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Structure of radial solutions to $\Delta u+\frac{1}{2}x\cdot\nabla u+\lambda u+|u|^{p-1}u=0$ in R^n

廣瀬 宗光 (早稲田大学 理工学部)
Munemitsu Hirose (Waseda University)

1. Introduction

In this talk, we will study the structure of positive solutions to the following initial value problem

$$\text{(IVP)}\qquad \begin{cases} u_{rr}+\dfrac{n-1}{r}u_r+\dfrac{r}{2}u_r+\dfrac{1}{p-1}u+|u|^{p-1}u=0, & r>0,\\ u(0)=\alpha \quad (0<\alpha<\infty), \end{cases}$$

where $n\geq 3$ and $p>1$. In [HaW], Haraux and Weissler have shown the non-uniqueness of solutions to semilinear heat equation,

$$\psi_t=\Delta\psi+|\psi|^{p-1}\psi,\ (t,x)\in(0,\infty)\times R^n. \tag{1.1}$$

In the proof, they have used some asymptotic properties of solutions to (IVP). When we discuss the following function, which is called a self-similar solution,

$$\psi(t,x):=t^{-\frac{1}{p-1}}u\left(\frac{x}{\sqrt{t}}\right),$$

it can be seen that ψ satisfies (1.1) if and only if $u(y):=u(x/\sqrt{t})$ satisfies

$$\Delta u+\frac{1}{2}y\cdot\nabla u+\frac{1}{p-1}u+|u|^{p-1}u=0,\ y\in R^n. \tag{1.2}$$

Moreover, if we set $r=|y|$, then $u=u(r)$ satisfies the equation of (IVP). Haraux and Weissler have obtained the following result on (IVP).

<u>THEOREM 1.1.</u> ([HaW]) If $1+2/n<p<(n+2)/(n-2)$, then there exists a positive number α_* such that $u(r;\alpha_*)$, which is a solution of (IVP) starting from $u(0)=\alpha_*$, satisfies the

following conditions:

(i) $u(r;\alpha_*) > 0$ for $r \geq 0$.

(ii) $\lim_{r\to\infty} r^{2/(p-1)} u(r;\alpha_*) = 0$.

(iii) For all $m > 0$, $\lim_{r\to\infty} r^m u(r;\alpha_*) = 0$ and $\lim_{r\to\infty} r^m u_r(r;\alpha_*) = 0$.

In view of Theorem 1.1, we can see that there exists a positive solution which decays rapidly as $r \to \infty$ in case $1+2/n < p < (n+2)/(n-2)$. Moreover, using above result, they have shown

<u>THEOREM 1.2.</u> ([HaW]) If $1+2/n < p < (n+2)/(n-2)$, then there exists a solution to (1.1) satisfying the following properties:

(i) $\psi(t,x) > 0$ for all $(t,x) \in (0,\infty) \times \mathrm{R}^n$.

(ii) If $1 \leq q < n(p-1)/2$, then $\lim_{t\to 0} \|\psi(t,\cdot)\|_{\mathrm{L}^q} = 0$.

In order to prove Theorem 1.2, put

$$\psi(t,x;\alpha_*) := t^{-\frac{1}{p-1}} u(|x|/\sqrt{t};\alpha_*),$$

where $u(r;\alpha_*)$ is the solution to (IVP) obtained in Theorem 1.1. Then we can see $\psi(t,x;\alpha_*) > 0$ for all $(t,x) \in (0,\infty) \times \mathrm{R}^n$ from (i) of Theorem 1.1. Moreover, in view of (iii) of Theorem 1.1,

$$\|\psi(t,\cdot\,;\alpha_*)\|_{\mathrm{L}^q} = t^{-\frac{1}{p-1}+\frac{n}{2q}} \|u(\cdot\,;\alpha_*)\|_{\mathrm{L}^q} \to 0 \text{ as } t \to 0,$$

because $\|u(\cdot\,;\alpha_*)\|_{\mathrm{L}^q} < \infty$ for all $q \geq 1$. Therefore, it is sufficient to take $\psi(t,x;\alpha_*)$ as a solution of (1.1)

In view of Theorem 1.2, initial value problem of heat equation

$$\begin{cases} \psi_t = \Delta\psi + |\psi|^{p-1}\psi, & (t,x) \in (0,\infty) \times \mathrm{R}^n, \\ \psi(t,\cdot) \to 0 \text{ as } t \to 0 \text{ in } \mathrm{L}^q(\mathrm{R}^n) \text{ with } q \in [1, n(p-1)/2), & \end{cases}$$

has at least three solutions, i.e., trivial solution, $\psi(t,x;\alpha_*)$ and $-\psi(t,x;\alpha_*)$, which is also a solution in view of the form of (1.1). Thus non-uniqueness of solutions to (1.1) has been shown.

