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ON EXISTENCE OF VISCOSITY SOLUTIONS AND WEAK SOLUTIONS TO THE CAUCHY PROBLEM FOR $u_t = u\Delta u - \gamma|\nabla u|^2$

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ABSTRACT. we consider the following Cauchy problem

$$\begin{aligned} &u_t = u\Delta u - \gamma|\nabla u|^2 \quad \text{in } Q_T, \\ &u(x,0) = u_0(x), \end{aligned}$$

where $N \geq 1$, $T > 0$, $\gamma \in \mathbb{R}$, $Q_T = \mathbb{R}^N \times (0,T)$ and u_0 is a nonnegative function on $\mathbb{R}^N$. We establish the existence theorems of nonnegative viscosity solutions under very weak assumptions on u_0 for any $\gamma \in \mathbb{R}$. We also investigate equivalence between viscosity solutions and weak solutions without SSH conditions.

1. INTRODUCTION

Consider the following Cauchy problem

$$u_t = u\Delta u - \gamma|\nabla u|^2 \quad \text{in } Q_T, \tag{1.1}$$

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where $N \geq 1$, $T > 0$, $\gamma \in \mathbb{R}$, $Q_T = \mathbb{R}^N \times (0,T)$ and u_0 is a nonnegative function on $\mathbb{R}^N$.

We define the upper and lower semicontinuous envelopes u^*, u_* of u by

$$u^*(z) = \lim_{r\to 0} \sup\{u(z');\ z' \in B_r(z)\},$$

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and

$$u_*(z) = \lim_{r\to 0} \inf\{u(z');\ z' \in B_r(z)\},$$

respectively,
where if $z \in \mathbb{R}^N$, $B_r(z) = \{z';\ |z-z'| < r\}$ and if $z = (x,t) \in \mathbb{R}^N \times [0,\infty)$, $B_r(z) = \{z' = (y,s);\ (|x-y|^2+|t-s|)^{1/2} < r\}$. Note that u^* is upper semicontinuous and if u is upper semicontinuous, $u = u^*$. Similarly, u_* is lower semicontinuous and if u is lower semicontinuous, $u = u_*$. We define viscosity solutions and weak solutions of (1.1) as follows:

Definition 1.1. *Let u be a locally bounded function on Q_T. we say that u is a viscosity subsolution of* (1.1) *in Q_T if u^* satisfies that for $(x,t) \in Q_T$ and $(a,p,X) \in \mathcal{P}^{2,+}u^*(x,t)$,*

$$a \le u^*(x,t)\mathrm{Tr}X - \gamma|p|^2.$$

u is a viscosity supersolution of (1.1) *in Q_T if u_* satisfies that for $(x,t) \in Q_T$ and $(a,p,X) \in \mathcal{P}^{2,-}u_*(x,t)$,*

$$a \ge u_*(x,t)\mathrm{Tr}X - \gamma|p|^2.$$

u is a viscosity solution of (1.1) *in Q_T if u is a viscosity supersolution and viscosity subsolution.*

Here, for $(x,t) \in \mathbb{R}^N \times [0,\infty)$

$$\begin{aligned}\mathcal{P}^{2,+}u^*(x,t) &= \{(a,p,X);\ u^*(y,s) \le u^*(x,t) + a(s-t) + \langle p, y-x\rangle \\ &\quad + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|y-x|^2+|s-t|) \\ &\qquad \text{as } (y,s) \to (x,t)\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}^{2,-}u_*(x,t) &= \{(a,p,X);\ u_*(y,s) \ge u_*(x,t) + a(s-t) + \langle p, y-x\rangle \\ &\quad + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|y-x|^2+|s-t|) \\ &\qquad \text{as } (y,s) \to (x,t)\},\end{aligned}$$

Definition 1.2. *$u \in L^\infty_{loc}(Q_T)$ is said to be a weak solution of* (1.1) *in Q_T if $\nabla u \in L^2_{loc}(Q_T)$ and it holds that*

$$\int_{Q_T} \Big[-u\psi_t + u\nabla u\cdot\nabla\psi + (\gamma+1)|\nabla u|^2\psi \Big]dxdt = 0,$$

for every $\psi \in C^1_0(Q_T)$.

The approach by "viscosity" solutions is difficult for this problem because of its degenerating property, that is, the coefficient u of the term Δu. Our first purpose is to establish the "viscosity" approach for it. On the other hand, "weak" solutions for the diffusion equations like this is studied by many authors. Our second purpose is to investigate the relation to our "viscosity" approach and the "weak" solutions.

Bertsch, D. Passo and Ughi has shown the existence of discontinuous "viscosity solutions" of (1.1)–(1.2) ([3]). They use "viscosity solutions" to indicate (weak or strong) solutions constructed by the method of vanishing viscosity. We only use the term here to indicate the one introduced by Crandall and Lions.

The existence of viscosity solutions and weak solutions with the semi superharmonic (SSH in short) condition is proved by [2] and [6]. Here, we say that a function u satisfies the SSH condition if $\Delta u \leq K$ in $\mathcal{D}'$ for some constants K (or if u is a viscosity subsolution of $K - \Delta u = 0$).

Theorem 1.3 ([6]). *Let $T > 0$, $N \geq 1$ and $u_0 \in C(\mathbb{R}^N)$ satisfy*

$$0 \leq u_0 \leq M(|x|^2 + 1), \tag{1.3}$$

$$|\nabla u_0| \leq K_1(|x| + 1) \quad a.e.\ in\ \mathbb{R}^N, \tag{1.4}$$

$$\Delta u_0 \leq K_2 \quad in\ \mathcal{D}', \tag{1.5}$$

for some constants $M > 0$, K_1 and $K_2 > 0$. If $\gamma \geq N/2$, then there is a nonnegative function $u \in C(\overline{Q_T})$ such that u is the unique nonnegative viscosity solution and weak solution of (1.1)–(1.2) *and satisfies that for some constants M_1, $K_3 > 0$,*

$$\begin{aligned} &0 \leq u(x,t) \leq M_1(|x|^2 + 1), \\ &|\nabla u| \leq K_3(|x| + 1) \qquad a.e.\ in\ \mathbb{R}^N, \\ &\Delta u \leq K_2 \qquad in\ \mathcal{D}', \end{aligned}$$

where M_1 depends only on M, N and T.

Theorem 1.4 ([2]). *Let $N \geq 1$ and $u_0 \in C(\mathbb{R}^N)$ satisfy* (1.3), (1.4) *and* (1.5). *Then there exist $T > 0$, $L_0 > 0$ and a nonnegative function $u \in C(\hat{Q}_T)$ such that u is the unique nonnegative viscosity solution of* (1.1)–(1.2) *and satisfies that*

$$\begin{aligned} &0 \leq u(x,t) \leq \Psi_{T,M}(t)(|x|^2 + 1), \\ &|\nabla u|^2 \leq \Psi_{T,L_0}(t)u \qquad s.e.\ in\ \mathbb{R}^N, \\ &\Delta u \leq (1 + \gamma_+)\Psi_{T,L_0}(t) \qquad in\ \mathcal{D}', \end{aligned}$$

where $\hat{Q}_T = \mathbb{R}^N \times [0,T)$, $\Psi_{T,C}(t) = C/(1 - T^{-1}t)$ for $C = M$ or L_0 and $\gamma_+ =$ $\max(\gamma, 0)$.

