).$$

[3]Here we restricted ourselves to a simple case mostly sufficient for the use in §5. To deal with the general case, we must employ Eisenstein series with congruence conditions. For details, see [Sh2].

Hence we have

$$L(s,\eta) = L_F(s,\chi), \tag{5.2}$$

where $L_F(s,\chi)$ denotes the Hecke L-function attached to χ regarded as the Hecke character of $F_A^\times$ corresponding to the quadratic extension K/F. The character χ_η of η is given by $\chi_\eta(g) = 2$, -2 or 0 according as $g = 1$, $g = \rho$ or $g \neq 1, \rho$. Hence Conjecture 3.1 for $L(s,\eta)$ is equivalent to

$$\exp\left(\frac{L'(0,\eta)}{L(0,\eta)}\right) \sim \pi^2 p_L(\mathrm{id},\mathrm{id})^4 \tag{5.3}$$

in view of Theorem S1, (3). By (5.2), (5.3) is equivalent to

$$\exp\left(\frac{L'_F(0,\chi)}{L_F(0,\chi)}\right) \sim \pi^2 p_L(\mathrm{id},\mathrm{id})^4. \tag{5.4}$$

On the other hand, in view of Theorem S1, Conjecture 3.4 is equivalent to

$$\exp\left(\frac{L'_F(0,\chi)}{L_F(0,\chi)}\right) \sim \pi^2 p_K(\mathrm{id},\mathrm{id})^2 p_K(\sigma,\sigma)^2. \tag{5.5}$$

Here we abbreviated $\sigma|K$ to σ. Similar notation will be used hereafter since no confusion is likely. Using Theorem S1, we see that (5.4) is equivalent to (5.5), i.e., Conjecture 3.1 for $L(s,\eta)$ and Conjecture 3.4 for $L_F(s,\chi)$ are equivalent. We note the following relations due to Shimura [Sh3], Proposition A.7.

$$h_K/h_F = h_{K'}/h_{F'}, \qquad N_{F/\mathbf{Q}}(D(K/F))D(F/\mathbf{Q}) = N_{F'/\mathbf{Q}}(D(K'/F'))D(F'/\mathbf{Q}). \tag{5.6}$$

For given F and χ, $\exp(L'_F(0,\chi)/L_F(0,\chi))$ can be calculated by Shintani's formulas ([Shi1], [Shi2]).[4] To compute CM-periods, we apply Theorem S2. For non-negative integers a and b, let $\lambda^{(1)}_{a,b}$ and $\lambda^{(2)}_{a,b}$ be Grössencharacters of conductor $\mathfrak{f}$ of $I(K)$ such that

$$\lambda^{(1)}_{a,b}((\alpha)) = \left(\frac{\alpha^\rho}{|\alpha|}\right)^a \left(\frac{\alpha^{\sigma\rho}}{|\alpha^\sigma|}\right)^b, \qquad \alpha \equiv 1 \mod {}^\times\mathfrak{f},$$

$$\lambda^{(2)}_{a,b}((\alpha)) = \left(\frac{\alpha^\rho}{|\alpha|}\right)^a \left(\frac{\alpha^{\sigma}}{|\alpha^{\sigma\rho}|}\right)^b, \qquad \alpha \equiv 1 \mod {}^\times\mathfrak{f}.$$

Similarly let $\mu^{(1)}_{a,b}$ and $\mu^{(2)}_{a,b}$ be Grössencharacters of conductor $\mathfrak{f}'$ of $I(K')$ such that

$$\mu^{(1)}_{a,b}((\alpha)) = \left(\frac{\alpha^\rho}{|\alpha|}\right)^a \left(\frac{\alpha^{\sigma\tau\rho}}{|\alpha^{\sigma\tau}|}\right)^b, \qquad \alpha \equiv 1 \mod {}^\times\mathfrak{f}',$$

$$\mu^{(2)}_{a,b}((\alpha)) = \left(\frac{\alpha^\rho}{|\alpha|}\right)^a \left(\frac{\alpha^{\sigma\tau}}{|\alpha^{\sigma\tau\rho}|}\right)^b, \qquad \alpha \equiv 1 \mod {}^\times\mathfrak{f}'.$$

---

[4]Shintani gave an arithmetic formula for $L_F(0,\chi)$ in [Shi1]. In [Shi2], he gave a closed formula for $L'_F(0,\chi)$ in terms of a double gamma function. Shintani's formulas for partial zeta functions (cf. (16), (17) of [Shi4]) have striking similarity to (2.8) and (2.9). For the evaluation of double gamma functions, we can efficiently use the asymptotic expansion given in [Shi2], p. 179.

We note that $L_K(1,\lambda_{a,b}^{(i)}) \neq 0$, $L_{K'}(1,\mu_{a,b}^{(i)}) \neq 0$ for every i and a, b. By Theorem S2, we have

$$\begin{aligned}
L_K(1,\lambda_{2,2}^{(1)}) &\sim \pi^4 p_K(2\cdot\mathrm{id}+2\cdot\sigma,\ \mathrm{id}+\sigma),\\
L_K(1,\lambda_{2,2}^{(2)}) &\sim \pi^4 p_K(2\cdot\mathrm{id}+2\cdot\sigma\rho,\ \mathrm{id}+\sigma\rho),\\
L_K(1,\lambda_{4,2}^{(1)}) &\sim \pi^5 p_K(4\cdot\mathrm{id}+2\cdot\sigma,\ \mathrm{id}+\sigma),\\
L_K(1,\lambda_{2,4}^{(1)}) &\sim \pi^5 p_K(2\cdot\mathrm{id}+4\cdot\sigma,\ \mathrm{id}+\sigma).
\end{aligned}$$

By Theorem S1, we get

$$L_K(1,\lambda_{2,2}^{(1)})L_K(1,\lambda_{2,2}^{(2)}) \sim \pi^8 p_K(\mathrm{id},\mathrm{id})^4 p_K(\sigma,\sigma)^4.$$

By (5.5), Conjecture 3.1 for $L(s,\eta)$ is equivalent to

$$\text{(5.7)}\qquad \pi^{-4}L_K(1,\lambda_{2,2}^{(1)})L_K(1,\lambda_{2,2}^{(2)})\exp\Big(-\frac{2L_F'(0,\chi)}{L_F(0,\chi)}\Big)\qquad \text{is algebraic.}$$

We see easily that (5.7) is also equivalent to

$$\text{(5.7}'\text{)}\qquad \pi^{-4}L_{K'}(1,\mu_{2,2}^{(1)})L_{K'}(1,\mu_{2,2}^{(2)})\exp\Big(-\frac{2L_F'(0,\chi)}{L_F(0,\chi)}\Big)\qquad \text{is algebraic.}$$

We shall derive another relation. By Theorems S1 and S2, we get

$$\text{(5.8)}\qquad \frac{L_K(1,\lambda_{4,2}^{(1)})}{L_K(1,\lambda_{2,2}^{(1)})} \sim \pi\, p_K(\mathrm{id},\ \mathrm{id}+\sigma)^2,\qquad \frac{L_{K'}(1,\mu_{2,4}^{(2)})}{L_{K'}(1,\mu_{2,2}^{(2)})} \sim \pi\, p_{K'}(\sigma\tau\rho,\ \mathrm{id}+\sigma\tau\rho)^2.$$

By Theorem S1, we obtain

$$\begin{aligned}
p_K(\mathrm{id},\mathrm{id}+\sigma) &\sim p_L(\mathrm{id},\mathrm{id})p_L(\mathrm{id},\sigma\tau)p_L(\mathrm{id},\sigma)p_L(\mathrm{id},\tau),\\
p_{K'}(\sigma\tau\rho,\mathrm{id}+\sigma\tau\rho) &\sim p_L(\sigma\tau\rho,\mathrm{id})p_L(\sigma\tau\rho,\tau)p_L(\sigma\tau\rho,\sigma\tau\rho)p_L(\sigma\tau\rho,\tau\sigma\tau\rho)\\
&\sim p_L(\mathrm{id},\sigma\tau)^{-1}p_L(\mathrm{id},\sigma)^{-1}p_L(\mathrm{id},\mathrm{id})p_L(\mathrm{id},\tau)^{-1},\\
p_K(\mathrm{id},\mathrm{id}+\sigma)p_{K'}(\sigma\tau\rho,\mathrm{id}+\sigma\tau\rho) &\sim p_L(\mathrm{id},\mathrm{id})^2.
\end{aligned}$$

Hence we have

$$\text{(5.9)}\qquad \frac{L_K(1,\lambda_{4,2}^{(1)})}{L_K(1,\lambda_{2,2}^{(1)})}\cdot\frac{L_{K'}(1,\mu_{2,4}^{(2)})}{L_{K'}(1,\mu_{2,2}^{(2)})} \sim \pi^2 p_L(\mathrm{id},\mathrm{id})^4.$$

Therefore Conjecture 3.1 for $L(s,\eta)$ is also equivalent to

$$\text{(5.10)}\qquad \frac{L_K(1,\lambda_{4,2}^{(1)})}{L_K(1,\lambda_{2,2}^{(1)})}\cdot\frac{L_{K'}(1,\mu_{2,4}^{(2)})}{L_{K'}(1,\mu_{2,2}^{(2)})}\cdot\exp\Big(-\frac{L_F'(0,\chi)}{L_F(0,\chi)}\Big)\qquad \text{is algebraic.}$$

For a real quadratic field k, we denote the archimedean places of k by ∞_1 and ∞_2. We choose ∞_1 as the place corresponding to the identity embedding of k into $\mathbf{R}$.

Example 1. We take $F = \mathbf{Q}(\sqrt{5})$, $K = \mathbf{Q}(\sqrt{\frac{13+\sqrt{5}}{2}}\, i)$. Then we have $F' = \mathbf{Q}(\sqrt{41})$, $K' = \mathbf{Q}(\sqrt{13+2\sqrt{41}}\, i)$. We have $h_F = h_{F'} = 1$. We have $(41) = (\frac{13+\sqrt{5}}{2})(\frac{13-\sqrt{5}}{2})$ in F. Put $\mathfrak{p} = (\frac{13+\sqrt{5}}{2})$, $z = \frac{1}{2}(\sqrt{\frac{13+\sqrt{5}}{2}}\, i + \frac{1-\sqrt{5}}{2})$. Then we see that z is integral over $\mathfrak{O}_F$. Hence we see that $\mathfrak{p}$ ramifies in K and all the other prime ideals of F are unramified in K. Furthermore we have $\mathfrak{O}_K = \mathfrak{O}_F z \oplus \mathfrak{O}_F$. We can also see easily that K is the maximal ray class field of conductor $\mathfrak{p}\infty_1\infty_2$ of F. In F', we have $(5) = (13+2\sqrt{41})(13-2\sqrt{41})$. Put $\mathfrak{p}' = (13+2\sqrt{41})$, $z' = \frac{1}{2}(\sqrt{13+2\sqrt{41}}\, i + 1)$. Similarly to the above, we see that $\mathfrak{p}'$ is the only prime ideal which ramifies in K', $\mathfrak{O}_{K'} = \mathfrak{O}_{F'} z' \oplus \mathfrak{O}_{F'}$ and that K' is the maximal ray class field of conductor $\mathfrak{p}'\infty_1\infty_2$ of F'.