As is already mentioned, Haraux and Weissler have shown the existence of a positive solution to (IVP) which decays rapidly as $r \to \infty$. Our aim of this talk is to get the uniqueness of this solution, i.e., to prove the uniqueness of $\alpha_* \in (0,\infty)$ which satisfies conditions of Theorem 1.1.

Moreover, we want to completely understand the behaviour of $u(r)$ for each $\alpha \in (0,\infty)$. In order to make this problem clear, we will classify the solutions to (IVP). For each $\alpha \in (0,\infty)$, (IVP) has a unique solution $u(r) \in C^2([0,\infty))$ with $u_r(r) = 0$, which is denoted by $u(r;\alpha)$. Furthermore, starting from initial value α, $u(\cdot\,;\alpha)$ decreases as long as positive. So first of all, we want to know whether $u(\cdot\,;\alpha)$ has a zero or not in $[0,\infty)$. Furthermore, if $u(\cdot\,;\alpha)$ does not have a zero, i.e., $u(\cdot\,;\alpha) > 0$ in $[0,\infty)$, then we also want to study asymptotic behaviour as $r \to \infty$. In this direction, Peletier Terman and Weissler [PTW] have obtained the following asymptotic properties.

THEOREM 1.3. ([PTW]) Set $\lambda = 1/(p-1)$ and $S := \lim_{r\to\infty} r^{2/(p-1)} u(r;\alpha)$. Then for all $\alpha \neq 0$, S exists and is finite. Moreover,

(i) If $S = 0$, then there exists some constant $R \neq 0$ such that

$$u(r;\alpha) = R r^{2\lambda - n} \exp\left(-\frac{r^2}{4}\right)\left\{1 + O\left(r^{-2}\right)\right\} \quad \text{as } r \to \infty. \tag{1.3}$$

(ii) If $S \neq 0$, then

$$u(r;\alpha) = S r^{-2\lambda} + o\left(r^{-2\lambda}\right) \quad \text{as } r \to \infty. \tag{1.4}$$

Theorem 1.3 says that the asymptotic behaviour of solutions to (IVP) is either (1.3) or (1.4). Now we will classify solutions of (IVP) as follows:

(i) $u(r;\alpha)$ is a crossing solution. $\underset{\text{def}}{\Leftrightarrow}$ $u(\cdot\,;\alpha)$ has a zero in $(0,\infty)$, i.e., there exists some $z \in (0,\infty)$ such that $u(z;\alpha)=0$.

(ii) $u(r;\alpha)$ is a rapidly decaying solution. $\underset{\text{def}}{\Leftrightarrow}$ $u(\cdot\,;\alpha)>0$ in $[0,\infty)$ and $u(r;\alpha)$ satisfies (1.3) with $R>0$.

(iii) $u(r;\alpha)$ is a slowly decaying solution. $\underset{\text{def}}{\Leftrightarrow}$ $u(\cdot\,;\alpha)>0$ in $[0,\infty)$ and $u(r;\alpha)$ satisfies (1.4) with $S>0$.

In view of the above classification, we want to decide completely whether $u(r;\alpha)$ is a crossing solution, a rapidly decaying solution or a slowly decaying solution for each initial value α. To our problem, we will summarize results in [HaW] as follows.

<u>THEOREM 1.4.</u> ([HaW])

(i) If $1<p\le 1+2/n$, then $u(r;\alpha)$ is a crossing solution for every $\alpha>0$.

(ii) If $p\ge (n+2)/(n-2)$, then $u(r;\alpha)$ is a slowly decaying solution for every $\alpha>0$.

(iii) Suppose $1+2/n<p<(n+2)/(n-2)$. Put

$$\alpha_* := \inf\{\alpha>0\ ;\ u(r;\alpha) \text{ is a crossing solution}\},$$

then $u(r;\alpha_*)$ is a rapidly decaying solution. Moreover, for sufficiently small $\alpha>0$ $u(r;\alpha)$ is a slowly decaying solution.

Although Haraux and Weissler [HaW] have not shown complete structure on case $1+2/n<p<(n+2)/(n-2)$, they have given the following conjecture:

<u>Conjecture by Haraux and Weissler [HaW]</u>

There exists a unique positive number α_* such that $u(r;\alpha_*)$ is a rapidly decaying solution. Moreover, $u(r;\alpha)$ is a crossing solution for every $\alpha \in (\alpha_*,\infty)$ and $u(r;\alpha)$ is a slowly decaying solution for every $\alpha \in (0,\alpha_*)$.