To prove theorem 1.3 and 1.4, we consider the following Cauchy problem

$$u_t = (u+\varepsilon)\Delta u - \gamma|\nabla u|^2 \quad \text{in } Q_T, \tag{1.6}$$

$$u(x,0) = u_0^\varepsilon(x) \qquad x \in \mathbb{R}^N, \tag{1.7}$$

where $\{u_0^\varepsilon\} \subset C^\infty(\mathbb{R}^N)$ such that $u_0^\varepsilon \to u_0$ in $C(\mathbb{R}^N)$ as $\varepsilon \downarrow 0$ and u_0^ε holds (1.3), (1.4) and (1.5) for $0 < \varepsilon < 1$. Then it is shown that the Cauchy problem (1.6)–(1.7) has a smooth solution u_ε for $0 < \varepsilon < 1$ and the sequence of smooth solutions $\{u_\varepsilon\}$ converges uniformly to the viscosity solution of (1.1)–(1.2) under the assumptions of theorem 1.3.

Below, section 2 is devoted to state our main results and to establish the existence of viscosity solutions of (1.1) which are lower semicontinuous and satisfy the initial condition

$$u(x,0) = u_{0*}(x) \quad x \in \mathbb{R}^N \tag{1.8}$$

instead of (1.2). For constructing solutions, we use the inf-convolution approximation of initial functions and apply the results of theorem 1.3 and 1.4 which establish the existence in a rather narrow function classes. Our method is not used before so long as we know.

In section 3, we discuss the behavior of viscosity (sub) solutions. We show that every viscosity solution u satisfies $\limsup_{t\to 0} u(x,t) \le u_0^*(x)$ for $x \in \mathbb{R}^N$. This implies that if u_0 is continuous, then the viscosity solution constructed in section 2 is really viscosity solution of (1.1)–(1.2) and satisfies $\lim_{t\to 0} u(x,t) = u_0(x)$ for any $x \in \mathbb{R}^N$. If u_0 is piecewise continuous, then $\lim_{t\to 0} u(x,t) = u_0(x)$ almost everywhere in $\mathbb{R}^N$.

In section 4, we consider equivalence between viscosity solutions and weak solutions.

2. Existence Results

In this section we assume that the initial function u_0 is just a real valued function and satisfies

$$0 \le u_0(x) \le M(|x|^2+1) \quad x \in \mathbb{R}^N, \tag{2.1}$$

for some constants $M > 0$. Note that we don't assume the continuity of u_0. Our purpose of this section is to construct the viscosity solutions and weak solutions of (1.1) with the initial condition (1.8) in $LSC(\hat{Q}_T)$, where $LSC(\hat{Q}_T)$ is a set of lower semicontinuous functions in $\hat{Q}_T$. The main results in this section are the following theorems.

Theorem 2.1. *Let $N = 1$ and u_0 satisfy* (2.1).

(i) *If* $\gamma \geq 1/2$, *for any* $T > 0$, *there exists* $u \in C(Q_T) \bigcap LSC(\overline{Q_T})$ *such that* u *is a nonnegative viscosity solution of* (1.1)–(1.8). *Moreover,* $\limsup_{t\to 0} u(x,t) \leq u_0^*(x)$ *for* $x \in \mathbb{R}$.

(ii) *If* $\gamma < 1/2$, *there exist a* $T > 0$ *and* $u \in C(Q_T) \bigcap LSC(\hat{Q}_T)$ *such that* u *is a nonnegative viscosity solution of* (1.1)–(1.8). *Moreover,* $\limsup_{t\to 0} u(x,t) \leq u_0^*(x)$ *for* $x \in \mathbb{R}$.

Theorem 2.2. *Let* $N \geq 2$ *and* $\gamma \geq N/2$. *If* u_0 *satisfies* (2.1) *then there exists* $u \in LSC(\overline{Q_T}) \bigcap L^\infty_{loc}(\overline{Q_T})$ *such that* u *is a nonnegative viscosity solution of* (1.1)–(1.8) *in* Q_T. *Moreover,* $\limsup_{t\to 0} u(x,t) \leq u_0^*(x)$ *for* $x \in \mathbb{R}^N$.

Remark 2.3. The viscosity solution u of (1.1)–(1.2) constructed in Theorem 2.1 or 2.2 is a weak solution and satisfies (i), (ii) and (iii) in proposition 2.4. Moreover, (iii) in proposition 2.4 implies that if $\gamma > N/2$ then u in theorem 2.2 is continuous.

The behavior near $t = 0$ is discussed in section 3 (theorem 3.2).

To prove the existence part of our theorems, we need the following notation: for $u \in LSC(\mathbb{R}^N)$,

$$u_\varepsilon(x) = \inf_{y\in\mathbb{R}^N} \left\{ u(y) + \frac{1}{2\varepsilon}|x-y|^2 \right\}.$$

This u_ε is called the inf-convolution of u. Then u_ε is semiconcave. This implies that $u_{0\varepsilon}$ satisfies (1.3), (1.4) and (1.5) (see remark 2.3 in [2]).

Now, let u_{0*} be the lower semicontinuous envelope of u_0 and $u_{0\varepsilon}$ be the inf-convolution of u_{0*} for any $\varepsilon > 0$. We consider the the Cauchy problem (1.1) with the initial condition

$$u(x,0) = u_{0\varepsilon}(x) \quad x \in \mathbb{R}^N. \tag{2.2}$$

By theorem 1.3 and 1.4, it has a viscosity and weak solution u_ε. We want to take the limit as ε to 0. However, the estimates of ∇u_ε and the terminal time T in theorem 1.4 depend on the constants K_1 and K_2 appeared in (1.4) and (1.5), which might be infinite as $\varepsilon \to 0$. Therefore, first we have estimate them so as not to depend on K_1 and K_2.

Proposition 2.4. *Let* $N \geq 1$, $\gamma \in \mathbb{R}$ *and* u *be the solution in theorem* 1.3 *and* 1.4. *Then the following properties hold.*

(i) $\Delta u \geq -\dfrac{1}{t}$ *in* $\mathcal{D}'$.

(ii) *For any* $R > 0$ *and* $0 < s < T$, *there exists a constant* $C > 0$ *such that*

$$\int_0^s \int_{B_R} |\nabla u|^2 \, dxdt \leq C,$$

where C *depends on* N, γ, M, T, R *and* s *only.*

(iii) *If* $\gamma > N/2$,

$$\Delta u \leq \frac{N}{(2\gamma - N)t} \quad \textit{in } \mathcal{D}',$$
$$|\nabla u|^2 \leq \frac{2u}{(2\gamma - N)t} \quad \textit{in } \mathcal{D}'.$$

Remark 2.5. (i) in proposition 2.4 implies that Δu is a Radon measure.

To prove this proposition, we show the following lemma.

Lemma 2.6. *Let $\varepsilon > 0$ and u^ε be a smooth solution of* (1.6). *Then the following properties hold.*

(i) $\Delta u^\varepsilon \geq -\dfrac{1}{t}$.