We can compute $L_F(0,\chi)$ and $L'_F(0,\chi)$ by the method of Shintani. We obtain

$$L_F(0,\chi) = 2, \tag{5.11}$$

$$L'_F(0,\chi) = -0.26558039348000766099171 65\cdots . \tag{5.12}$$

Since $E_K = E_F$, $E_{K'} = E_{F'}$, we have $h_K = h_{K'} = 1$ by (4.4), (5.6) and (5.11).

For every non-negative even integers a and b, there exist (unique) Grössencharacters $\lambda^{(1)}_{a,b}$, $\lambda^{(2)}_{a,b}$ (resp. $\mu^{(1)}_{a,b}$, $\mu^{(2)}_{a,b}$) of $I(K)$ (resp. $I(K')$) of conductor (1). By (4.10), we have[5]

$$E^F_{2,0}(z) = \frac{\sqrt{5}}{2^3\cdot 3\cdot 5^3}\{1 + 120 \sum_{(\sqrt{5})^{-1}|\nu\gg 0} \sigma_1(\nu) e^{2\pi i S(\nu z)}\}, \qquad F = \mathbf{Q}(\sqrt{5}), \tag{5.13}$$

$$E^{F'}_{2,0}(z) = \frac{\sqrt{41}}{3\cdot 41^2}\{1 + 3 \sum_{(\sqrt{41})^{-1}|\nu\gg 0} \sigma_1(\nu) e^{2\pi i S(\nu z)}\}, \qquad F' = \mathbf{Q}(\sqrt{41}). \tag{5.14}$$

By (4.8), we have

$$\begin{aligned} &L_K(1,\lambda^{(1)}_{2,2}) = (2\pi i)^4 E^F_{2,0}(\omega), \qquad L_K(1,\lambda^{(1)}_{4,2}) = (2\pi i)^5 \sqrt{\frac{13+\sqrt{5}}{2}}\, i\, E^F_{2,\{1,0\}}(\omega), \\ &L_K(1,\lambda^{(1)}_{2,4}) = (2\pi i)^5 \sqrt{\frac{13-\sqrt{5}}{2}}\, i\, E^F_{2,\{0,1\}}(\omega), \end{aligned} \tag{5.15}$$

[5] We write $E_{k,r}(z)$ for F as $E^F_{k,r}(z)$ to indicate the dependence on F.

where

$$\omega = \left(\frac{1}{2}\left(\sqrt{\frac{13+\sqrt{5}}{2}}\, i + \frac{1-\sqrt{5}}{2}\right),\ \frac{1}{2}\left(\sqrt{\frac{13-\sqrt{5}}{2}}\, i + \frac{1+\sqrt{5}}{2}\right)\right) \in \mathfrak{H}^2.$$

We get

$L_K(1,\lambda_{2,2}^{(1)}) = 1.13172003415883621168\cdots, \quad L_K(1,\lambda_{4,2}^{(1)}) = 1.06744092000442833016\cdots,$
$L_K(1,\lambda_{2,4}^{(1)}) = 1.40377042189535030964\cdots.$

Next evaluating Eisenstein series at a CM-point

$$\left(\frac{1}{2}\left(\sqrt{\frac{13+\sqrt{5}}{2}}\, i + \frac{1-\sqrt{5}}{2}\right)\cdot \frac{1+\sqrt{5}}{2},\ \frac{1}{2}\left(-\sqrt{\frac{13-\sqrt{5}}{2}}\, i + \frac{1+\sqrt{5}}{2}\right)\cdot \frac{1-\sqrt{5}}{2}\right) \in \mathfrak{H}^2,$$

we obtain

$L_K(1,\lambda_{2,2}^{(2)}) = 1.05594274348607867398\cdots, \quad L_K(1,\lambda_{4,2}^{(2)}) = 1.48897355341553581717\cdots,$
$L_K(1,\lambda_{2,4}^{(2)}) = 1.36329307657845702302\cdots.$

In this way, we obtain

$$x(1) = 0.985042941589540335081 3407\cdots,$$

where

$$x(1) = \frac{L_K(1,\lambda_{4,2}^{(1)})}{L_K(1,\lambda_{2,2}^{(1)})} \cdot \frac{L_{K'}(1,\mu_{2,4}^{(2)})}{L_{K'}(1,\mu_{2,2}^{(2)})} \cdot \frac{1}{\exp(L_F'(0,\chi)/2)}.$$

the quantity in (5.10). It would be very hard to identify $x(1)$ with an algebraic number by just looking the numerical value. However, regarding $\exp(L_F'(0,\chi)/2)$ as "absolute period", we let $\mathrm{Gal}(L/\mathbf{Q})$ act formally on Grössencharacters and will make conjugates of $x(1)$. For $\sigma \in \mathrm{Gal}(L/\mathbf{Q})$, CM-types $\{\mathrm{id}, \sigma|K\}$ and $\{\mathrm{id}, \sigma\tau\rho|K'\}$ of K and K' change to

$$\{\sigma|K, \sigma^2|K\} = \{\sigma\rho|K, \mathrm{id}\}^\rho, \qquad \{\sigma|K', \sigma\tau\rho\sigma|K'\} = \{\sigma\tau|K', \mathrm{id}\}^\rho.$$

Hence, by the action of σ, $\lambda_{2,2}^{(1)}$, $\lambda_{4,2}^{(1)}$, $\mu_{2,2}^{(2)}$, $\mu_{2,4}^{(2)}$ are transformed to $\overline{\lambda_{2,2}^{(2)}}$, $\overline{\lambda_{2,4}^{(2)}}$, $\overline{\mu_{2,2}^{(1)}}$, $\overline{\mu_{4,2}^{(1)}}$ respectively. Here we note that $\overline{L_K(1,\lambda_{a,b}^{(i)})} = L_K(1,\lambda_{a,b}^{(i)})$ and $\overline{L_K(1,\mu_{a,b}^{(i)})} = L_K(1,\mu_{a,b}^{(i)})$. Put

$$x(2) = \frac{L_K(1,\lambda_{2,4}^{(2)})}{L_K(1,\lambda_{2,2}^{(2)})} \cdot \frac{L_{K'}(1,\mu_{4,2}^{(1)})}{L_{K'}(1,\mu_{2,2}^{(1)})} \cdot \frac{1}{\exp(L_F'(0,\chi)/2)}.$$

We have

$$x(2) = 9.86656529268701848014 16328\cdots.$$

We can regard $x(2)$ as a "conjugate" of $x(1)$ under σ. Similarly for $\tau \in \mathrm{Gal}(L/\mathbf{Q})$, we get

$$x(3) = \frac{L_K(1,\lambda_{2,4}^{(1)})}{L_K(1,\lambda_{2,2}^{(1)})} \cdot \frac{L_{K'}(1,\mu_{2,4}^{(1)})}{L_{K'}(1,\mu_{2,2}^{(1)})} \cdot \frac{1}{\exp(L'_F(0,\chi)/2)},$$
$$x(3) = 2.32749724236461357673638 64\cdots.$$

For $\sigma\tau \in \mathrm{Gal}(L/\mathbf{Q})$, we get[6]

$$x(4) = \frac{L_K(1,\lambda_{4,2}^{(2)})}{L_K(1,\lambda_{2,2}^{(2)})} \cdot \frac{L_{K'}(1,\mu_{4,2}^{(2)})}{L_{K'}(1,\mu_{2,2}^{(2)})} \cdot \frac{1}{\exp(L'_F(0,\chi)/2)},$$
$$x(4) = 1.6125611900254942747073081\cdots.$$

Now we compute the polynomial

$$f(T) = \prod_{j=1}^{4} (T - 2^3 \cdot 3^2 \cdot 41\ x(j))$$

using these numerical values of $x(j)$. We obtain

$$\begin{aligned} f(T) =& T^4 - 43665.\overbrace{00\cdots0}^{20}45\cdots T^3 + 489989565.\overbrace{00\cdots0}^{15}12\cdots T^2 \\ &- 2032811110800.\overbrace{00\cdots0}^{12}89\cdots T + 2770077684038400.\overbrace{00\cdots0}^{8}16\cdots. \end{aligned}$$

Therefore it is very plausible that

$$f(T) = T^4 - 43665T^3 + 489989565T^2 - 2032811110800T + 2770077684038400.$$

The roots of this polynomial are

$$\frac{43665}{4} + \frac{12669}{4}\sqrt{5} + \frac{3765}{4}\sqrt{41} + \frac{1425}{4}\sqrt{205}$$

and its conjugates. Comparing with numerical values, we obtain identifications:

$$\begin{aligned} 2^3 \cdot 3^2 \cdot 41\ x(1) &= \frac{43665}{4} - \frac{12669}{4}\sqrt{5} - \frac{3765}{4}\sqrt{41} + \frac{1425}{4}\sqrt{205}, \\ 2^3 \cdot 3^2 \cdot 41\ x(2) &= \frac{43665}{4} + \frac{12669}{4}\sqrt{5} + \frac{3765}{4}\sqrt{41} + \frac{1425}{4}\sqrt{205}, \\ 2^3 \cdot 3^2 \cdot 41\ x(3) &= \frac{43665}{4} + \frac{12669}{4}\sqrt{5} - \frac{3765}{4}\sqrt{41} - \frac{1425}{4}\sqrt{205}, \\ 2^3 \cdot 3^2 \cdot 41\ x(4) &= \frac{43665}{4} - \frac{12669}{4}\sqrt{5} + \frac{3765}{4}\sqrt{41} - \frac{1425}{4}\sqrt{205}. \end{aligned}$$

[6] By Theorems S1 and S2, we can verify $x(1) \sim x(2) \sim x(3) \sim x(4)$.