To this conjecture, I [Hi] have shown that their conjecture is correct in the special case

$$p = 2 \text{ and } 3 \le n < 6.$$

Recently, Yanagida [Ya] has also shown the affirmative answer to the conjecture in case

$$1 + 2/n < p \le n/(n-2).$$

In addition, if $n/(n-2) \le p < (n+2)/(n-2)$, as a joint work with Claus Dohmen (University of Bonn) I also prove

THEOREM A. ([DHi]) Suppose

$$n \ge 3 \text{ and } n/(n-2) \le p < (n+2)/(n-2).$$

Then the conjecture by Haraux and Weissler is correct.

This thorem is proved by using the structure thorem by Yanagida and Yotsutani (see [YaYo] or [Yo]). Thus we have complete information for the structure of positive solutions to (IVP) for $n \ge 3$ and $p > 1$. (See Section 2.)

Moreover, in (p,α)-plane we will define the following domains:

$$\begin{cases} D_C := \{(p,\alpha) \in (1,\infty)\times(0,\infty) \mid u(r;\alpha) \text{ is a crossing solution}\}, \\ D_R := \{(p,\alpha) \in (1,\infty)\times(0,\infty) \mid u(r;\alpha) \text{ is a rapidly decaying solution }\}, \\ D_S := \{(p,\alpha) \in (1,\infty)\times(0,\infty) \mid u(r;\alpha) \text{ is a slowly decaying solution }\}. \end{cases}$$

According to this definition, we want to investigate the relation of D_C, D_R and D_S in (p,α)-plane. To this problem, as a joint work with Eiji Yanagida (Tokyo Institute of Technology) I have

THEOREM B. ([HiYa]) For $1 + 2/n < p < (n+2)/(n-2)$, D_R is a unique C^1-class curve in (p,α)-plane. If we define D_R by

$$\alpha = \alpha_*(p) \quad \text{for } p \in \left(1 + \frac{2}{n}, \frac{n+2}{n-2}\right),$$

then $\alpha_*(p)$ satisfies

$$\alpha_*(p) \to 0 \text{ as } p \to 1+\frac{2}{n} \text{ and } \alpha_*(p) \to \infty \text{ as } p \to \frac{n+2}{n-2}.$$

Moreover, in (p,α)-plane, domain D_C is in the left side of curve D_R and domain D_S is in the right side of curve D_R. (See Fig.1.)

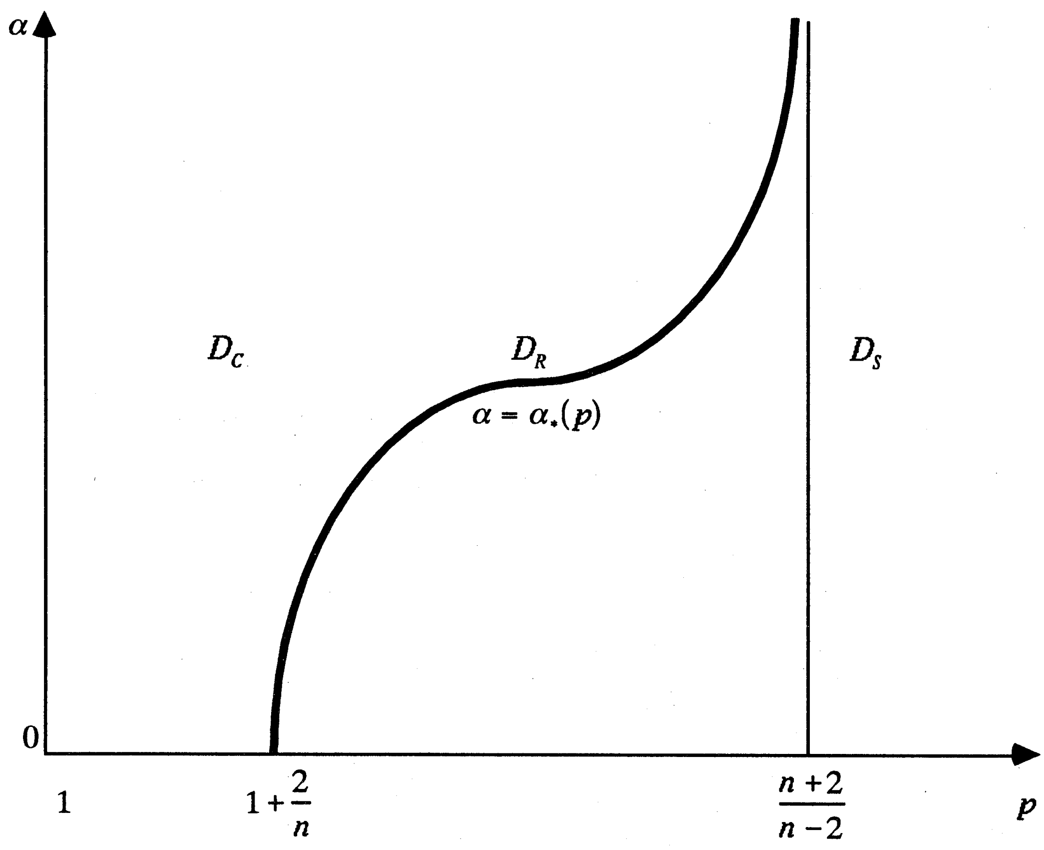

Fig.1

2. Proof of Theorem A

In Sections 2 and 3, we will define

$$\lambda := \frac{1}{p-1}.$$

As is already stated, in order to prove Theorem A, we will apply the structure theorem by Yanagida and Yotsutani. We will explain their result for the following initial value problem