(ii) *Let $R > 0$ be any fixed number, $0 \leq s < T$ and $M_{R,s}$ be a positive number such that $0 \leq u^\varepsilon \leq M_{R,s}$ holds on $B_{R+1} \times [0, s]$ for any $\varepsilon > 0$. Then for $0 < \alpha < 1$ with $\alpha \neq \gamma + 1$, there exists a constant $C > 0$ such that*

$$\int_0^s \int_{B_R} \frac{|\nabla u^\varepsilon|^2}{(u^\varepsilon + \varepsilon)^\alpha}\, dxdt \leq C, \tag{2.3}$$

where C depend on N, γ, R, s, $M_{R,s}$ and α only.

(iii) *If* $\gamma > N/2$,

$$\Delta u^\varepsilon \leq \frac{N}{(2\gamma - N)t} \quad \textit{in } \mathcal{D}',$$
$$|\nabla u^\varepsilon|^2 \leq \frac{2u^\varepsilon}{(2\gamma - N)t} \quad \textit{in } \mathcal{D}'.$$

Proof. (i) for $\gamma \geq 0$ and (iii) is proved by [4]. Moreover, by applying the maximum principle we can prove (i) for $\gamma < 0$ in an analogous way to that for the porous medium equation (see [1]).

Finally, we prove (ii). Let $v^\varepsilon = u^\varepsilon + \varepsilon$ and $\phi \in C_0^\infty(B_{R+1})$ satisfy that $0 \leq \phi \leq 1$ in B_{R+1}, $\phi = 1$ in B_R and $|\Delta\phi| \leq 1$ in B_{R+1}. Then

$$\begin{aligned} 0 &= \int_0^s \int_{B_{R+1}} \Big\{ v_t^\varepsilon - v^\varepsilon \Delta v^\varepsilon + \gamma |\nabla v^\varepsilon|^2 \Big\} (v^\varepsilon)^{-\alpha} \phi \, dxdt \\ &= \int_0^s \int_{B_{R+1}} \Big\{ \frac{((v^\varepsilon)^{1-\alpha})_t}{1-\alpha} \phi + (1 - \alpha + \gamma) \frac{|\nabla v^\varepsilon|^2}{(v^\varepsilon)^\alpha} \phi + (v^\varepsilon)^{1-\alpha} \nabla v^\varepsilon \cdot \nabla\phi \Big\} dxdt \\ &= \int_0^s \int_{B_{R+1}} \Big\{ \frac{((v^\varepsilon)^{1-\alpha})_t}{1-\alpha} \phi + (1 - \alpha + \gamma) \frac{|\nabla v^\varepsilon|^2}{(v^\varepsilon)^\alpha} \phi - \frac{(v^\varepsilon)^{2-\alpha}}{2-\alpha} \Delta\phi \Big\} dxdt. \end{aligned}$$

Hence,

$$|1-\alpha+\gamma|\int_0^s\int_{B_R}\frac{|\nabla v^\varepsilon|^2}{(v^\varepsilon)^\alpha}dxdt \;\le\; \Big(\frac{2M_{R,s}}{1-\alpha}+\frac{M_{R,s}^{2-\alpha}\cdot s}{2-\alpha}\Big)m(B_{R+1}),$$

where $m(B_{R+1})$ is Lebesgue measure of B_{R+1}. □

Proof of proposition 2.4. (i) follows from lemma 2.6 (i). By lemma 2.6 and theorem 1.3, we have

$$\int_0^s\int_{B_R}|\nabla u^\varepsilon|^2\,dxdt \le \begin{cases}(M_1(R^2+1))^\alpha C, & \gamma\ge N/2,\\ (\Psi_{T,M}(s)(R^2+1))^\alpha C & \gamma<N/2,\end{cases} \tag{2.4}$$

which implies that (ii) holds valid. By (2.4) we see that $|\nabla u^\varepsilon|^2\to|\nabla u|^2$ in $\mathcal{D}'$. Indeed, since $\nabla u^\varepsilon\rightharpoonup\nabla u$ weakly in $L^2_{loc}(Q_T)$, $\Delta u^\varepsilon\to\Delta u$ in $\mathcal{D}'$ and $u^\varepsilon\to u$ uniformly in $\overline{Q_T}$ or $\hat{Q}_T$, we have

$$\begin{aligned}\int_0^s\int_{B_R}|\nabla u^\varepsilon|^2\phi\,dxdt &= -\int_0^s\int_{B_R}u^\varepsilon\nabla u^\varepsilon\cdot\nabla\phi\,dxdt-\int_0^s\int_{B_R}\Delta u^\varepsilon u^\varepsilon\phi\,dxdt\\ &\to -\int_0^s\int_{B_R}u\nabla u\cdot\nabla\phi\,dxdt-\langle\Delta u,u\phi\rangle=\int_0^s\int_{B_R}|\nabla u|^2\phi\,dxdt,\end{aligned}$$

for any $\phi\in C_0^\infty(B_R\times(0,s))$. Hence (iii) holds. □

Next, when $N=1$ we prove that the terminal time T is independent of K_1 and K_2 in (1.4) and (1.5) for $\gamma<1/2$.

Theorem 2.7. *Let $N=1$ and $\gamma<1/2$. Assume that $u_0\in C(\mathbb{R})$ satisfies* (1.3) *and* (1.5). *Then there exist $T=T(M,\gamma)>0$ and $u\in C(\hat{Q}_T)$ such that u is a nonnegative viscosity solution of* (1.1)-(1.2) *and satisfies*

$$0\le u(x,t)\le\Psi_{T,M}(t)(|x|^2+1)\quad (x,t)\in\hat{Q}_T,$$

(i) *and* (ii) *in proposition* 2.4.

Proof. Let $T=[2M(1-2\gamma_-)]^{-1}$, where $\gamma_-=\min(\gamma,0)$ and u^ε be a smooth solution of (1.6)-(1.7). Then, since $\Psi_{T,M}(t)(|x|^2+1)$ is a supersolution of (1.1) and $\Psi_{T,M}(0)(|x|^2+1)=M(|x|^2+1)\ge u_0^\varepsilon\ge\varepsilon$, by the maximum principle, we have

$$\varepsilon\le u^\varepsilon+\varepsilon\le\Psi_{T,M}(t)(|x|^2+1)\quad\text{in }\hat{Q}_T.$$

Moreover, u^ε satisfies (i) and (ii) in Lemma 2.6 for any $\varepsilon>0$.

Next, put $p^\varepsilon=u^\varepsilon_{xx}$. Then p^ε is a solution of

$$p_t-u^\varepsilon p_{xx}-2(1-\gamma)u^\varepsilon_x p_x-(1-2\gamma)p^2=0. \tag{2.5}$$

It can be easily seen that $\Psi_{K,S}(t)$ is a supersolution of (2.5) in $\hat{Q}_S$, where $S=[K(1-2\gamma)]^{-1}$. Hence, by (1.5), $u^\varepsilon_{xx}\le\Psi_{K,S}(t)$ in $\hat{Q}_S$. Therefore u^ε is uniformly semiconcave in $\mathbb{R} \times [0,\delta)$ and is uniformly semiconvex in $\mathbb{R} \times [\delta/2, T)$ for some $0 < \delta < S$. This yields that it is locally equicontinuous in $\mathbb{R}^N \times [0,T)$. Therefore, there exists a subsequence $\{\varepsilon_i\}$ converging to 0 as $i \to \infty$ and $u \in C(\hat{Q}_T)$ such that $u^{\varepsilon_i} \to u$ uniformly in $\hat{Q}_T$ as $i \to \infty$ and $\nabla u^{\varepsilon_i} \rightharpoonup \nabla u$ weakly in $L^2_{loc}(Q_T)$. This u satisfies our requirement. $\square$

Remark 2.8. In theorem 2.7, since u^ε satisfies (i) in lemma 2.6 for any $\varepsilon > 0$, we get $|\nabla u^{\varepsilon_i}|^2 \rightharpoonup |\nabla u|^2$ weakly in the sense of measure, i.e.,

$$\int_{Q_T} |\nabla u^{\varepsilon_i}|^2 \psi \, dxdt \to \int_{Q_T} |\nabla u|^2 \psi \, dxdt \quad \text{for any } \psi \in C_0(Q_T).$$

Therefore, u in theorem 2.7 is a weak solution.