The numerical values of $x(j)$ coincide with these algebraic values up to the 23-rd decimal place. By (5.1), we also have the compatibility with

$$x(1)^{\sigma} = x(2), \qquad x(1)^{\tau} = x(3), \qquad x(1)^{\sigma\tau} = x(4).$$

Theorem S2 tells that

$$A = \frac{\pi^2 L_K(1,\lambda_{4,2}^{(1)}) L_K(1,\lambda_{2,4}^{(1)})}{L_K(1,\lambda_{2,2}^{(1)})^3}, \qquad B = \frac{\pi^2 L_K(1,\lambda_{4,2}^{(2)}) L_K(1,\lambda_{2,4}^{(2)})}{L_K(1,\lambda_{2,2}^{(2)})^3},$$
$$A' = \frac{\pi^2 L_{K'}(1,\mu_{4,2}^{(1)}) L_{K'}(1,\mu_{2,4}^{(1)})}{L_{K'}(1,\mu_{2,2}^{(1)})^3}, \qquad B' = \frac{\pi^2 L_{K'}(1,\mu_{4,2}^{(2)}) L_{K'}(1,\mu_{2,4}^{(2)})}{L_{K'}(1,\mu_{2,2}^{(2)})^3}$$

are algebraic. For the sake of completeness, we computed these values numerically and found that

$$A = \frac{\sqrt{5}}{2^6 \cdot 3}\left(\frac{-585 + 365\sqrt{41}}{2}\right), \qquad B = \frac{\sqrt{5}}{2^6 \cdot 3}\left(\frac{585 + 365\sqrt{41}}{2}\right),$$
$$A' = \frac{\sqrt{41}}{2^4 \cdot 3}(693 + 334\sqrt{5}), \qquad B' = \frac{\sqrt{41}}{2^4 \cdot 3}(-693 + 334\sqrt{5}).$$

(The coincidence is up to the 22-nd decimal place.)

We carried out the second calculation with higher precision[7] and found that the coincidence of $x(j)$ with algebraic values was improved to the 64-th decimal place.

We presented above a roundabout way to study (5.4) numerically, since this example shows intricate relations among various quantities which appear in examples in this section. Now we shall show more direct way. Set

$$Q = \pi^{-4} L_K(1,\lambda_{2,2}^{(1)}) L_K(1,\lambda_{2,2}^{(2)}) \exp(-L_F'(0,\chi)),$$
$$R = \pi^{-4} L_{K'}(1,\mu_{2,2}^{(1)}) L_{K'}(1,\mu_{2,2}^{(2)}) \exp(-L_F'(0,\chi)),$$

the quantities in (5.7) and (5.7′). For Q and R, $\mathrm{Gal}(L/\mathbf{Q})$ "acts trivially". Hence we can guess that Q and R are rational numbers. In fact we find

$$Q = \frac{2}{5^3}, \qquad R = \frac{2^4}{41^2}.$$

Numerical coincidence is to the 67-th decimal place.

Example 2. We take $F = \mathbf{Q}(\sqrt{17})$, $K = \mathbf{Q}(\sqrt{9+2\sqrt{17}}\, i)$. Then we have $F' = \mathbf{Q}(\sqrt{13})$, $K' = \mathbf{Q}(\sqrt{\frac{9+\sqrt{13}}{2}}\, i)$. We have $h_F = h_{F'} = 1$. We have

$$\mathfrak{O}_K = \mathfrak{O}_F \cdot \frac{1}{2}(\sqrt{9+2\sqrt{17}}\, i + 1) \oplus \mathfrak{O}_F$$

[7] For these calculations, we employed "UBASIC " created by Y. Kida and a desk computer PC9821-Ap2.

and that K is the maximal ray class field of conductor $\mathfrak{p}\infty_1\infty_2$ of F. Here $\mathfrak{p} = (9+2\sqrt{17})$. We also have

$$\mathfrak{O}_{K'} = \mathfrak{O}_{F'} \cdot \frac{1}{2}\left(\sqrt{\frac{9+\sqrt{13}}{2}}\, i + \frac{-1+\sqrt{13}}{2}\right) \oplus \mathfrak{O}_{F'}$$

and that K' is the maximal ray class field of conductor $\mathfrak{p}'\infty_1\infty_2$ of F', where $\mathfrak{p}' = (\frac{9+\sqrt{13}}{2})$. We have

$$L_F(0,\chi) = 2, \qquad L'_F(0,\chi) = -0.4238901837952971056559698\cdots.$$

Hence $h_K = h_{K'} = 1$. Let Q and R be defined in the completely same manner as in Example 1. We find

$$Q = \frac{2^4}{3 \cdot 17^2}, \qquad R = \frac{2}{13^2}.$$

The coincidence is up to the 35-th decimal place.

Example 3. We take $F = \mathbf{Q}(\sqrt{5})$, $K = \mathbf{Q}(\sqrt{7+2\sqrt{5}}\, i)$. Then we have $F' = \mathbf{Q}(\sqrt{29})$, $K' = \mathbf{Q}(\sqrt{\frac{7+\sqrt{29}}{2}}\, i)$. We have $h_F = h_{F'} = 1$. In F, we have $(29) = (7+2\sqrt{5})(7-2\sqrt{5})$. Put $\mathfrak{p} = (7+2\sqrt{5})$ and let M be the maximal ray class field of conductor $(4)\mathfrak{p}\infty_1\infty_2$ of F. We have $[M : F] = 16$ and $K \subseteq M$. We find

$$L_F(0,\chi) = 4, \qquad L'_F(0,\chi) = -11.95434037167457903017\cdots.$$

Hence $h_K = h_{K'} = 2$. We have

$$\mathfrak{O}_K = \mathfrak{O}_F \cdot \sqrt{7+2\sqrt{5}}\, i \oplus \mathfrak{O}_F.$$

The prime ideal (2) of F ramifies in K. Put $(2) = \mathfrak{P}_2^2$ in K. We can easily verify that $\mathfrak{P}_2$ is not a principal ideal and that

$$\mathfrak{P}_2 = \mathfrak{O}_F \cdot (\sqrt{7+2\sqrt{5}}\, i + 1) \oplus 2\mathfrak{O}_F.$$

Similarly (2) ramifies in K'. Let $\mathfrak{P}'_2$ be the prime factor of (2) in K'. Then $\mathfrak{P}'_2$ is not principal.

For every non-negative even integers a and b, there exist two Grössencharacters $\lambda_{a,b}^{(1)}$, $\lambda_{a,b}^{(2)}$ (resp. $\mu_{a,b}^{(1)}$, $\mu_{a,b}^{(2)}$) of $I(K)$ (resp. $I(K')$) of conductor (1). They are determined by $\lambda_{a,b}^{(i)}(\mathfrak{P}_2) = 1$ or -1 (resp. $\mu_{a,b}^{(i)}(\mathfrak{P}'_2) = 1$ or -1).

First let Q and R be the quantities of (5.7) and (5.7$'$) determined by $\lambda_{2,2}^{(1)}(\mathfrak{P}_2) = \lambda_{2,2}^{(2)}(\mathfrak{P}_2) = 1$, $\mu_{2,2}^{(1)}(\mathfrak{P}'_2) = \mu_{2,2}^{(2)}(\mathfrak{P}'_2) = 1$. Then we find

$$Q = \frac{2^4 \cdot 13}{5^2}, \qquad R = \frac{2^4 \cdot 3^2 \cdot 5 \cdot 11}{29^2}.$$

Similarly define Q and R by choosing $\lambda^{(1)}_{2,2}(\mathfrak{P}_2) = \lambda^{(2)}_{2,2}(\mathfrak{P}_2) = -1$, $\mu^{(1)}_{2,2}(\mathfrak{P}'_2) = \mu^{(2)}_{2,2}(\mathfrak{P}'_2) = -1$. Then we find

$$Q = \frac{2^4}{5}, \qquad R = \frac{2^4 \cdot 11}{29}.$$

The numerical coincidence is up to the 30-th decimal place.[8]

To clarify the relation with the Stark-Shintani conjecture, let us examine this example more closely. Let C denote the ideal class group of conductor $(4)\mathfrak{p}\infty_1\infty_2$. We have $C \cong \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \cong \mathrm{Gal}(M/F)$. Let α, β, γ be the classes of ideals (89), (59), $(106+\sqrt{5})$ in C respectively. Then $\alpha^4 = \beta^2 = \gamma^2 = 1$ and $C = \langle\alpha\rangle \oplus \langle\beta\rangle \oplus \langle\gamma\rangle$. Let D be the character group of C. Define ω, ψ, $\eta \in D$ by

$$\omega(\alpha) = \sqrt{-1}, \quad \omega(\beta) = \omega(\gamma) = 1, \qquad \psi(\alpha) = \psi(\gamma) = 1, \quad \psi(\beta) = -1,$$
$$\eta(\alpha) = \eta(\beta) = 1, \quad \eta(\gamma) = -1.$$

For $\mu \in D$, let $\mathfrak{f}(\mu)$ denote the conductor of μ. We have

$$\mathfrak{f}(\omega) = \mathfrak{p}\infty_2, \quad \mathfrak{f}(\psi) = (4)\infty_1, \quad \mathfrak{f}(\eta) = (4)\infty_1\infty_2.$$

The Hecke character χ, regarded as a character of C, is equal to $\omega^2\eta$.

For $c \in C$, let $\zeta_F(s,c) = \sum_{\mathfrak{a}\in c} N(\mathfrak{a})^{-s}$ be the partial zeta function of the class c. Let D_0, D_1 and D_2 be the subgroups of D consisting of all characters whose conductors divide $(4)\mathfrak{p}$, $(4)\mathfrak{p}\infty_1$ and $(4)\mathfrak{p}\infty_2$ respectively. Let C_i be the annihilator of D_i for $i = 0, 1, 2$. We have

$$C_0 = \{1, \beta\gamma, \alpha^2\beta, \alpha^2\gamma\}, \qquad C_1 = \{1, \alpha^2\gamma\}, \qquad C_2 = \{1, \beta\gamma\}.$$

From the functional equation, we see easily that

$$\sum_{c\in C} \mu(c)\zeta_F(0,c) = \sum_{c\in C} \mu(c)\zeta'_F(0,c) = 0$$

for every $\mu \in D_0$. Hence we obtain

$$\sum_{c_0\in C_0} \zeta'_F(0, cc_0) = 0 \qquad \text{for every } c \in C.$$

If we admit the Stark-Shintani conjecture, $\exp(2(\zeta'_F(0,c) + \zeta'_F(0,c\alpha^2\gamma)))$, $c \in C$ is a unit of the maximal ray class field of conductor $(4)\mathfrak{p}\infty_2$. In fact, following the procedure described in [St], III, our numerical computation suggests overwhelmingly that $\exp(2(\zeta'_F(0,c)+\zeta'_F(0,c\beta\gamma))) + \exp(-2(\zeta'_F(0,c)+\zeta'_F(0,c\beta\gamma)))$ is a root of the polynomial

$$X^4 - (31+13\sqrt{5})X^3 + \frac{1143+513\sqrt{5}}{2}X^2 - (3711+1661\sqrt{5})X + (7649+3420\sqrt{5})$$

[8]In general, Q and R are not rational numbers. For example, let $K = \mathbf{Q}(\sqrt{3+\sqrt{2}}\,i)$ and $\mathfrak{P}_2$ be the unique prime divisor of (2) in K. We have $h_K = 2$. We find $Q = \frac{3}{2^2}$ or $Q = \frac{\sqrt{2}}{2^2\cdot 3}$ according as the assignment that both of $\lambda^{(1)}_{2,2}(\mathfrak{P}_2)$ and $\lambda^{(2)}_{2,2}(\mathfrak{P}_2)$ are 1 or -1.

for every $c \in C$. Similarly it is very plausible that $\exp(2(\zeta'_F(0,c)+\zeta'_F(0,c\alpha^2\gamma)))$ $+\exp(-2(\zeta'_F(0,c)+\zeta'_F(0,c\alpha^2\gamma)))$ is a root of the polynomial

$$X^4-(476+212\sqrt{5})X^3+(18584+8312\sqrt{5})X^2-(168576+75392\sqrt{5})X+(441344+197376\sqrt{5})$$

for every $c \in C$.