(2.1)
$$\begin{cases} (g(r)v_r)_r + g(r)K(r)(v^+)^p = 0, \ r > 0, \\ v(0) = \alpha \in (0,\infty), \end{cases}$$

where $v^+ = \max\{v,0\}$. We suppose that $g(r)$ and $K(r)$ satisfy

(g)
$$\begin{cases} g(r) \in C^1([0,\infty)); \\ g(r) > 0 \ \text{in} \ (0,\infty); \\ 1/g(r) \notin L^1(0,1); \\ 1/g(r) \in L^1(1,\infty), \end{cases}$$

and

(K)
$$\begin{cases} K(r) \in C(0,\infty); \\ K(r) \geq 0 \ \text{and} \ K(r) \not\equiv 0 \ \text{in} \ (0,\infty); \\ h(r)K(r) \in L^1(0,1); \\ h(r)\{h(r)/g(r)\}^p K(r) \in L^1(1,\infty), \end{cases}$$

where

$$h(r) := g(r)\int_r^\infty \{1/g(s)\}ds.$$

Moreover, define the following functions

$$G(r) := \frac{2}{p+1} g(r)h(r)K(r) - \int_0^r g(s)K(s)ds,$$

$$H(r) := \frac{2}{p+1} h(r)^2 \left\{\frac{h(r)}{g(r)}\right\}^p K(r) - \int_r^\infty h(s)\left\{\frac{h(s)}{g(s)}\right\}^p K(s)ds,$$

and set

$$r_G := \inf\{r \in (0,\infty); \ G(r) < 0\}, \quad r_H := \sup\{r \in (0,\infty); \ H(r) < 0\}.$$

<u>THEOREM 2.1.</u> ([YaYo] or [Yo]) Suppose that $G(r) \not\equiv 0$ in $[0,\infty)$ and let $v(r;\alpha)$ be the solution of (2.1). If $0 < r_H \leq r_G < \infty$, then there exists a unique positive number α_* such that

(i) For every $\alpha \in (\alpha_*,\infty)$, $v(\cdot\,;\alpha)$ has a zero in $[0,\infty)$.

(ii) If $\alpha = \alpha_*$, then $v(\cdot\,;\alpha_*) > 0$ in $[0,\infty)$ and

(2.2) $$0 < \lim_{r\to\infty}\left\{\frac{g(r)}{h(r)}\right\}v(r;\alpha_*) < \infty.$$

(iii) For every $\alpha \in (0,\alpha_*)$, $v(\cdot\,;\alpha) > 0$ in $[0,\infty)$ and

(2.3) $$\lim_{r\to\infty}\left\{\frac{g(r)}{h(r)}\right\}v(r;\alpha_*) = \infty.$$

In order to apply Theorem 2.1 to (IVP), put

$$u(r) := v(r)\varphi(r).$$

Then the equation of (IVP) is rewritten as

$$v_{rr} + \left(2\frac{\varphi_r}{\varphi} + \frac{n-1}{r} + \frac{r}{2}\right)v_r + |\varphi|^{p-1}|v|^{p-1}v + \left\{\frac{\varphi_{rr}}{\varphi} + \left(\frac{n-1}{r} + \frac{r}{2}\right)\frac{\varphi_r}{\varphi} + \lambda\right\}v = 0.$$

Therefore, if we take $\varphi(r)$ which satisfies the following initial value problem

(2.4) $$\begin{cases} \varphi_{rr} + \left(\dfrac{n-1}{r} + \dfrac{r}{2}\right)\varphi_r + \lambda\varphi = 0, \quad r > 0, \\ \varphi(0) = 1, \end{cases}$$

then $v(r)$ must satisfy

(2.5) $$\begin{cases} (g(r)v_r)_r + g(r)K(r)|v|^{p-1}v = 0, \quad r > 0, \\ v(0) = \alpha \in (0,\infty), \end{cases}$$

where $g(r) := r^{n-1}\exp(r^2/4)\varphi^2$ and $K(r) := |\varphi|^{p-1}$. On initial value problem (2.4), we obtain the following properties.

<u>PROPOSITION 2.2.</u>

(i) There exists a unique solution $\varphi(r) = C^2([0,\infty))$ of (2.4) with $\varphi_r(0) = 0$.