Proof of theorem 2.1. Let $u_{0\varepsilon}$ be the inf-convolution of u_{0*} for any $\varepsilon > 0$.

Let $\gamma \geq 1/2$ and $T > 0$ be arbitrary. By theorem 1.3, there exists a viscosity and weak solution $u_\varepsilon \in C(\overline{Q_T})$ satisfying the assertions in theorem 1.3 and proposition 2.4 for any $\varepsilon > 0$. This yields that $\{u^\varepsilon\}$ is locally uniform bounded and equicontinuous in $\mathbb{R} \times [\delta, T)$ for any $\delta > 0$. Therefore, there exist a subsequence $\{\varepsilon_i\}$ converging to 0 as $i \to \infty$ and $u \in C(\mathbb{R} \times [\delta, T))$ such that $u_{\varepsilon_i} \to u$ uniformly in $\mathbb{R} \times [\delta, T)$ as $i \to \infty$ for any $\delta > 0$. That is, $u \in C(Q_T)$ and it is a viscosity solution of (1.1). Further, $u_{0\varepsilon}(x) \to u_{0*}(x)$ as $\varepsilon \downarrow 0$ for $x \in \mathbb{R}$. This implies that $u(x,0) = u_{0*}(x)$ for $x \in \mathbb{R}$.

In the case $\gamma < 1/2$, by theorem 2.7, there is $T = T(M,\gamma) > 0$ and $u_\varepsilon \in C(\hat{Q}_T)$ such that u_ε is a viscosity and weak solution of (1.1)–(2.2) and satisfies the assertions in proposition 2.4 and theorem 2.7 for any $\varepsilon > 0$. In the same manner, there exists $u \in C(Q_T)$ such that it is a viscosity solution of (1.1) with initial data $u(x,0) = u_{0*}(x)$.

Finally, since $u_* \in LSC(\hat{Q}_T)$ and $u(x,0) = u_{0*}(x) \in LSC(\mathbb{R})$, for any $\eta > 0$ there exist $\delta_\eta > 0$ and $t_\eta > 0$ such that

$$u(y,s) \geq u_*(y,s) > u_*(x,0) - \eta = u(x,0) - \eta,$$

for any $x \in \mathbb{R}$, $y \in B_{\delta_\eta}(x)$ and $0 \leq t < t_\eta$. Hence, $u \in LSC(\hat{Q}_T) \cap C(Q_T)$. In particular, if $u_0 \in LSC(\mathbb{R})$ then $u(x,0) = u_{0*}(x) = u_0(x)$. $\square$

Proof of theorem 2.2. Let $u_{0\varepsilon}$ be the inf-convolution of u_{0*} for any $\varepsilon > 0$ and $T > 0$ be arbitrary. Then, by theorem 1.3, there is $u_\varepsilon \in C(\overline{Q_T})$ such that it is a viscosity solution and of (1.1)–(2.2).

We set $u(x,t) = \sup_{\varepsilon > 0} u_\varepsilon(x,t)$. Then $u \in LSC(\overline{Q_T}) \cap L^\infty_{loc}(\overline{Q_T})$, $u(x,0) = u_{0*}(x)$ for $x \in \mathbb{R}^N$. Perron's method yields that u^* is a viscosity subsolution of (1.1). Further, by the comparison theorem for viscosity solutions under the SSH condition (Theorem 3.1 in [6]), we see that $u_\varepsilon \uparrow u$ as $\varepsilon \downarrow 0$.

Finally, we prove by the contradiction that u is a viscosity supersolution of (1.1). Assume that there exists $(x_0,t_0)\in Q_T$ and $(a,p,X)\in \mathcal{P}^{2,-}_{Q_T}u(x_0,t_0)$ such that

$$a-u(x_0,t_0)\mathrm{Tr}X+\gamma|p|^2<0.$$

For μ and $\nu>0$, we set

$$\begin{aligned}u_{\mu,\nu}(x,t):=&\ u(x_0,t_0)+\mu+a(t-t_0)+\langle p,x-x_0\rangle\\ &+\frac{1}{2}\langle X(x-x_0),x-x_0\rangle-\nu(|x-x_0|^2+|t-t_0|).\end{aligned}$$

Then $u_{\mu,\nu}$ is a viscosity subsolution of (1.1) in $B_r(x_0,t_0)$ and $B_r(x_0,t_0)\subset Q_T$ for small enough μ, ν and $r>0$. Now, $u_{\mu,\nu}(x_0,t_0)=u(x_0,t_0)+\mu>u(x_0,t_0)$. And since

$$\begin{aligned}u(x,t)\ \ \geq&\ \ u(x_0,t_0)+a(t-t_0)+\langle p,x-x_0\rangle\\ &+\frac{1}{2}\langle X(x-x_0),x-x_0\rangle+o(|x-x_0|^2+|t-t_0|)\\ =&\ \ u_{\mu,\nu}(x,t)-\mu+\nu(|x-x_0|^2+|t-t_0|)+o(|x-x_0|^2+|t-t_0|),\end{aligned}$$

if $r>0$ is sufficiently small and $\mu=\dfrac{\nu}{8}r^2$ then there exists $d>0$ such that

$$u_{\mu,\nu}(x,t)+d\leq u(x,t)\quad (x,t)\in B=\overline{B_r(x_0,t_0)\setminus B_{r/2}(x_0,t_0)}. \tag{2.6}$$

Moreover, there exists $\varepsilon_1>0$ such that

$$u_{\mu,\nu}(x,t)\leq u_{\varepsilon_1}(x,t)\quad (x,t)\in B.$$

Indeed, assume that for any $\varepsilon>0$ there exists $(x_\varepsilon,t_\varepsilon)\in B$ such that

$$u_{\mu,\nu}(x_\varepsilon,t_\varepsilon)>u_\varepsilon(x_\varepsilon,t_\varepsilon).$$

We may also assume that there exist the subsequence $\{\varepsilon_i\}$ converging 0 as $i\to\infty$ and $(x_0,t_0)\in B$ such that $(x_{\varepsilon_i},t_{\varepsilon_i})\to(x_0,t_0)$. Since $u_{\mu,\nu}$ is continuous, by (2.6), there exists $\varepsilon_d>0$ such that for any $\varepsilon_i\in(0,\varepsilon_d)$,

$$\begin{aligned}u(x_0,t_0)\ \ \geq&\ \ u_{\mu,\nu}(x_0,t_0)+d\\ >&\ \ u_{\mu,\nu}(x_{\varepsilon_i},t_{\varepsilon_i})+\frac{d}{2}>u_{\varepsilon_i}(x_{\varepsilon_i},t_{\varepsilon_i})+\frac{d}{2}.\end{aligned}$$

Since $u\in LSC(\overline{Q_T})$, we have

$$u(x_0,t_0)\geq\liminf_{i\to\infty}u_{\varepsilon_i}(x_{\varepsilon_i},t_{\varepsilon_i})+\frac{d}{2}\geq u(x_0,t_0)+\frac{d}{2}.$$

This is a contradiction.