Put $z(c) = \exp(\zeta'_F(0,c))$, $c \in C$. Admitting the Stark-Shintani conjecture, we have $\prod_{c_i\in C_i} z(cc_i) \sim 1$ for every $i = 0, 1, 2$ and every $c \in C$. Hence we see easily that every $z(c)$ can be written as a monomial of $z(1)$, $z(\alpha)$, $z(\alpha^2)$ and $z(\alpha^3)$ up to algebraic numbers. We have $F(i) \subseteq M$ and Conjecture 3.1 holds for $F(i)$ by Theorem 2.7. Let M_0 be the maximal CM-subfield of M. We have $\mathrm{Gal}(M/M_0) = \{1, \alpha^2\beta\}$, $[M_0 : F] = 8$. From abelian L-functions attached to the characters η, $\omega^2\eta$, $\omega\psi$ and $\omega^3\psi$ for M_0/F, we can obtain four Artin L-functions for $\widetilde{M_0}/\mathbf{Q}$ with odd representations, where $\widetilde{M_0}$ denotes the normal closure of M_0 over $\mathbf{Q}$. If we trust Conjecture 3.2, all of $z(1)$, $z(\alpha)$, $z(\alpha^2)$ and $z(\alpha^3)$ can be expressed by CM-periods.

§6. Absolute CM-periods

6.1. The numerical examples presented in §5 suggest the possibility to make conjectures in §3 in more precise forms, i.e., to formulate them in covariant forms under the action of $\mathrm{Aut}(\mathbf{C})$. In this section, we shall discuss this problem in certain simple cases.

Let K be a CM-field and F be its maximal real subfield. Let χ be the Hecke character of $F_A^\times$ which corresponds to the quadratic extension K/F and let $L_F(s,\chi)$ be the L-function attached to χ. We put

$$P = \exp\Big(\frac{L'_F(0,\chi)}{L_F(0,\chi)}\Big).$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be Grössencharacters of $I(K)$ of conductor $\mathfrak{f}$. We assume that there exists a CM-type Φ_i and non-negative integers $t_\sigma^{(i)}$, $\sigma \in \Phi_i$ for every i such that

$$\lambda_i((\alpha)) = \prod_{\sigma\in\Phi_i} (\alpha^{\sigma\rho}/|\alpha^\sigma|)^{t_\sigma^{(i)}} \qquad \text{if} \quad \alpha \equiv 1 \mod {}^\times\mathfrak{f}.$$

Assume that there exists an integer m which satisfies

$$m - t_\sigma^{(i)} \in 2\mathbf{Z} \quad \text{and} \quad -t_\sigma^{(i)} < m \le t_\sigma^{(i)}$$

for every i and every $\sigma \in \Phi_i$. By Theorem S2, we have

$$L(m/2,\lambda_i) \sim \pi^{e_i/2} p_K\Big(\sum_{\sigma\in\Phi_i} t_\sigma^{(i)}\cdot\sigma, \Phi_i\Big), \quad e_i = m[F:\mathbf{Q}] + \sum_{\sigma\in\Phi_i} t_\sigma^{(i)}.$$

Let L be the normal closure of K. For every i, we take $\Phi_i^0 \in I_L$ so that $\mathrm{Res}_{L/K}(\Phi_i^0) = \Phi_i$ and put $\eta_i = \mathrm{Inf}_{L/K}(\sum_{\sigma\in\Phi_i} t_\sigma^{(i)}\cdot\sigma)$. By Theorem S1, (4), (6), we have

$$p_K\Big(\sum_{\sigma\in\Phi_i} t_\sigma^{(i)}\cdot\sigma, \Phi_i\Big) \sim p_L(\eta_i, \Phi_i^0) \sim p_L(\mathrm{id}, \Phi_i^0\eta_i^{-1}),$$

where $\Phi_i^0\eta_i^{-1} = \sum_{\gamma\in\Phi_i^0,\,\delta\in\eta_i}\gamma\delta^{-1}$. Take a CM-type S of L. We can write

$$\Phi_i^0\eta_i^{-1} = \sum_{\sigma\in S} l_\sigma^{(i)}\cdot\sigma + m_\sigma^{(i)}\cdot\sigma\rho.$$

By Theorem S1, (3), we have

$$(6.1)\qquad \prod_{i=1}^n p_K(\sum_{\sigma\in\Phi_i} t_\sigma^{(i)}\cdot\sigma,\Phi_i)\sim p_L(\mathrm{id},\sum_{\sigma\in S} n_\sigma\cdot\sigma),\quad n_\sigma=\sum_{i=1}^n(l_\sigma^{(i)}-m_\sigma^{(i)}).$$

Conjecture 3.4 states

$$(6.2)\qquad P\sim\pi^{[K:\mathbf{Q}]/2}\prod_{\sigma\in J_K}p_K(\sigma,\sigma).$$

Regard χ as the non-trivial character of $\mathrm{Gal}(K/F)$ and lift χ to the character $\widetilde{\chi}$ of $\mathrm{Gal}(L/F)$. Put $\psi=\mathrm{Ind}(\widetilde{\chi};\mathrm{Gal}(L/F)\longrightarrow\mathrm{Gal}(L/\mathbf{Q}))$. By Theorem S1, (3) and (3.7), (6.2) is equivalent to

$$(6.3)\qquad P\sim\pi^{[K:\mathbf{Q}]/2}\prod_{\sigma\in S}p_L(\mathrm{id},2\chi_\psi(\sigma)\cdot\sigma).$$

We assume that there exists $e\in 2^{-1}\mathbf{Z}$ such that

$$(6.4)\qquad \sum_{\sigma\in S}n_\sigma\cdot\sigma = e\sum_{\sigma\in S}2\chi_\psi(\sigma)\cdot\sigma.$$

Put

$$(6.5)\qquad A=\sum_{i=1}^n e_i - e\cdot\frac{[K:\mathbf{Q}]}{2}.$$

By Theorem S2, Conjecture 3.4 is equivalent to

$$(6.6)\qquad \prod_{i=1}^n L(m/2,\lambda_i)\sim\pi^A P^e.$$

Now we can state a precise version of Conjecture 3.4.

Conjecture 6.1. *Let $h_\mathfrak{f}$ be the order of the ideal class group modulo $\mathfrak{f}$ of K. Then we have*

$$\{(\frac{\prod_{i=1}^n L(m/2,\lambda_i)}{\pi^A P^e})^{wh_\mathfrak{f}}\}^\sigma = (\frac{\prod_{i=1}^n L(m/2,\lambda_i^\sigma)}{\pi^A P^e})^{wh_\mathfrak{f}}$$

for every $\sigma\in Aut(\mathbf{C})$. Here w is a certain positive integer depending on F.

6.2. We shall prove Conjecture 6.1 for imaginary quadratic fields in more precise form. The proof will be completed in §6.4. In this section, we shall treat the simplest case.

Let K be an imaginary quadratic field with discriminat $-d$. Let χ be the Dirichlet character which corresponds to the quadratic extension $K/\mathbf{Q}$. Let h be the class number of K and w be the number of roots of unity contained in K. By (2.11) and (4.2), we have

$$P = \exp\left(\frac{L'(0,\chi)}{L(0,\chi)}\right) = \frac{1}{d}\prod_{a=1}^{d-1}\Gamma\left(\frac{a}{d}\right)^{w\chi(a)/2h}. \tag{6.7}$$

Let λ be a Gössencharacter of conductor $\mathfrak{f}$ of K which satisfies

$$\lambda((\alpha)) = \left(\frac{\alpha^{\rho}}{|\alpha|}\right)^k, \qquad \alpha \equiv 1 \mod {}^{\times}\mathfrak{f}.$$

For an integer m such that $m-k \in 2\mathbf{Z}$ and $-k < m \leq k$, Conjecture 6.1 states

$$\left\{\left(\frac{L(m/2,\lambda)}{\pi^{m/2}P^{k/2}}\right)^{12h_{\mathfrak{f}}}\right\}^{\sigma} = \left(\frac{L(m/2,\lambda^{\sigma})}{\pi^{m/2}P^{k/2}}\right)^{12h_{\mathfrak{f}}}$$

for every $\sigma \in \mathrm{Aut}(\mathbf{C})$. (We take $w = 12$.)

We assume that k is an even integer greater than 2. Let

$$E_k(z) = (2\pi i)^{-k}\sum_{(c,d)/\sim}(cz+d)^{-k}$$

be the Eisenstein series of weight k with respect to $SL(2,\mathbf{Z})$. Here $\sum_{(c,d)/\sim}$ has the same meaning as in §4. Let $\mathfrak{a} = \mathbf{Z}\omega_1 \oplus \mathbf{Z}\omega_2$ be a fractional ideal of K. If $\Im(\omega_1/\omega_2) > 0$, we have

$$E_k\left(\frac{\omega_1}{\omega_2}\right) = \frac{w}{2}\cdot(2\pi i)^{-k}\omega_2^{-k}\sum_{(\alpha)\subseteq\mathfrak{a}}\left(\frac{\alpha^{\rho}}{|\alpha|}\right)^k N(\alpha)^{-k/2}$$

by a direct computation. Let $c_1, c_2, \cdots, c_h$ be the ideal classes of K. For every i, we choose a fractional ideal $\mathfrak{a}_i$ in the class c_i so that $\mathfrak{a}_i = \mathbf{Z}\omega_{1,i}\oplus\mathbf{Z}\omega_{2,i}$, $\Im(\omega_{1,i}/\omega_{2,i}) > 0$. We have

$$\sum_{\mathfrak{a}\in c_i^{-1},\mathfrak{a}:\text{integral}}\lambda(\mathfrak{a})N(\mathfrak{a})^{-s} = N(\mathfrak{a}_i)^s\lambda(\mathfrak{a}_i)^{-1}\sum_{(\alpha)\subseteq\mathfrak{a}_i}\lambda((\alpha))N(\alpha)^{-s}.$$