(ii) Let $L := \lim_{r\to\infty} r^{2\lambda}\varphi(r)$. If $0 < \lambda < n/2$ ($\Leftrightarrow p > 1 + 2/n$), then $\varphi(r) > 0$ in $[0,\infty)$ and

$0 < L < \infty$, i.e.,

(2.6) $$\varphi(r) = Lr^{-2\lambda} + o(r^{-2\lambda}) \text{ as } r \to \infty.$$

(iii) If $0 < \lambda \le (n-2)/2$ ($\Leftrightarrow p \ge n/(n-2)$), then

(2.7) $$-2\lambda < \frac{r\varphi_r}{\varphi} < 0 \text{ in } [0,\infty).$$

Therefore, in order to know whether u has a zero or not, it is sufficient to investigate whether v has a zero or not. Since it is possible to verify that $g(r) = r^{n-1}\exp(r^2/4)\varphi^2$ and $K(r) = \varphi^{p-1}$ satisfy (g) and (K), respectively, we can use Theorem 2.1 to (2.5). In order to apply Theorem 2.1, we must know the location of r_G and r_H. For this purpose, we will investigate the profiles of $G(r)$ and $H(r)$. First, differentiating $G(r)$ and $H(r)$, we obtain

(2.8) $$G'(r) = \frac{2}{p+1} g(r)K(r)\left\{\Phi(r) - \frac{p+3}{2}\right\} = \left(\int_r^\infty \frac{1}{g(s)}ds\right)^{-p-1} H'(r),$$

where

$$\Phi(r) := r^{n-2}\exp\left(\frac{r^2}{4}\right)\varphi(r)^2\left\{r^2 + 2(n-1) + (p+3)\frac{r\varphi_r(r)}{\varphi(r)}\right\}\int_r^\infty s^{1-n}\exp\left(-\frac{s^2}{4}\right)\varphi(s)^{-2}ds.$$

In view of (2.8), it is important to study the relation between $\Phi(r)$ and $(p+3)/2$. Using (2.7), we get the following

PROPOSITION 2.3. Suppose $n \ge 3$ and $(n-2) \le p < (n+2)/(n-2)$. Let $\hat{r}$ and $\tilde{r}$ be some positive and finite numbers satisfying

$$\Phi(\hat{r}) = \frac{p+3}{2} \text{ and } \Phi'(\hat{r}) < 0$$

and

$$\Phi(\tilde{r}) = \frac{p+3}{2} \text{ and } \Phi'(\tilde{r}) = 0,$$

respectively. Then the relation between $q = \Phi(r)$ and $q = (p+3)/2$ in (r,q)-plane is one of the following:

(a) $\Phi(r) > \frac{p+3}{2}$ in $[0,\hat{r})$ and $\Phi(r) < \frac{p+3}{2}$ in $(\hat{r},\infty)$.

(b) $\Phi(r) > \frac{p+3}{2}$ in $[0,\tilde{r})$ and $\Phi(r) < \frac{p+3}{2}$ in $(\tilde{r},\infty)$.

(c) $\Phi(r) > \frac{p+3}{2}$ in $[0,\hat{r})$ and $\Phi(r) < \frac{p+3}{2}$ in $(\hat{r},\infty)\setminus\tilde{r}$.

Therefore, in view of Proposition 2.3 and (2.8) there is exactly one point where $G'(r)$ and $H'(r)$ change their signs from positive to negative. Thus we obtain

PROPOSITION 2.4. Suppose $n \geq 3$ and $(n-2) \leq p < (n+2)/(n-2)$. Then there exists a unique number $r_* \in (0,\infty)$ such that

(i) For $r \in [0, r_*)$, $G(r)$ and $H(r)$ are increasing.

(ii) For $r \in (r_*, \infty)$, $G(r)$ and $H(r)$ are decreasing.

Moreover, we will determine the behaviour of $G(r)$ and $H(r)$ near $r = 0$ and $r = \infty$.

PROPOSITION 2.5. Suppose $n \geq 3$ and $(n-2) \leq p < (n+2)/(n-2)$. Then

(i) $\lim_{r \to \infty} G(r) = -\infty$.

(ii) $\lim_{r \to 0} G(r) = 0$.

(iii) $\liminf_{r \to \infty} H(r) \geq 0$.

(iv) $\limsup_{r \to 0} H(r) < 0$.

In view of Propositions 2.4 and 2.5, we can draw the graphs of $q = G(r)$ and $q = H(r)$ in (r,q)-plane as Fig.2. Then we obtain $0 < r_H < r_* < r_G < \infty$. Therefore, using Theorem 2.1, we have the following result:

PROPOSITION 2.6. Suppose $n \geq 3$ and $(n-2) \leq p < (n+2)/(n-2)$. Then

(i) For $\alpha \in (\alpha_*, \infty)$, $v(\cdot\,;\alpha)$ has a zero in $(0,\infty)$, i.e., $u(\cdot\,;\alpha)$ has a zero in $(0,\infty)$.