Then

$$U_{\varepsilon_1}(x,t)=\begin{cases}\max\{u_{\varepsilon_1}(x,t),u_{\mu,\nu}(x,t)\} & (x,t)\in B_r(x_0,t_0),\\ u_{\varepsilon_1}(x,t) & \text{otherwise},\end{cases}$$

is a viscosity subsolution of (1.1) with initial data

$$u(x,0) = u_{0\varepsilon_1}(x) \quad x \in \mathbb{R}^N$$

and $u_{\varepsilon_1} < U_{\varepsilon_1}$. This is a contradiction since the comparison theorem for viscosity solutions under the SSH condition holds valid. □

Proof of remark 2.3. Let u_ε be defined as in theorem 2.1 or 2.2. Then, since u_ε is a weak solution and satisfies (i) and (ii) in proposition 2.4, there is a subsequence $\{\varepsilon_i\}$ such that

$$\begin{aligned} \nabla u_{\varepsilon_i} &\rightharpoonup \nabla u && \text{weakly in } L^2_{loc}(Q_T), \\ u_{\varepsilon_i}\nabla u_{\varepsilon_i} &\rightharpoonup u\nabla u && \text{weakly in } L^2_{loc}(Q_T), \\ |\nabla u_{\varepsilon_i}|^2 &\rightharpoonup |\nabla u|^2 && \text{weakly in the sense of measure}, \\ \Delta u_{\varepsilon_i} &\to \Delta u && \text{in } \mathcal{D}'. \end{aligned}$$

as $\varepsilon_i \to 0$. Hence, the proof of (ii) is complete. □

3. The Behavior near $t = 0$

The purpose of this section is to establish the behavior for the viscosity (sub) solutions of (1.1)–(1.2) near $t = 0$. The viscosity solutions constructed in section 2 is lower semicontinuous at $t = 0$ and satisfies $u(x,0) = u_{0*}(x)$ for $x \in \mathbb{R}^N$. This implies that for any $y \in \mathbb{R}^N$

$$\liminf_{t\to 0} u(y,t) \geq u_{0*}(y).$$

We consider the estimates of $\limsup_{t\to 0} u(y,t)$ for any $y \in \mathbb{R}^N$. To do it, we prove

Theorem 3.1. *Let $N \geq 1$, $\gamma \in \mathbb{R}$, v be locally Lipschitz continuous in $\mathbb{R}^N$ and w be a viscosity subsolution of* (1.1). *Then, if $w(y,0) \leq v(y)$ for $y \in \mathbb{R}^N$,*

$$\limsup_{t\to 0} w(y,t) \leq v(y) \quad \textit{for } y \in \mathbb{R}^N. \tag{3.1}$$

Thus, we have

Theorem 3.2. *Let $N \geq 1$, $\gamma \in \mathbb{R}$, u_0 be locally bounded and u be a viscosity solution of* (1.1)-(1.2). *Then,*

$$\limsup_{t\to 0} u(y,t) \leq u_0^*(y) \quad \textit{for } y \in \mathbb{R}^N. \tag{3.2}$$

Proof. We set

$$u_0^\delta(x) = \sup_{y\in\mathbb{R}^N} \left\{ u_0(y) - \frac{1}{2\delta}|x-y|^2 \right\}.$$

Then, u_0^δ is locally Lipschitz continuous for any $\delta > 0$ and $u_0^\delta(x) \downarrow u_0^*(x)$ as $\delta \downarrow 0$ for $x \in \mathbb{R}^N$. Hence, from theorem 3.1, we see that for any $y \in \mathbb{R}^N$

$$\limsup_{t\to 0} u(y,t) \le u_0^\delta(y).$$

Letting $\delta \to 0$, we have (3.2). $\square$

Note that This theorem holds for all viscosity solutions of (1.1)–(1.2). Moreover, for the viscosity solutions constructed in section 2, the following holds.

Corollary 3.3. *Let $N \ge 1$, $\gamma \in \mathbb{R}$, u_0 be continuous and satisfy* (2.1). *Then, the viscosity solutions u constructed in section* 2 *satisfy*

$$\lim_{t\to 0} u(y,t) = u_0(y) \quad \text{for } y \in \mathbb{R}^N, \tag{3.3}$$

i.e., u is continuous at $t = 0$. Moreover, if u_0 is piecewise continuous, (3.3) *holds almost everywhere in $\mathbb{R}^N$.*

Proof. If u_0 is continuous at $y \in \mathbb{R}^N$, $u_0(y) = u_{0*}(y) = u_0^*(y)$. $\square$

Proof of theorem 3.1. We use the comparison theorem for viscosity solutions of (1.1) on bounded domains which can be established in a exactly analogous way to [2] and [6]. For sake of brevity, we don't state it here in a precise form.

Let $y \in \mathbb{R}^N$, $r_0 > 0$, $t \in (0,T)$ be fixed. We set

$$h_{y,\varepsilon}(x,t) = v(y) + \varepsilon + \frac{L_{r_0}^2}{\varepsilon}|x-y|^2 + e^{\lambda_\varepsilon t} - 1,$$

where $L_{r_0} = \|\nabla v\|_{L^\infty(B_{r_0}(y))}$, $0 < \varepsilon < 1$ and $\lambda_\varepsilon > 0$.

When $\gamma \ge N/2$,

$$\begin{aligned}
&(h_{y,\varepsilon})_t - h_{y,\varepsilon}\Delta h_{y,\varepsilon} + \gamma|\nabla h_{y,\varepsilon}|^2 \\
&= \lambda_\varepsilon e_{\lambda_\varepsilon} - \frac{2L_{r_0}^2 N}{\varepsilon}(v(y) + \varepsilon + e^{\lambda_\varepsilon} - 1) + \frac{2L_{r_0}^4}{\varepsilon^2}|x-y|^2(2\gamma - N) \\
&\ge e^{\lambda_\varepsilon}\Big[\lambda_\varepsilon - \frac{2L_{r_0}^2 N}{\varepsilon}(v(y)+1)\Big].
\end{aligned}$$

Hence, if $\lambda_\varepsilon = 2L_{r_0}^2\varepsilon^{-1}N(\|v\|_{L^\infty(B_{r_0}(y))} + 1)$, $h_{y,\varepsilon}$ is a classical supersolution of (1.1) in $B_{r_0}(y) \times (0,T)$. Now, for $x \in \partial B_{r_0}(y)$,

$$\begin{aligned}
w(x,0) - h_{y,\varepsilon}(x,0) &\le v(x) - h_{y,\varepsilon}(x,0) \\
&\le (v(x) - v(y)) - (\varepsilon + \frac{L_{r_0}^2}{\varepsilon}|x-y|^2) \\
&\le L_{r_0}|x-y| - (\varepsilon + \frac{L_{r_0}^2}{\varepsilon}|x-y|^2) \le 0.
\end{aligned}$$