We further assume that the conductor of λ is (1). Then, by (6.9), we get

$$\sum_{\mathfrak{a}\in c_i^{-1},\mathfrak{a}:\text{integral}}\lambda(\mathfrak{a})N(\mathfrak{a})^{-s} = N(\mathfrak{a}_i)^{k/2}\lambda(\mathfrak{a}_i)^{-1}\left(\frac{w}{2}\right)^{-1}(2\pi i)^k\omega_{2,i}^{-k}E_k\left(\frac{\omega_{1,i}}{\omega_{2,i}}\right).$$

Therefore we obtain

$$L\left(\frac{k}{2},\lambda\right) = \frac{2}{w}\cdot(2\pi i)^k\sum_{i=1}^{h}\lambda(\mathfrak{a}_i)^{-1}N(\mathfrak{a}_i)^{k/2}\omega_{2,i}^{-k}E_k\left(\frac{\omega_{1,i}}{\omega_{2,i}}\right). \tag{6.10}$$

Let $S_k(SL(2,\mathbf{Z}))$ be the space of holomorphic modular cusp forms of weight k with respect to $SL(2,\mathbf{Z})$. Let $f \in S_k(SL(2,\mathbf{Z}))$. For a lattice L in $\mathbf{C}$, we choose ω_1, ω_2 so that $L = \mathbf{Z}\omega_1 \oplus \mathbf{Z}\omega_2$, $\Im(\omega_1/\omega_2) > 0$ and put

$$f(L) = f(\frac{\omega_1}{\omega_2})\omega_2^{-k}.$$

We can easily verify that this is well defined and that

$$f(\alpha L) = \alpha^{-k} f(L), \qquad \alpha \in \mathbf{C}^\times.$$

Let

$$\eta(z) = e^{2\pi i z/24}\prod_{n=1}^{\infty}(1 - e^{2\pi i n z})$$

be the familiar η-function. Put $f = \eta^{24} \in S_k(SL(2,\mathbf{Z}))$. Then the exact Chowla-Selberg formula gives (cf. Weil [W1], p. 92)

$$\prod_{i=1}^{h} N(\mathfrak{a}_i)^{12}|f(\mathfrak{a}_i)|^2 = (2\pi)^{-12h}P^{12h}. \tag{6.11}$$

Now we further assume that $k = 24h$. By (6.10) and (6.11), we obtain

$$\frac{L(12h,\lambda)}{(2\pi)^{12h}P^{12h}} = \frac{2}{w}\cdot\frac{\sum_{i=1}^{h}\lambda(\mathfrak{a}_i)^{-1}N(\mathfrak{a}_i)^{12h}\omega_{2,i}^{-24h}E_{24h}(\omega_{1,i}/\omega_{2,i})}{\prod_{i=1}^{h}N(\mathfrak{a}_i)^{12}|f(\mathfrak{a}_i)|^2}.$$

We put

$$g = E_{24h}, \qquad Q = \frac{\sum_{i=1}^{h}\lambda(\mathfrak{a}_i)^{-1}N(\mathfrak{a}_i)^{12h}g(\mathfrak{a}_i)}{\prod_{i=1}^{h}N(\mathfrak{a}_i)^{12}f(\mathfrak{a}_i)^2}. \tag{6.12}$$

Then we have

$$\frac{L(12h,\lambda)}{\pi^{12h}P^{12h}} = 2^{12h}\cdot\frac{2}{w}\cdot Q\cdot\prod_{i=1}^{h}\frac{\overline{f(\mathfrak{a}_i)}}{f(\mathfrak{a}_i)}. \tag{6.13}$$

We are going to determine the action of $\mathrm{Aut}(\mathbf{C})$ on Q using Proposition 6.36 of Shimura [Sh1]. Let $G = GL(2)$ regarded as an algebraic group defined over $\mathbf{Q}$. Set

$$G_{\mathbf{Q}+} = \{g \in G_{\mathbf{Q}} \mid \det g > 0\}, \qquad G_{A+} = \{g \in G_A \mid (\det g)_\infty > 0\}.$$

Let $\mathfrak{O}_K$ be the ring of integers of K. We choose ω so that $\mathfrak{O}_K = \mathbf{Z}\omega \oplus \mathbf{Z}$, $\Im(\omega) > 0$. We put

$$\begin{pmatrix}\omega_{1,i}\\ \omega_{2,i}\end{pmatrix} = \begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}\begin{pmatrix}\omega\\ 1\end{pmatrix}, \qquad \alpha_i = \begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}. \tag{6.14}$$

Then we have $\alpha_i \in G_{\mathbf{Q}+}$, $N(\mathfrak{a}_i) = \det \alpha_i$. We define an algebra homomorphism q of K into $M(2, \mathbf{Q})$ by

$$q(a)\begin{pmatrix} \omega \\ 1 \end{pmatrix} = \begin{pmatrix} a\omega \\ a \end{pmatrix}, \qquad a \in K. \tag{6.15}$$

Clearly we have $q(K^\times) \subset G_{\mathbf{Q}+}$, $q(\mathfrak{O}_K) \subset M(2, \mathbf{Z})$. We extend q to the homomorphism from $K_A^\times$ to G_{A+} and denote it by the same letter q. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{R})$, $\det \alpha > 0$. For $z \in \mathfrak{H}$, we put $j(\alpha, z) = cz + d$. For a function h on $\mathfrak{H}$ and an integer k, we put $(h|[\alpha]_k)(z) = (\det \alpha)^{k/2} h(\alpha(z)) j(\alpha, z)^{-k}$. We put

$$f_i = f|[\alpha_i]_{12}, \qquad g_i = g|[\alpha_i]_{24h}.$$

Then we have

$$N(\mathfrak{a}_i)^6 f(\mathfrak{a}_i) = f_i(\omega), \qquad N(\mathfrak{a}_i)^{12h} f(\mathfrak{a}_i) = f_i(\omega).$$

Hence we obtain

$$Q = \frac{\sum_{i=1}^h \lambda(\mathfrak{a}_i)^{-1} g_i(\omega)}{\prod_{i=1}^h f_i(\omega)^2}. \tag{6.16}$$

We may and shall assume that $\mathfrak{a}_i$ are integral ideals. Let N be the least common multiple of $N(\mathfrak{a}_i)$, $1 \le i \le h$. Let C_N be the maximal ray class field modulo N of K. The we have

$$\frac{f_i(\omega)}{f(\omega)}, \qquad \frac{g_i(\omega)}{f(\omega)^{2h}} \in C_N \tag{6.17}$$

by [Sh1], Prop. 6.36. (Note that $f(z)$ does not vanish on $\mathfrak{H}$.) Let $\sigma = (\frac{C_N/K}{\mathfrak{b}})$ where $\mathfrak{b}$ is a fractional ideal of K such that $(\mathfrak{b}, N) = 1$. We can determine the images of quantities in (6.17) in the following way. Let

$$\mathfrak{b}^{-1} = \mathbf{Z}\omega_1' \oplus \mathbf{Z}\omega_2', \qquad \Im(\omega_1'/\omega_2') > 0.$$

Take $\xi \in G_{\mathbf{Q}+}$ so that

$$\xi^{-1}\begin{pmatrix} \omega \\ 1 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}.$$

Choose $s \in K_A^\times$ so that $s\mathfrak{O}_K = \mathfrak{b}$, $q(s)_p = 1$ for all $p \mid N$, and put $t = q(s^{-1})\xi \in G_{A+}$. We can find $\gamma_i \in SL(2, \mathbf{Z})$ so that $\mathbf{Z}^2 \alpha_i t = \mathbf{Z}^2 \alpha_i \gamma_i$. Put $\eta_i = \gamma_i \xi^{-1}$. Then we have

$$\left(\frac{f_i(\omega)}{f(\omega)}\right)^\sigma = \left(\frac{f_i(\eta_i(\omega))}{f(\eta_i(\omega))}\right), \quad \left(\frac{g_i(\omega)}{f(\omega)^{2h}}\right)^\sigma = \left(\frac{g_i(\eta_i(\omega))}{f(\eta_i(\omega))^{2h}}\right), \quad 1 \le i \le h. \tag{6.18}$$

Therefore we obtain

$$\left(\frac{g_i(\omega)}{\prod_{i=1}^h f_i(\omega)^2}\right)^\sigma = \frac{g_i(\eta_i(\omega))}{\prod_{i=1}^h f_i(\eta_i(\omega))^2} \times \frac{\prod_{i=1}^h f(\eta_i(\omega))^2}{f(\eta_i(\omega))^{2h}}.$$

By definition, we have

$$
\begin{aligned}
f(\eta_i(\omega)) &= f(\xi^{-1}(\omega))j(\gamma_i, \xi^{-1}(\omega))^{12}, \\
g_i(\eta_i(\omega)) &= (\det \alpha_i)^{12h} g(\alpha_i\eta_i(\omega))j(\alpha_i, \eta_i(\omega))^{-24h}, \\
f_i(\eta_i(\omega)) &= (\det \alpha_i)^{6} f(\alpha_i\eta_i(\omega))j(\alpha_i, \eta_i(\omega))^{-12}, \\
j(\alpha_i, \eta_i(\omega))j(\gamma_i, \xi^{-1}(\omega)) &= j(\alpha_i\eta_i, \omega)j(\xi^{-1}, \omega)^{-1}.
\end{aligned}
$$

Take $a_i \in K_A^\times$ so that $\mathfrak{a}_i = a_i\mathfrak{O}_K$. Then we have

$$\mathbf{Z}^2\alpha_i\eta_i = \mathbf{Z}^2\alpha_i\gamma_i\xi^{-1} = \mathbf{Z}^2\alpha_i q(s^{-1}) = \mathbf{Z}^2 q(a_i s^{-1}).$$

Hence we see that $\alpha_i\eta_i \begin{pmatrix} \omega \\ 1 \end{pmatrix}$ gives a basis of the ideal $\mathfrak{a}_i\mathfrak{b}^{-1}$. Then we have

$$f(\alpha_i\eta_i(\omega))j(\alpha_i\eta_i, \omega)^{-12} = f(\mathfrak{a}_i\mathfrak{b}^{-1}), \quad g(\alpha_i\eta_i(\omega))j(\alpha_i\eta_i, \omega)^{-24h} = g(\mathfrak{a}_i\mathfrak{b}^{-1}).$$

Therefore we obtain

$$(6.20) \qquad \left(\frac{N(\mathfrak{a}_i)^{12h} g(\mathfrak{a}_i)}{\prod_{i=1}^h N(\mathfrak{a}_i)^{12} f(\mathfrak{a}_i)^2}\right)^\sigma = \frac{N(\mathfrak{a}_i)^{12h} g(\mathfrak{a}_i\mathfrak{b}^{-1})}{\prod_{i=1}^h N(\mathfrak{a}_i)^{12} f(\mathfrak{a}_i\mathfrak{b}^{-1})^2}, \quad 1 \le i \le h.$$