(ii) For $\alpha \in [0, \alpha_*]$, $v(\cdot\,;\alpha) > 0$ in $(0,\infty)$, i.e., $u(\cdot\,;\alpha) > 0$ in $(0,\infty)$.

Finally, on the asymptotic behaviour we get the following result by noting (2.6) and

$$\int_r^\infty g(s)^{-1} ds = 2r^{-n} \exp\left(-\frac{r^2}{4}\right) \varphi(r)^{-2}(1 + o(1)) \quad \text{as } r \to \infty.$$

PROPOSITION 2.7. The following equivalence relations hold between $u(r;\alpha)$ and $v(r;\alpha)$:

(i) $u(r;\alpha)$ satisfies (1.3).⇔ $v(r;\alpha)$ satisfies (2.2).

(ii) $u(r;\alpha)$ satisfies (1.4).⇔ $v(r;\alpha)$ satisfies (2.3).

From Proposition 2.7, $u(r;\alpha_*)$ satisfies (1.3) and for $\alpha \in (0,\alpha_*)$ $u(r;\alpha)$ satisfies (1.4). Thus, combining Proposition 2.6, we complete the proof of Theorem A.

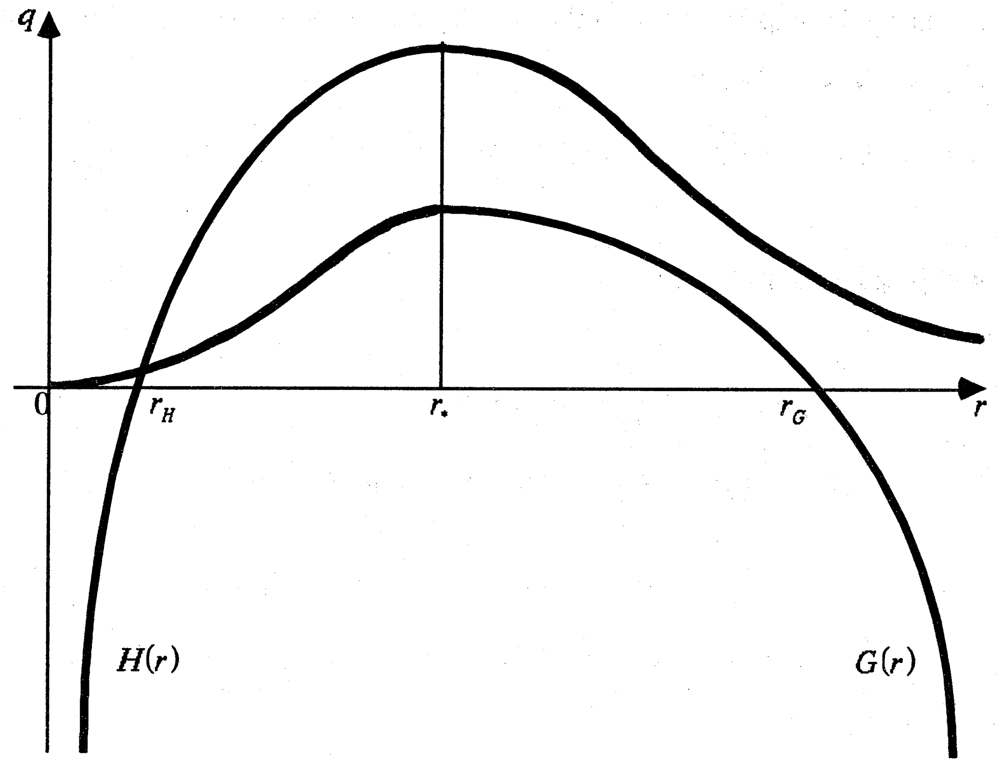

Fig.2

3. Proof of Theorem B

In Theorem A, we have already proved $D_R = \{(p,\alpha_*(p)) \mid p \in (1+2/n,(n+2)/(n-2))\}$. In order to show that domain D_R is a C^1-class curve $\alpha = \alpha_*(p)$ in (p,α)-plane, we will take the following steps:

I. Using the implicit function theorem, we will show that there exists a unique branch of class C^1 in a neighbourhood of $(p,\alpha) = (1+2/n,0)$.

II. Moreover, using the implicit function theorem again, we will show that this branch can be extended up to $p = (n+2)/(n-2)$.