Let $\varepsilon_0 = L_{r_0}^2 r_0^{1/2} M_0^{-1}$ and $r_\varepsilon = (\varepsilon M_0)^{1/2} L_{r_0}^{-1}$, which $M_0 = \|w\|_{L^\infty(B_{r_0}(y)\times[0,t_0])}$. Then, for any $0 < \varepsilon < \varepsilon_1 := \min(\varepsilon_0, 1)$,

$$\begin{aligned} h_{y,\varepsilon}(x,t) &\geq v(y) + \varepsilon + \frac{L_{r_0}^2}{\varepsilon} \cdot r_\varepsilon^2 + e^{\lambda_\varepsilon t} - 1 \\ &\geq M_0 \geq u(x,t), \end{aligned}$$

for $(x,t) \in \partial B_{r_0}(y) \times [0,t_0]$. Moreover, $\Delta h_{y,\varepsilon} = 2L_{r_0}^2 \varepsilon^{-1} N < +\infty$. Hence, by comparison theorem for viscosity solutions on bounded domains, we see that $w \leq h_{y,\varepsilon}$ in $B_{r_0}(y) \times [0,t_0]$ for $0 < \varepsilon < \varepsilon_1$. This implies that

$$w(x,t) \leq \inf_{0<\varepsilon<\varepsilon_1} h_{y,\varepsilon}(x,t) \quad (x,t) \in B_{r_0}(y) \times [0,t_0].$$

In particular, for $t \in [0,t_0]$,

$$\begin{aligned} w(y,t) &\leq \inf_{0<\varepsilon<\varepsilon_1} h_{y,\varepsilon}(y,t) \\ &= v(y) - 1 + \inf_{0<\varepsilon<\varepsilon_1} (\varepsilon + e^{\lambda_\varepsilon t}). \end{aligned}$$

Therefore, we have (3.1).

When $\gamma < N/2$,

$$\begin{aligned} &(h_{y,\varepsilon})_t - h_{y,\varepsilon}\Delta h_{y,\varepsilon} + \gamma|\nabla h_{y,\varepsilon}|^2 \\ &\geq e^{\lambda_\varepsilon}\Big[\lambda_\varepsilon - \frac{2L_{r_0}^2}{\varepsilon}\Big(N(v(y)+1) + \frac{L_{r_0}^2}{\varepsilon} \cdot r_0^2(N-2\gamma)\Big)\Big]. \end{aligned}$$

Hence, if $\lambda_\varepsilon = 2L_{r_0}^2 \varepsilon^{-1}[N(\|v\|_{L^\infty(B_{r_0}(y))} + 1) + L_{r_0}^2 r_0^2 \varepsilon^{-1}(N-2\gamma)]$, $h_{y,\varepsilon}$ is a viscosity supersolution of (1.1) in $B_{r_0}(y) \times (0,T)$. Therefore, in the same manner as the case $\gamma \geq N/2$, we can prove this theorem. □

4. Viscosity Solutions and Weak Solutions

First, we give the definition of weak subsolutions and supersolutions of (1.1).

Definition 4.1. *$u \in L^\infty_{loc}(Q_T)$ is a weak subsolution (resp. supersolution) of (1.1) in Q_T if $\nabla u \in L^2_{loc}(Q_T)$ and it holds that*

$$\int_{Q_T} \Big[- u\psi_t + u\nabla u \cdot \nabla\psi + (\gamma+1)|\nabla u|^2\psi \Big] dxdt \leq 0,$$

$$(\textit{resp.} \quad \int_{Q_T} \Big[- u\psi_t + u\nabla u \cdot \nabla\psi + (\gamma+1)|\nabla u|^2\psi \Big] dxdt \geq 0,)$$

for every $\psi \in C_0^1(Q_T)$ so that $\psi \geq 0$.

Theorem 4.2. *Let $N \geq 1$. If $u \in L^\infty_{loc}(Q_T) \bigcap USC(Q_T)$ is a viscosity subsolution of* (1.1), *$\nabla u \in L^2_{loc}(Q_T)$ and satisfies*

$$\Delta u \geq -\frac{1}{t} \quad \textit{in viscosity sense,}$$

then u is a weak subsolution of (1.1).

Theorem 4.3. *Let $N \geq 1$. If $u \in L^\infty_{loc}(Q_T) \bigcap LSC(Q_T)$ is a viscosity supersolution of* (1.1), *$\nabla u \in L^2_{loc}(Q_T)$ and satisfies*

$$\Delta u \leq f \quad \textit{in viscosity sense,}$$

for some nonnegative functions $f \in L^1_{loc}(Q_T)$, then u is a weak supersolution of (1.1).

Proof of theorem 4.2. Let Q be a bounded subset of Q_T with $\overline{Q} \subset\subset Q_T$. We choose $\lambda > 0$ and $\varepsilon > 0$ so that $Q_\lambda = \{(x,t); \mathrm{dist}((x,t),Q) \leq \lambda\} \subset Q_T$ and $\lambda > 2(\varepsilon L)^{1/2}$, where $L = \sup\limits_{\overline{Q_\lambda}}|u|$. We may assume that $Q \subset \mathbb{R}^N \times [\delta, T)$ for some $\delta > 0$, $u \in L^\infty(Q_\lambda)$, $\nabla u \in L^2(Q)$ and

$$\Delta u \geq -\frac{1}{\delta} \quad \text{in viscosity sense on } Q. \tag{4.1}$$

Let u^ε be the sup-convolution of u, i.e.

$$u^\varepsilon(x,t) = \sup_{(y,s)\in Q_\lambda}\left\{u(y,s) - \frac{1}{2\varepsilon}(|x-y|^2 + |t-s|^2)\right\}.$$

Then there exist $M^\varepsilon \in L^1(Q; \mathcal{S}(N))$ and $\mathcal{S}(N)$-valued measure Γ^ε on Q such that

$$\nabla^2 u^\varepsilon = M^\varepsilon + \Gamma^\varepsilon, \quad \Gamma^\varepsilon \geq 0 \quad \text{in } \mathcal{D}', \tag{4.2}$$

$$(u^\varepsilon_t(x,t), \nabla u^\varepsilon(x,t), M^\varepsilon(x,t)) \in \mathcal{P}^{2,+}u(y^\varepsilon, s^\varepsilon) \quad \text{a.e. in } Q, \tag{4.3}$$

$$u^\varepsilon(x,t) = u(y^\varepsilon, s^\varepsilon) - \frac{\varepsilon}{2}(|\nabla u^\varepsilon(x,t)|^2 + |u^\varepsilon_t(x,t)|^2), \tag{4.4}$$

where $y^\varepsilon = y^\varepsilon(x,t) = x + \varepsilon\nabla u^\varepsilon(x,t)$, $s^\varepsilon = s^\varepsilon(x,t) = t + \varepsilon u^\varepsilon_t(x,t)$ and $\mathcal{S}(N)$ is a set of $N \times N$ symmetric matrices (see [5] and [7]). Since u is a viscosity subsolution of (1.1), for almost all $(x,t) \in Q$,

$$u^\varepsilon_t(x,t) - u(y^\varepsilon, s^\varepsilon)\mathrm{Tr}M^\varepsilon(x,t) - \gamma|\nabla u^\varepsilon(x,t)|^2 \leq 0.$$