Since $\sigma|K = \mathrm{id}$, we see easily that (6.20) holds without assuming that $\mathfrak{a}_i$ are integral ideals. Since the right hand side is invariant when we replace $\mathfrak{b}$ by $(\alpha)\mathfrak{b}$, $\alpha \in K^\times$, we see that

$$\frac{N(\mathfrak{a}_i)^{12h} g(\mathfrak{a}_i)}{\prod_{i=1}^h N(\mathfrak{a}_i)^{12} f(\mathfrak{a}_i)^2} \in H$$

and that (6.20) holds for $\sigma = (\frac{H/K}{\mathfrak{b}})$ for any fractional ideal $\mathfrak{b}$. Here H denotes the Hilbert class field of K. By (6.16), we obtain

$$(6.21) \qquad \begin{gathered} Q^\sigma = \frac{\sum_{i=1}^h \lambda_{12h}^\sigma(\mathfrak{a}_i)^{-1} N(\mathfrak{a}_i)^{12h} g(\mathfrak{a}_i\mathfrak{b}^{-1})}{\prod_{i=1}^h N(\mathfrak{a}_i)^{12} f(\mathfrak{a}_i\mathfrak{b}^{-1})^2}, \\ \sigma \in \mathrm{Aut}(\mathbf{C}) \text{ such that } \sigma|H = (\frac{H/K}{\mathfrak{b}}). \end{gathered}$$

Going back to the previous situation, we assume that $\mathfrak{a}_i$ are integral ideals and let $\sigma = (\frac{C_N/K}{\mathfrak{b}})$. By (6.18), we get

$$
\begin{aligned}
\left(\frac{\prod_{i=1}^h f_i(\omega)}{f(\omega)^h}\right)^\sigma &= \frac{\prod_{i=1}^h f_i(\eta_i(\omega))}{\prod_{i=1}^h f(\eta_i(\omega))} \\
&= \frac{\prod_{i=1}^h N(\mathfrak{a}_i)^6 f(\alpha_i\eta_i(\omega))j(\alpha_i, \eta_i(\omega))^{-12}}{f(\xi^{-1}(\omega))^h \prod_{i=1}^h j(\gamma_i, \xi^{-1}(\omega))^{12}} = \frac{\prod_{i=1}^h N(\mathfrak{a}_i)^6 f(\mathfrak{a}_i\mathfrak{b}^{-1})}{f(\xi^{-1}(\omega))^h j(\xi^{-1}, \omega)^{-12h}}.
\end{aligned}
$$

Therefore we obtain

$$(6.22)\qquad \left(\frac{\prod_{i=1}^{h} f_i(\omega)}{f(\omega)^h}\right)^{\sigma} = \frac{\prod_{i=1}^{h} N(\mathfrak{a}_i)^6 f(\mathfrak{a}_i\mathfrak{b}^{-1})}{f(\mathfrak{b}^{-1})^h}.$$

Similarly to the above, we see that $\left(\frac{\prod_{i=1}^{h} f_i(\omega)}{f(\omega)^h}\right)^{\sigma} \in H$ and that (6.22) holds for $\sigma = \left(\frac{H/K}{\mathfrak{b}}\right)$ for any fractional ideal $\mathfrak{b}$. We have $\rho\sigma\rho = \left(\frac{H/K}{\mathfrak{b}^{\rho}}\right)$. Hence, by (6.22), we get

$$(6.23)\qquad \left(\frac{\prod_{i=1}^{h} \overline{f_i(\omega)}}{\overline{f(\omega)}^h}\right)^{\sigma} = \frac{\prod_{i=1}^{h} N(\mathfrak{a}_i)^6 \overline{f(\mathfrak{a}_i(\mathfrak{b}^{\rho})^{-1})}}{\overline{f((\mathfrak{b}^{\rho})^{-1})}^h}.$$

Since $\overline{f(-\bar{z})} = f(z)$ and $(\mathfrak{b}^{\rho})^{-1} = \mathbf{Z}(-\bar{\omega}_1') \oplus \mathbf{Z}\bar{\omega}_2'$, we have $\overline{f((\mathfrak{b}^{\rho})^{-1})} = f(\mathfrak{b}^{-1})$. We see easily that $f(\omega) \in \mathbf{R}$. Hence, by (6.22) and (6.23), we have

$$(6.24)\qquad \left(\frac{\prod_{i=1}^{h} f(\mathfrak{a}_i)}{\prod_{i=1}^{h} \overline{f(\mathfrak{a}_i)}}\right)^{\sigma} = \frac{\prod_{i=1}^{h} f(\mathfrak{a}_i\mathfrak{b}^{-1})}{\prod_{i=1}^{h} \overline{f(\mathfrak{a}_i(\mathfrak{b}^{\rho})^{-1})}}.$$

Combining (6.13), (6.21) and (6.24), we obtain

$$(6.25)\quad \left(\frac{L(12h,\lambda)}{\pi^{12h}P^{12h}}\right)^{\sigma} = 2^{12h}\cdot\frac{2}{w}\cdot\frac{\sum_{i=1}^{h}\lambda^{\sigma}(\mathfrak{a}_i)^{-1}N(\mathfrak{a}_i)^{12h}g(\mathfrak{a}_i\mathfrak{b}^{-1})}{\prod_{i=1}^{h} N(\mathfrak{a}_i)^{12} f(\mathfrak{a}_i\mathfrak{b}^{-1})^2} \times \frac{\prod_{i=1}^{h} f(\mathfrak{a}_i\mathfrak{b}^{-1})}{\prod_{i=1}^{h} \overline{f(\mathfrak{a}_i(\mathfrak{b}^{\rho})^{-1})}}.$$

Put $\mathfrak{a}_i' = \mathfrak{a}_i\mathfrak{b}^{-1}$, $\mathfrak{a}_i'' = \mathfrak{a}_i(\mathfrak{b}^{\rho})^{-1}$, $1 \le i \le h$. The the right hand side of (6.25) is equal to

$$2^{12h}\cdot\frac{2}{w}\cdot\frac{\lambda^{\sigma}(\mathfrak{b})^{-1}\sum_{i=1}^{h}\lambda^{\sigma}(\mathfrak{a}_i')^{-1}N(\mathfrak{a}_i')^{12h}g(\mathfrak{a}_i')}{\prod_{i=1}^{h} N(\mathfrak{a}_i')^{12}|f(\mathfrak{a}_i')|^2} \times \frac{\prod_{i=1}^{h} \overline{f(\mathfrak{a}_i')}}{\prod_{i=1}^{h} \overline{f(\mathfrak{a}_i'')}}.$$

By (6.13), we obtain

$$\left(\frac{L(12h,\lambda)}{\pi^{12h}P^{12h}}\right)^{\sigma} = \zeta\left(\frac{L(12h,\lambda^{\sigma})}{\pi^{12h}P^{12h}}\right), \qquad \zeta = \lambda^{\sigma}(\mathfrak{b})^{-1} \times \frac{\prod_{i=1}^{h} \overline{f(\mathfrak{a}_i')}}{\prod_{i=1}^{h} \overline{f(\mathfrak{a}_i'')}}.$$

We have $\mathfrak{a}_i' = \mathfrak{a}_i''\mathfrak{b}^{\rho}\mathfrak{b}^{-1}$. For every i, we can find $j(i)$ so that $\mathfrak{a}_i''\mathfrak{b}^{\rho}\mathfrak{b}^{-1} = \mathfrak{a}_{j(i)}''(\mu_i)$, $\mu_i \in K^{\times}$. Then $i \longrightarrow j(i)$ gives a permutation on h letters. We have

$$\frac{\prod_{i=1}^{h} \overline{f(\mathfrak{a}_i')}}{\prod_{i=1}^{h} \overline{f(\mathfrak{a}_i'')}} = \prod_{i=1}^{h} \bar{\mu}_i^{-12}.$$

Let $\mathfrak{b}^h = (\mu)$, $\mu \in K^{\times}$. Then we have $(\mathfrak{b}^{\rho}\mathfrak{b}^{-1})^h = (\bar{\mu}/\mu)$ and $\prod_{i=1}^{h} \bar{\mu}_i^{-12} = (\bar{\mu}/\mu)^{12}$. Then we obtain

$$(6.27)\qquad \zeta = \lambda^{\sigma}(\mathfrak{b})^{-1}(\bar{\mu}/\mu)^{12}, \qquad \mathfrak{b}^h = (\mu).$$

We have

$$\zeta^h = \lambda^{\sigma}(\mu)^{-1}(\bar{\mu}/\mu)^{12h} = (\bar{\mu}/|\mu|)^{-24h}(\bar{\mu}/\mu)^{12h} = 1.$$

Thus we have shown the following Proposition.

Proposition 6.2. *Let λ be a Grössencharacter of K of conductor* (1) *which satisfies*

$$\lambda((\alpha)) = (\frac{\alpha^\rho}{|\alpha|})^{24h}, \qquad \alpha \in K^\times.$$

Then, for $\sigma \in Aut(\mathbf{C})$ such that $\sigma|H = (\frac{H/K}{\mathfrak{b}})$, we have

$$(\frac{L(12h,\lambda)}{\pi^{12h}p^{12h}})^\sigma = \zeta \cdot (\frac{L(12h,\lambda^\sigma)}{\pi^{12h}p^{12h}}) \qquad \text{with} \quad \zeta^h = 1.$$

We have $\zeta = \lambda^\sigma(\mathfrak{b})^{-1}(\bar{\mu}/\mu)^{12}$ if $\mathfrak{b}^h = (\mu)$.

6.3. Proposition 6.2 solves Conjecture 6.1 for imaginary quadratic fields only for a very special case. Though general case can be shown in similar way, we shall derive it from considerations on motives. Since Blasius [B] showed that critical values of L-functions attached to Grössencharacters of A_0-type behave according to Deligne's conjecture, there is no harm to deal the problem in this way.