III. Finally, we will prove $\alpha_*(p) \to \infty$ as $p \to (n+2)/(n-2)$.

STEP I. We will prepare the following problem

(B)
$$\begin{cases} w_{rr} + \left(\dfrac{n-1}{r} + \dfrac{r}{2}\right) w_r + \lambda w + |w|^{p-1} w = 0, \\ r^{n-2\lambda} \exp(r^2/4) w(r) \to \beta \in (0,\infty) \text{ as } r \to \infty. \end{cases}$$

It can be seen that (B) has unique global solution $w(r) \in C^2((0,\infty))$, and we will denote this solution by $w(r;\beta)$. In view of (IVP) and (B), $u(r;\alpha)$, a solution of (IVP), is a rapidly decaying solution if and only if

(3.1) $$u(1;\alpha) = w(1;\beta) \text{ and } u_r(1;\alpha) = w_r(1;\beta)$$

hold for some $\beta \in (0,\infty)$. Then we will define the following functions:

(3.2)
$$\begin{cases} f(\alpha,\beta,p) := u(1;\alpha) - w(1;\beta), \\ g(\alpha,\beta,p) := u_r(1;\alpha) - w_r(1;\beta). \end{cases}$$

Clearly, $u(r;\alpha)$ is a rapidly decaying solution if and only if one can find β satisfying $f = g = 0$. In fact, we will prove that $f = g = 0$ holds around $(\alpha,\beta,p) = (0,\ 0,\ 1+2/n)$. First, the following proposition is important.

PROPOSITION 3.1. If $p = 1 + 2/n$ ($\Leftrightarrow \lambda = n/2$), then

$$\varphi(r) = \exp\left(-\frac{r^2}{4}\right)$$

is a unique solution of (2.4).

Now set

$$u = \alpha\,\bar{u}, \quad w = \beta\,\bar{w}, \quad \beta = t\alpha,$$

then (IVP) and (B) are respectively rewritten by

$$\text{(IVP)'} \qquad \begin{cases} \bar{u}_{rr} + \left(\dfrac{n-1}{r} + \dfrac{r}{2}\right)\bar{u}_r + \lambda\bar{u} + \bar{\alpha}|\bar{u}|^{p-1}\bar{u} = 0, \\ \bar{u}(0) = 1, \end{cases}$$

and

$$\text{(B)'} \qquad \begin{cases} \bar{w}_{rr} + \left(\dfrac{n-1}{r} + \dfrac{r}{2}\right)\bar{w}_r + \lambda\bar{w} + t^{p-1}\bar{\alpha}|\bar{w}|^{p-1}\bar{w} = 0, \\ \lim_{r\to\infty} r^{n-2\lambda}\exp(r^2/4)\bar{w}(r) = 1, \end{cases}$$

where $\bar{\alpha} = \alpha^{p-1}$. Moreover, since

$$\begin{cases} f(\alpha,\beta,p) = \alpha\ \{\bar{u}(1;\bar{\alpha},p) - t\bar{w}(1;\bar{\alpha},t,p)\}, \\ g(\alpha,\beta,p) = \alpha\ \{\bar{u}_r(1;\bar{\alpha},p) - t\bar{w}_r(1;\bar{\alpha},t,p)\}, \end{cases}$$

we will study

$$(3.3) \qquad \begin{cases} \bar{f}(\bar{\alpha},t,p) := \bar{u}(1;\bar{\alpha},p) - t\bar{w}(1;\bar{\alpha},t,p), \\ \bar{g}(\bar{\alpha},t,p) := \bar{u}_r(1;\bar{\alpha},p) - t\bar{w}_r(1;\bar{\alpha},t,p), \end{cases}$$

instead of (3.2). Noting Proposition 3.1 and putting $(\bar{\alpha},t,p) = (0,\ 1,\ 1+2/n)$ in (3.3), we get

$$\bar{f}(0,\ 1,\ 1+2/n) = \bar{g}(0,\ 1,\ 1+2/n) = 0.$$

Furthermore, in view of (IVP)' and (B)', we obtain

$$\det\begin{pmatrix} \frac{\partial \bar{f}}{\partial t}\left(0,\ 1,\ 1+\frac{2}{n}\right) & \frac{\partial \bar{f}}{\partial p}\left(0,\ 1,\ 1+\frac{2}{n}\right) \\ \frac{\partial \bar{g}}{\partial t}\left(0,\ 1,\ 1+\frac{2}{n}\right) & \frac{\partial \bar{g}}{\partial p}\left(0,\ 1,\ 1+\frac{2}{n}\right) \end{pmatrix} \neq 0.$$

Therefore, applying the implicit function theorem to (3.3), we have

PROPOSITION 3.2. In a neighbourhood of $(\bar{\alpha},t,p)=(0,\ 1,\ 1+2/n)$, there exist C^1-class functions $t(\bar{\alpha})$ and $p(\bar{\alpha})$ such that $\bar{f}(\bar{\alpha},t(\bar{\alpha}),p(\bar{\alpha}))=\bar{g}(\bar{\alpha},t(\bar{\alpha}),p(\bar{\alpha}))=0$ and $(t(0),p(0))=(1,1+2/n)$.