For the sake of simplicity we omit the independent variables (x,t) henceforth. Let $\psi \in C_0^\infty(Q)$ so that $\psi \geq 0$. Since, $\Gamma^\varepsilon \geq 0$ in $\mathcal{D}'$,

$$\int_Q u^\varepsilon_t \psi\, dxdt - \langle u(y^\varepsilon, s^\varepsilon)\Delta u^\varepsilon, \psi\rangle + \gamma\int_Q |\nabla u^\varepsilon|^2\psi\, dxdt \leq 0.$$

Here, we remark that Δu^ε is Radon measure for any $\varepsilon > 0$. Moreover, we have

$$\int_Q |\nabla u^\varepsilon|^2 \psi\, dxdt = -\langle u^\varepsilon \Delta u^\varepsilon, \psi\rangle + \frac{1}{2}\int_Q (u^\varepsilon)^2 \Delta\psi\, dxdt.$$

Hence,

$$\int_Q (-u^\varepsilon)\psi_t dxdt - \langle (u(y^\varepsilon, s^\varepsilon) + \gamma u^\varepsilon)\Delta u^\varepsilon, \psi\rangle + \frac{\gamma}{2}\int_Q (u^\varepsilon)^2 \Delta\psi\, dxdt \le 0. \tag{4.5}$$

Moreover, by (4.1),

$$\Delta u^\varepsilon \ge \mathrm{Tr} M^\varepsilon \ge -\frac{1}{\delta} \quad \text{in } \mathcal{D}'. \tag{4.6}$$

We choose $\lambda' > 0$ so that $Q_{\lambda'} = \{(x,t); \mathrm{dist}((x,t), Q_\lambda) \le \lambda'\} \subset Q_T$ and fix any $\varepsilon' > 0$ so that $\lambda' > 2(\varepsilon' L')^{1/2}$, where $L' = \sup_{\overline{Q_{\lambda'}}} |u|$. Since $u^{\varepsilon'}$ is locally Lipschitz continuous, $u^{\varepsilon'}(y^\varepsilon, s^\varepsilon) \to u^{\varepsilon'}$ uniformly in Q as $\varepsilon \to 0$. This implies that for any $\eta > 0$, $u^{\varepsilon'}(y^\varepsilon, s^\varepsilon) \le u^{\varepsilon'} + \eta$ in Q for small enough ε. Then, since $u^\varepsilon(x,t) \le u(y^\varepsilon, s^\varepsilon)$ and $u \le u^{\varepsilon'}$ in Q_λ, if $\gamma \ge 0$,

$$\begin{aligned}
&\langle (u(y^\varepsilon, s^\varepsilon) + \gamma u^\varepsilon)\Delta u^\varepsilon, \psi\rangle \\
= \;& \langle (u(y^\varepsilon, s^\varepsilon) + \gamma u^\varepsilon)(\Delta u^\varepsilon + \frac{1}{\delta}), \psi\rangle - \frac{1}{\delta}\int_Q (u(y^\varepsilon, s^\varepsilon) + \gamma u^\varepsilon)\psi\, dxdt \\
\le \;& (1+\gamma)\langle u^{\varepsilon'}(y^\varepsilon, s^\varepsilon)(\Delta u^\varepsilon + \frac{1}{\delta}), \psi\rangle - \frac{1+\gamma}{\delta}\int_Q u^\varepsilon \psi\, dxdt \\
\le \;& (1+\gamma)\langle (u^{\varepsilon'} + \eta)\Delta u^\varepsilon, \psi\rangle + \frac{1+\gamma}{\delta}\int_Q (u^{\varepsilon'} + \eta - u^\varepsilon)\psi\, dxdt.
\end{aligned}$$

Hence, by $\Delta u^\varepsilon \to \Delta u$ in $\mathcal{D}'$ and (4.5),

$$\begin{aligned}
\int_Q (-u\psi_t)\, dxdt - (1+\gamma)\langle (u^{\varepsilon'} + \eta)\Delta u, \psi\rangle + \frac{\gamma}{2}\int_Q u^2 \Delta\psi\, dxdt \\
- \frac{1+\gamma}{\delta}\int_Q (u^{\varepsilon'} + \eta - u)\psi\, dxdt \le 0.
\end{aligned}$$

By letting $\eta \to 0$ and $\varepsilon' \to 0$,

$$\int_Q (-u\psi_t)\, dxdt - (1+\gamma)\langle u\Delta u, \psi\rangle + \frac{\gamma}{2}\int_Q u^2\Delta\psi\, dxdt \le 0.$$

Since $\nabla u \in L^2(Q)$, we conclude that

$$\int_Q \Big(-u\psi_t + u\nabla u \cdot \nabla\psi + (\gamma+1)|\nabla u|^2 \psi\Big)\, dxdt \le 0. \tag{4.7}$$

When $\gamma < 0$, let $0 < \varepsilon < \varepsilon'$. Then, since $u^{\varepsilon'} \le u^\varepsilon$,

$$\begin{aligned}
&\langle (u(y^\varepsilon, s^\varepsilon) + \gamma u^\varepsilon)\Delta u^\varepsilon, \psi\rangle \\
\le \;& \langle ((1+\gamma)u^{\varepsilon'} + \eta)(\Delta u^\varepsilon + \frac{1}{\delta}), \psi\rangle - \frac{1+\gamma}{\delta}\int_Q u^\varepsilon \psi\, dxdt,
\end{aligned}$$

for small enough ε. Therefore, by the same manner as the case of $\gamma \geq 0$, by letting $\varepsilon \to 0$, $\eta \to 0$ and $\varepsilon' \to 0$, we have (4.7). This completes the proof. $\square$

Proof of theorem 4.3. Let Q, λ, ε, λ' and ε' be defined as in the proof of theorem 4.2.

Let u_ε be the inf-convolution of u, i.e.

$$u_\varepsilon(x,t) = \inf_{(y,s)\in Q_\lambda} \left\{ u(y,s) + \frac{1}{2\varepsilon}(|x-y|^2 + |t-s|^2) \right\}.$$

Then there exist $M_\varepsilon \in L^1(Q;\mathcal{S}(N))$ and $\mathcal{S}(N)$-valued measure Γ_ε on Q such that

$$\begin{gathered} \nabla^2 u_\varepsilon = M_\varepsilon + \Gamma_\varepsilon, \quad \Gamma_\varepsilon \leq 0 \quad \text{in } \mathcal{D}', \\ (u_{\varepsilon t}(x,t), \nabla u_\varepsilon(x,t), M_\varepsilon(x,t)) \in \mathcal{P}^{2,-}u(y_\varepsilon, s_\varepsilon) \quad \text{a.e. in } Q, \\ u_\varepsilon(x,t) = u(y_\varepsilon, s_\varepsilon) + \frac{\varepsilon}{2}(|\nabla u_\varepsilon(x,t)|^2 + |u_{\varepsilon t}(x,t)|^2), \end{gathered}$$

where $y_\varepsilon = y_\varepsilon(x,t) = x - \varepsilon\nabla u_\varepsilon(x,t)$, $s_\varepsilon = s_\varepsilon(x,t) = t - \varepsilon u_{\varepsilon t}(x,t)$. Since u is a viscosity supersolution of (1.1), for almost all $(x,t) \in Q$,

$$u_{\varepsilon t}(x,t) - u(y_\varepsilon, s_\varepsilon)\mathrm{Tr} M_\varepsilon(x,t) - \gamma|\nabla u_\varepsilon(x,t)|^2 \geq 0.$$