Let K be a CM-field and E be an algebraic number field. Let M be a motive over K with coefficients in E. We assume that M is of rank 1, i.e., $H_{DR}(M)$ is a free $E \otimes_\mathbf{Q} K$-module of rank 1. Let

$$I^+ : H_B^+(R_{K/\mathbf{Q}}(M)) \otimes_\mathbf{Q} \mathbf{C} \cong H_{DR}^+(R_{K/\mathbf{Q}}(M)) \otimes_\mathbf{Q} \mathbf{C}$$

be the canonical isomorphism as $E \otimes_\mathbf{Q} \mathbf{C}$-modules. We assume that $H_{DR}(M)$ is *not* a free $E \otimes_\mathbf{Q} K$-module. We know (cf. [Y], §2.3) that $H_{DR}^+(M) \oplus \rho H_{DR}^+(M)$ is a free $E \otimes_\mathbf{Q} K$-module. For every $\tau \in J_K^0$, we have

$$I_\tau^+ : (H_{\tau,B}(M) \oplus H_{\rho\tau,B}(M))^+ \otimes_\mathbf{Q} \mathbf{C} \cong (H_{DR}^+(M) \oplus \rho H_{DR}^+(M)) \otimes_{K,\tau} \mathbf{C}$$

as $E \otimes_\mathbf{Q} \mathbf{C}$-modules. Fix a basis of $H_{DR}^+(M) \oplus \rho H_{DR}^+(M)$ as a free $E \otimes_\mathbf{Q} K$-module of rank 1. The covariantly defined τ-period $c_\tau^+(M)$ is defined as the determinant of I_τ^+ with respect to this basis and a basis of $(H_{\tau,B}(M) \oplus H_{\rho\tau,B}(M))^+$ as an E-module. We shall show

$$c^+(R_{K/\mathbf{Q}}(M)) = (1 \otimes D_F^{1/2}) \prod_{\tau \in J_K^0} c_\tau^+(M) \qquad \text{mod } E^\times, \tag{6.28}$$

where F is the maximal real subfield of K and D_F is the discriminant of F. Put $V = H_{DR}^+(M)$. We may identify V as a direct factor of $E \otimes_\mathbf{Q} K$. Let $E \otimes_\mathbf{Q} K = A_1 \oplus A_2$, $A_1 \cong V$ as $E \otimes_\mathbf{Q} K$-modules. Let ϵ_1 and ϵ_2 be the idempotents corresponding to A_1 and A_2. Let v be a generator of $E \otimes_\mathbf{Q} K$. Let $\omega_1, \omega_2, \cdots, \omega_n$ be a basis of F over $\mathbf{Q}$, where $n = [F : \mathbf{Q}]$. Then

$$(1 \otimes \omega_1)\epsilon_1 v, \quad (1 \otimes \omega_2)\epsilon_1 v, \cdots, (1 \otimes \omega_n)\epsilon_1 v$$

give a basis of V as an E-module. In $(V \oplus \rho V) \otimes_{K,\tau} \mathbf{C}$, we have $(1 \otimes \omega_i)v = v \otimes \omega_i^\tau$. Since $\det(\omega_i^\tau)_{1 \le i \le n, \tau \in J_K^0} = D_F^{1/2}$, we obtain (6.28).

Now let N be a motive over K with coefficients in E. We assume that N is of rank 1. We shall determine $c^+(R_{K/\mathbf{Q}}(M \otimes N))$. We pose an assumption

$$F^-(M \otimes N) = F^-(M) \otimes N. \tag{6.29}$$

For $\tau \in J_K$, let

$$I_\tau : H_{\tau,B}(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{K,\tau} \mathbf{C}, \quad J_\tau : H_{\tau,B}(N) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(N) \otimes_{K,\tau} \mathbf{C}$$

be the canonical isomorphisms as $E \otimes_{\mathbf{Q}} \mathbf{C}$-modules. Let t and u be generators of $H_{\tau,B}(M)$ and $H_{\tau,B}(N)$ as E-modules respectively. Let v_1 and v_2 be generators of $H_{DR}(M)$ and $\rho H_{DR}(M)$ as free $E \otimes_{\mathbf{Q}} K$-modules respectively. We may put

$$I_\tau(t) = P_\tau v_1, \qquad I_{\rho\tau}(F_{\infty_\tau} t) = \bar{P}_\tau v_2 \tag{6.30}$$

where P_τ, $\bar{P}_\tau \in E \otimes_{\mathbf{Q}} \mathbf{C}$ and $\bar{P}_\tau$ is obtained by letting the complex conjugation act on the second argument of $E \otimes_{\mathbf{Q}} \mathbf{C}$. Since we take a basis v_1, we may identify $H_{DR}(M)$ with $E \otimes_{\mathbf{Q}} K$. Let ϵ_2 be the idempotent of $E \otimes_{\mathbf{Q}} K$ such that

$$F^-(M) = (E \otimes_{\mathbf{Q}} K)\epsilon_2 v_1.$$

Then we have

$$\rho F^-(M) = (E \otimes_{\mathbf{Q}} K)\epsilon_1 v_2, \qquad \epsilon_1 = 1 - \epsilon_2.$$

Hence we get

$$(P_\tau v_1, \bar{P}_\tau v_2) = c_\tau^+(M)(\epsilon_1 v_1 + \epsilon_2 v_2) \qquad \mathrm{mod}\ (E \otimes_{\mathbf{Q}} \mathbf{C})\epsilon_2 v_1 + (E \otimes_{\mathbf{Q}} \mathbf{C})\epsilon_1 v_2. \tag{6.31}$$

Let w be a generator of $H_{DR}(N)$ as a free $E \otimes_{\mathbf{Q}} K$-module. We have

$$J_\tau(u) = q_\tau w, \qquad J_{\rho\tau}(F_{\infty_\tau} u) = \bar{q}_\tau w \tag{6.32}$$

with q_τ, $\bar{q}_\tau \in E \otimes_{\mathbf{Q}} \mathbf{C}$, and

$$q_\tau \bar{q}_\tau = \delta_\tau(N). \tag{6.33}$$

Now we have

$$\begin{aligned} &((I_\tau \otimes J_\tau)(t \otimes u), ((I_{\rho\tau} \otimes J_{\rho\tau})(F_{\infty_\tau} t \otimes F_{\infty_\tau} u) \\ &= c_\tau^+(M) q_\tau(\epsilon_1 v_1 \otimes w) + c_\tau^+(M)\bar{q}_\tau(\epsilon_2 v_2 \otimes w) \\ &\quad \mathrm{mod}\ (E \otimes_{\mathbf{Q}} \mathbf{C})(\epsilon_2 v_1 \otimes w) + (E \otimes_{\mathbf{Q}} \mathbf{C})(\epsilon_1 v_2 \otimes w). \end{aligned}$$

Let $\epsilon_{i,\tau}$ be the image of ϵ_i in $E \otimes_{\mathbf{Q}} \mathbf{C} \cong (E \otimes_{\mathbf{Q}} K \otimes_{K,\tau} \mathbf{C})$. By the above formula and (6.29), we get

$$c_\tau^+(M \otimes N) = c_\tau^+(M)(q_\tau \epsilon_{1,\tau} + \bar{q}_\tau \epsilon_{2,\tau}). \tag{6.34}$$

Using (6.28), we obtain

$$c^+(R_{K/\mathbf{Q}}(M\otimes N))=(\prod_{\tau\in J_K^0}(q_\tau\epsilon_{1,\tau}+\bar{q}_\tau\epsilon_{2,\tau}))c^+(R_{K/\mathbf{Q}}(M)). \tag{6.35}$$

Let us consider $c^+(R_{K/\mathbf{Q}}(M^{\otimes l}))$. Assume that

$$F^-(M^{\otimes l})=\epsilon_2 H_{DR}(M^{\otimes l}). \tag{6.36}$$

Then we find easily that

$$\begin{aligned}(P_\tau v_1)^{\otimes l}&\equiv c_\tau^+(M)^l\epsilon_1 v_1^{\otimes l} &&\mod (E\otimes_{\mathbf{Q}}\mathbf{C})\epsilon_2 v_1^{\otimes l},\\ (\bar{P}_\tau v_1)^{\otimes l}&\equiv c_\tau^+(M)^l\epsilon_2 v_2^{\otimes l} &&\mod (E\otimes_{\mathbf{Q}}\mathbf{C})\epsilon_1 v_2^{\otimes l}.\end{aligned}$$

Hence we obtain

$$c_\tau^+(M^{\otimes l})=c_\tau^+(M)^l, \tag{6.37}$$

$$c^+(R_{K/\mathbf{Q}}(M^{\otimes l}))=c^+(R_{K/\mathbf{Q}}(M))^l(1\otimes D_F^{1/2})^{-(l-1)}. \tag{6.38}$$

We also have

$$\begin{aligned}c_\tau^+(M)(\epsilon_1-\epsilon_2)\epsilon_1 v_1&\equiv P_\tau v_1 &&\mod (E\otimes_{\mathbf{Q}}\mathbf{C})\epsilon_2 v_1,\\ c_\tau^-(M)(\epsilon_1-\epsilon_2)\epsilon_2 v_2&\equiv -\bar{P}_\tau v_2 &&\mod (E\otimes_{\mathbf{Q}}\mathbf{C})\epsilon_1 v_2.\end{aligned}$$

Assume

$$F^-(M)=F^+(M). \tag{6.39}$$

Then we obtain

$$c_\tau^-(M)=c_\tau^+(M)(\epsilon_{1,\tau}-\epsilon_{2,\tau}), \tag{6.40}$$

$$c^-(R_{K/\mathbf{Q}}(M))=\prod_{\tau\in J_K^0}(\epsilon_{1,\tau}-\epsilon_{2,\tau})c^+(R_{K/\mathbf{Q}}(M)). \tag{6.41}$$

6.4. Let K be an imaginary quadratic field. Let λ be a Grössencharacter of conductor $\mathfrak{f}$ of K which satisfies

$$\lambda((\alpha))=(\frac{\alpha^\rho}{|\alpha|})^k,\qquad \alpha\equiv 1 \mod {}^\times\mathfrak{f}.$$

We put $\lambda_0(\mathfrak{a})=\lambda(\mathfrak{a})N(\mathfrak{a})^{k/2}$ for an integral ideal $\mathfrak{a}$ of K. Then λ_0 is a Grössencharacter of conductor $\mathfrak{f}$ of K such that

$$\lambda_0((\alpha))=(\alpha^\rho)^k,\qquad \alpha\equiv 1 \mod {}^\times\mathfrak{f}.$$

We have $L(s,\lambda) = L(s+\frac{k}{2},\lambda_0)$. Let E be the algebraic number field generated by the values of λ_0. There exists a motive of rank 1 $M(\lambda_0)$ over K with coefficients in E such that $L(s,M(\lambda_0)) = (L(s,\lambda_0^\sigma))_{\sigma\in J_E}$. By Deligne's conjecture proved by Blasius [B] in this case, we have

$$(6.42) \qquad L(m,M(\lambda_0)) \equiv c^+(R_{K/\mathbf{Q}}(M(\lambda_0)\otimes T(m))) \qquad \mod E^\times$$

for $m\in\mathbf{Z}$, $1\le m\le k$ where $T(m)$ denotes the Tate motive. In the notation of (6.32), (6.33), we have $q_\tau = 1\otimes(2\pi\sqrt{-1})^m$ for every τ taking $N=T(m)$. If m is even, we have

$$(6.43\text{a}) \qquad c^+(R_{K/\mathbf{Q}}(M(\lambda_0)\otimes T(m))) = c^+(R_{K/\mathbf{Q}}(M(\lambda_0)))(1\otimes(2\pi\sqrt{-1})^m),$$

by (6.28) and (6.34). If m is odd, we have

$$(6.43\text{b}) \qquad c^+(R_{K/\mathbf{Q}}(M(\lambda_0)\otimes T(m))) = c^-(R_{K/\mathbf{Q}}(M(\lambda_0)))(1\otimes(2\pi\sqrt{-1})^m),$$

by (6.28), (6.35) and (6.41).

We are going to determine idempotents ϵ_1 and ϵ_2. First we note that $E\supseteq K$. Hence $E\otimes_{\mathbf{Q}} K = A_1\oplus A_2$, where A_i, $i=1,2$ are fields. We see that $\frac{1}{2}(1\otimes 1\pm\frac{1}{d}\sqrt{-d}\otimes\sqrt{-d})$ give the idempotents in $E\otimes_{\mathbf{Q}} K$. We have

$$E\otimes_{\mathbf{Q}} K \cong E[X]/(X^2+d)\cong E[X]/(X-\sqrt{-d})\oplus E[X]/(X+\sqrt{-d})\cong E\oplus E.$$

We may assume that $A_1 = E[X]/(X-\sqrt{-d})$, $A_2 = E[X]/(X+\sqrt{-d})$. The projection from $E[X]/(X^2+d)$ to A_1 is obtained by sending X to $\sqrt{-d}$. The structure function φ of A_1 is given by $\varphi(g)=1$ if $g|K=\text{id}$, $\varphi(g)=0$ if $g|K=\rho$, We also see easily that

$$A_1 = \frac{1}{2}(1\otimes 1-\frac{1}{d}\sqrt{-d}\otimes\sqrt{-d})(E\otimes_{\mathbf{Q}} K).$$

Let $F^m(M(\lambda_0))$ be the Hodge filtration of $H_{DR}(M(\lambda_0))$. We have $F^m(M(\lambda_0))=\{0\}$ if $m>k$. For $1\le m\le k$, let ψ_m be the structure function of $F^m(M(\lambda_0))$. Then we have

$$\varphi(g)=1 \quad\text{if}\quad g|K=\rho, \qquad \varphi(g)=0\quad\text{if}\quad g|K=\text{id}.$$

Hence we have $F^m(M(\lambda_0)) = A_2$ for $1\le m\le k$. Put

$$\epsilon_1 = \frac{1}{2}(1\otimes 1-\frac{1}{d}\sqrt{-d}\otimes\sqrt{-d}), \qquad \epsilon_2 = \frac{1}{2}(1\otimes 1+\frac{1}{d}\sqrt{-d}\otimes\sqrt{-d}).$$

For $\sigma\in\text{Aut}(\mathbf{C})$, we understand that λ^σ is defined by $\lambda^\sigma(\mathfrak{a}) = (\lambda_0(\mathfrak{a}))^\sigma N(\mathfrak{a})^{k/2}$. We put

$$(6.44) \qquad \left(\frac{L(m/2,\lambda)}{\pi^{m/2}P^{k/2}\sqrt{d}^{(k-m)/2}}\right)^\sigma = \zeta(\lambda,\sigma,m)\frac{L(m/2,\lambda^\sigma)}{\pi^{m/2}P^{k/2}\sqrt{d}^{(k-m)/2}}, \qquad \sigma\in\text{Aut}(\mathbf{C})$$

when $L(m/2,\lambda)\ne 0$. We note that $L(m/2,\lambda)\ne 0$ is equivalent to $L(m/2,\lambda^\sigma)\ne 0$ if $m/2$ is critical, by (6.42) for example.

Lemma 6.3. *Assume $L(m/2,\lambda) \neq 0$. Then for $m' \in \mathbf{Z}$, $m' \equiv k \mod 2$, $-k < m' \leq k$, we have*

$$\left(\frac{L(m'/2,\lambda)}{\pi^{(m'-m)/2}\sqrt{d}^{(m'-m)/2}L(m/2,\lambda)}\right)^{\sigma} = \frac{L(m'/2,\lambda^{\sigma})}{\pi^{(m'-m)/2}\sqrt{d}^{(m'-m)/2}L(m/2,\lambda^{\sigma})}$$

for $\sigma \in Aut(\mathbf{C})$ such that $\sigma|K = id$.

Proof. Assume $m \equiv m' \mod 4$. By (6.43a, b), we have

$$\frac{c^{+}(R_{K/\mathbf{Q}}(M(\lambda_0)) \otimes T((k+m')/2))}{c^{+}(R_{K/\mathbf{Q}}(M(\lambda_0)) \otimes T((k+m)/2))} \equiv 1 \otimes (2\pi\sqrt{-1})^{(m'-m)/2} \qquad \mod E^{\times}.$$

By (6.42), the assertion follows. Next assume $4 \nmid m - m'$. Then we have

$$\frac{c^{+}(R_{K/\mathbf{Q}}(M(\lambda_0)) \otimes T((k+m')/2))}{c^{+}(R_{K/\mathbf{Q}}(M(\lambda_0)) \otimes T((k+m)/2))} \equiv (\epsilon_{1,\tau} - \epsilon_{2,\tau})(1 \otimes (2\pi\sqrt{-1})^{(m'-m)/2})$$

$$\equiv -\frac{1}{d}(\sqrt{-d} \otimes \sqrt{d}\, i)(1 \otimes (2\pi\sqrt{-1})^{(m'-m)/2}) \equiv (\sqrt{-d} \otimes \pi^{(m'-m)/2}\sqrt{d}) \qquad \mod E^{\times}.$$

The σ-component of $(\sqrt{-d} \otimes \sqrt{d})(1 \otimes \pi^{(m'-m)/2})$ in $E \otimes_{\mathbf{Q}} \mathbf{C} \cong (\mathbf{C})_{\sigma \in J_E}$ does not depend on σ if $\sigma|K = \mathrm{id}$. Hence the assertions follows from (6.42). This completes the proof.

Corollary 6.4. *If $L(m/2,\lambda) \neq 0$, then (6.44) holds for m' with $\zeta(\lambda,\sigma,m') = \zeta(\lambda,\sigma,m)$ for $\sigma \in Aut(\mathbf{C})$ such that $\sigma|K = id$.*

Lemma 6.5. *Assume $L(m/2,\lambda) \neq 0$. Let χ be a character of the ideal class group modulo $\mathfrak{f}'$ of K. Then we have*

$$\left(\frac{L(m/2,\lambda \otimes \chi)}{L(m/2,\lambda)}\right)^{\sigma} = \chi^{\sigma}(\sigma)^{-1} \cdot \frac{L(m/2,\lambda^{\sigma} \otimes \chi^{\sigma})}{L(m/2,\lambda^{\sigma})}$$

for $\sigma \in Aut(\mathbf{C})$ such that $\sigma|K = id$. Here we identify χ as the character of $Gal(\overline{\mathbf{Q}}/K)$ and put $\chi^{\sigma}(\sigma) = \chi^{\sigma}(\sigma|\overline{\mathbf{Q}})$.

Proof. There exists a finite abelian extension L of K such that χ factors through $\mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(L/K)$. Enlarging E if necessary, we may assume that χ takes values in E. Let $M(\chi)$ be the motive over K with coefficients in E attached to χ. We have

$$H_{\tau,B}(M(\chi)) = E, \qquad H_{DR}(M(\chi)) = (E \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^{\mathrm{Gal}(\overline{\mathbf{Q}}/K)}.$$

Here $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$ acts on E by χ and acts on $\overline{\mathbf{Q}}$ as the Galois group. We have $(E \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^{\mathrm{Gal}(\overline{\mathbf{Q}}/K)} = (E \otimes_{\mathbf{Q}} L)^{\mathrm{Gal}(L/K)}$. Put $G = \mathrm{Gal}(L/K)$. We can find an $\omega \in L$ so that $\sum_{\mu \in G} \chi(\mu) \otimes \mu(\omega) \neq 0$ and that this element gives a generator of $H_{DR}(M(\chi))$ as a free $E \otimes_{\mathbf{Q}} K$-module of rank 1. Taking $N = M(\chi)$, q_τ defined in §6.3 is given by

$$q_\tau = \sum_{\mu \in G} \chi(\mu) \otimes \tau\mu(\omega) \in E \otimes_{\mathbf{Q}} \mathbf{C}, \qquad \tau \in J_K.$$

Here we use the same letter τ for its extension to Aut(**C**); $q_\tau \mod E^\times$ does not depend on this extension.

By (6.34), we have

$$\begin{aligned} c_\tau^+(M(\lambda_0)\otimes T(m)\otimes M(\chi)) &= c_\tau^+(M(\lambda_0)\otimes T(m)) \\ &\times \{\sum_{\mu\in G}\chi(\mu)\otimes\mu(\omega)\times\frac{1}{2}(1\otimes 1-\frac{1}{d}\sqrt{-d}\otimes\sqrt{d}\,i) \\ &+\sum_{\mu\in G}\chi(\mu)\otimes\rho\mu(\omega)\times\frac{1}{2}(1\otimes 1+\frac{1}{d}\sqrt{-d}\otimes\sqrt{d}\,i)\}, \end{aligned}$$

for $\tau = \mathrm{id}$. For $\sigma\in J_E$, $\sigma|K=\mathrm{id}$, the σ-component of $q_\tau\epsilon_{1,\tau}$ is

$$\sum_{\mu\in G}\chi^\sigma(\mu)\cdot\mu(\omega)\times\frac{1}{2}(1-\frac{1}{d}\sqrt{d}\,i\times\sqrt{d}\,i)=\chi^\sigma(\sigma)\sum_{\mu\in G}\chi^\sigma(\mu)\sigma\mu(\omega).$$

We see that the σ-component of $\bar{q}_\tau\epsilon_{2,\tau}$ is 0. Hence the assertion follows. (We should take the inverse of $\chi^\sigma(\sigma)$.)

Corollary 6.6. *If $L(m/2,\lambda)\neq 0$, then (6.44) holds for $\lambda\otimes\chi$ with $\zeta(\lambda\otimes\chi,\sigma,m) = \chi^\sigma(\sigma)^{-1}\zeta(\lambda,\sigma,m)$ for $\sigma\in Aut(\mathbf{C})$ such that $\sigma|K=id$.*

Lemma 6.7. *Assume $L(m/2,\lambda)\neq 0$. Let l be a positive integer. Then we have*

$$\left(\frac{L(lm/2,\lambda^l)}{L(m/2,\lambda)^l}\right)^\sigma=\frac{L(lm/2,(\lambda^\sigma)^l)}{L(m/2,\lambda^\sigma)^l}$$

for $\sigma\in Aut(\mathbf{C})$ such that $\sigma|K=id$.

Proof. We have $(M(\lambda_0)\otimes T(m))^{\otimes l}\cong M(\lambda_0^l)\otimes T(m)$. Now the assertion follows from (6.38) and (6.42).

Corollary 6.8. *If $L(m/2,\lambda)\neq 0$, then (6.44) holds for λ^l with $\zeta(\lambda^l,\sigma,lm)=\zeta(\lambda,\sigma,m)^l$ for $\sigma\in Aut(\mathbf{C})$ such that $\sigma|K=id$.*

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