We omit (x,t) henceforth. Let $\psi \in C_0^\infty(Q)$ so that $\psi \geq 0$. Then,

$$\int_Q (-u_\varepsilon)\psi_t dxdt - \langle (u(y_\varepsilon, s_\varepsilon) + \gamma u_\varepsilon)\Delta u_\varepsilon, \psi\rangle + \frac{\gamma}{2}\int_Q (u_\varepsilon)^2 \Delta\psi \, dxdt \geq 0.$$

Now, if $\gamma \geq -1$,

$$\begin{aligned} &\langle (u(y_\varepsilon, s_\varepsilon) + \gamma u_\varepsilon)\Delta u_\varepsilon, \psi\rangle \\ = \;& \langle (u(y_\varepsilon, s_\varepsilon) + \gamma u_\varepsilon)(\Delta u_\varepsilon - f), \psi\rangle + \int_Q (u(y_\varepsilon, s_\varepsilon) + \gamma u_\varepsilon) f\psi \, dxdt \\ \geq \;& (1+\gamma)\langle u(\Delta u_\varepsilon - f), \psi\rangle + (1+\gamma)\int_Q u(y_\varepsilon, s_\varepsilon) f\psi \, dxdt \\ & \qquad - \int_Q (u - u(y_\varepsilon, s_\varepsilon)) f\psi \, dxdt, \end{aligned}$$

where $f = \max(f, 0)$. By letting $\varepsilon \to 0$,

$$\int_Q (-u\psi_t)\, dxdt - (1+\gamma)\langle u\Delta u, \psi\rangle + \frac{\gamma}{2}\int_Q u^2 \Delta\psi \, dxdt \geq 0. \tag{4.8}$$

If $\gamma < -1$, since for any $\eta > 0$, $u_{\varepsilon'}(y_\varepsilon, s_\varepsilon) \geq u_{\varepsilon'} - \eta$ for small enough ε,

$$\begin{aligned}
&\langle (u(y_\varepsilon, s_\varepsilon) + \gamma u_\varepsilon)\Delta u_\varepsilon, \psi\rangle \\
= \;& \langle (u(y_\varepsilon, s_\varepsilon) + \gamma u_\varepsilon)(\Delta u_\varepsilon - f), \psi\rangle + \int_Q (u(y_\varepsilon, s_\varepsilon) + \gamma u_\varepsilon) f\psi \, dxdt \\
\geq \;& (1+\gamma)\langle (u^{\varepsilon'} + \eta)(\Delta u_\varepsilon - f), \psi\rangle + (1+\gamma)\int_Q uf\psi \, dxdt \\
&\qquad + \gamma \int_Q (u - u(y_\varepsilon, s_\varepsilon)) f\psi \, dxdt.
\end{aligned}$$

Hence, by letting $\varepsilon \to 0$, $\eta \to 0$ and $\varepsilon' \to 0$, we have (4.8). This completes the proof. □

We have the reverse assertion of theorems 4.2 and 4.3 as follows.

Theorem 4.4. *If the comparison principle for weak solutions holds, then the weak subsolution (resp. supersolution) of* (1.1) *is a viscosity subsolution (resp. supersolution).*

Proof may be done in the same manner as in the proof of theorem 4.5 in [6].

Theorem 4.5. *Let Q be an open set such that $\overline{Q} \subset\subset \mathbb{R}^N \times (0,T)$ and let u and v be a weak subsolution and supersolution, respectively. Assume that $u \leq v$ on ∂Q.*

(i) *When $\gamma \geq -2/3$, if u and v satisfy the SSH condition, then $u \leq v$ in $\overline{Q}$.*
(ii) *When $\gamma < -2/3$, if u and v satisfy that Δu, $\Delta v \geq -1/t$ in $\mathcal{D}'$, then $u \leq v$ in $\overline{Q}$.*

Proof. Let $w = u - v$ and $\zeta = u + v$. As in the proof of lemma 4.6 in [6], we have

$$\int_Q \Big[-\frac{\lambda}{2}(w_+)^2 + \frac{1}{2}\zeta|\nabla w_+|^2 - \frac{1}{2}\Big(\gamma + \frac{3}{2}\Big)\Delta\zeta(w_+)^2\Big] e^{\lambda t} \, dxdt \leq 0,$$

for any $\lambda \in \mathbb{R}$.

In the case (i), since $\Delta\zeta \leq K$ for some $K > 0$, we have

$$\int_Q (w_+)^2 e^{\lambda t} \, dxdt \leq 0, \tag{4.9}$$

for $\lambda < -(2\gamma + 3)K$. This implies that $u \leq v$ in $\overline{Q}$.

In the case (ii), we can assume that $\Delta\zeta \geq -2/\delta$ for some $\delta > 0$. Hence, we have (4.9) for $\lambda < 2(2\gamma + 3)/\delta$. □

Remark 4.6. We say that $\Delta u \geq -\frac{1}{t}$ holds in viscosity sense on Q if u is a viscosity subsolution of $-\Delta u - \frac{1}{t} = 0$ in Q. Then, if it holds in viscosity sense, it holds in distribution sense. Indeed, by (4.2) and (4.3),

$$\Delta u^\varepsilon \geq \mathrm{Tr} M^\varepsilon \geq -\frac{1}{t} \quad \text{in } \mathcal{D}'.$$

Therefore, since $\Delta u^\varepsilon \to \Delta u$ in $\mathcal{D}'$, $\Delta u \geq -\frac{1}{t}$ in $\mathcal{D}'$. Moreover, if u is continuous, the converse statement holds. Similarly, we define that $\Delta u \leq f$ holds in viscosity sense on Q if u is a viscosity supersolution of $-\Delta u + f = 0$ in Q. Then, if it holds in viscosity sense, it holds in distribution sense. Moreover, if u is continuous, the converse statement holds.

References

1. D. G. Aronson and P. Bénilan, Régularité des solutions de milieux poreux dans $\mathbb{R}^N$, C. R. Acad. Sc. Paris, **288** (1979), Série A 103–105.
2. K. Anada, I. Fukuda and M. Tsutsumi, Local existence and uniqueness of viscosity solutions for $u_t = u\Delta u - \gamma|\nabla u|^2 + f(t,u)$, preprint.
3. M. Bertsch, R. D. Passo and M. Ughi, Discontinuous "viscosity" solutions of a degenerate parabolic equation, Trans. Amer. Math. Soc., **320** (1990), 779–798.
4. M. Bertsch and M. Ughi, Positivity properties of viscosity solutions of a degenerate parabolic equation, Nonlinear Analysis, T.M.A., **14** (1990), 571–592.
5. M. G. Crandall, H. Ishii and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc., **27** (1992), 1-67.
6. I. Fukuda, H. Ishii and M. Tsutsumi, Uniqueness of solutions to the Cauchy problem for $u_t = u\Delta u - \gamma|\nabla u|^2$, Diff. Integral Equ., **6** (1993),
7. R. Jensen, The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations, Arch. Rat. Mech. Anal., **101** (1988), 1–27.