In addition, expanding $\bar{f}$ and $\bar{g}$ around $(\bar{\alpha},t,p)=(0,\ 1,\ 1+2/n)$, we get

PROPOSITION 3.3. In a neighbourhood of $(\bar{\alpha},t,p)=(0,\ 1,\ 1+2/n)$, $p=p(\bar{\alpha})$ can be expressed by

$$p-\left(1+\frac{2}{n}\right)=C\bar{\alpha}+o\left(\bar{\alpha}^2,p^2\right),$$

where C is some positive constant.

Therefore, noting $\bar{\alpha}=\alpha^{p-1}$, we can draw a figure of a branch $p=p(\bar{\alpha})$ in (p,α)-plane as follows:

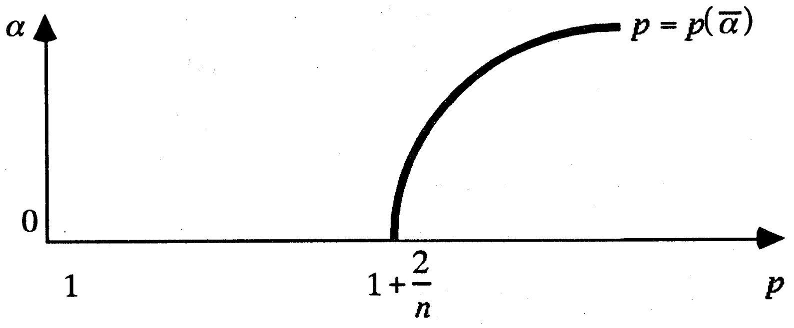

Fig.3

We will denote this branch by $\alpha = \alpha_*(p)$ below.

STEP II. Using the implicit function theorem again and noting the uniqueness of rapidly decaying solution for $p \in (1+2/n, (n+2)/(n-2))$, we conclude the following result.

PROPOSITION 3.4. Branch $\alpha = \alpha_*(p)$ can be extended up to $p = (n+2)/(n-2)$ as a unique C^1-class curve.

STEP III. If $p = (n+2)/(n-2)$, then for every $\alpha \in (0,\infty)$ $u(r;\alpha)$ is a slowly decaying solution. Therefore, $\alpha_*(p)$ satisfies either

(3.4) $$\alpha_*(p) \to 0 \text{ as } p \to (n+2)/(n-2)$$

or

(3.5) $$\alpha_*(p) \to +\infty \text{ as } p \to (n+2)/(n-2).$$

But (3.4) is impossible: Suppose that (3.4) holds. Then as $\overline{\alpha} = \alpha_*(p)^{p-1} \to 0$ and $p \to (n+2)/(n-2)$, the solution of (IVP)' converges to a solution of

$$\begin{cases} \overline{u}_{rr} + \left(\dfrac{n-1}{r} + \dfrac{r}{2}\right)\overline{u}_r + \dfrac{n-2}{4}\overline{u} = 0, \\ \overline{u}(0) = 1. \end{cases}$$

Thus in view of (2.6), if $p \to (n+2)/(n-2)$, then $\overline{u}(r;\overline{\alpha},p)$ converges a solution satisfying

$$\overline{u}(r;\overline{\alpha},p) = Lr^{-2\lambda} + o\left(r^{-2\lambda}\right) \text{ as } r \to \infty.$$

But this is a contradiction since $u(r;\alpha_*) = \alpha_* \overline{u}(r;\overline{\alpha},p)$ is a rapidly decaying solution. Therefore,

PROPOSITION 3.5. $\alpha_*(p) \to +\infty$ as $p \to (n+2)/(n-2)$.

Thus we can show that domaim D_R consists of a C^1-class curve $\alpha = \alpha_*(p)$ in (p,α)-plane. In addition, it follows from Theorem A that the left side and the right side of D_R are D_C and D_S, respectively.

4. Generalization

In this section, we will study

$$(4.1)\qquad \begin{cases} u_{rr} + \dfrac{n-1}{r}u_r + \dfrac{r}{2}u_r + \lambda u + |u|^{p-1}u = 0, \ r > 0, \\ u(0) = \alpha \ \ (0 < \alpha < \infty), \end{cases}$$

where λ is a positive parameter and does not depend on p. For (4.1) also, the asymptotic behaviour of the solution of (4.1) is either (1.3) or (1.4). Therefore, we can classify solutions of (4.1) as well as (IVP). Moreover, we will define three types of structure of solutions as follows:

(i) TypeC $\underset{\text{def}}{\Leftrightarrow}$ For every $\alpha \in (0,\infty)$, $u(r;\alpha)$ is a crossing solution.

(ii) TypeS $\underset{\text{def}}{\Leftrightarrow}$ For every $\alpha \in (0,\infty)$, $u(r;\alpha)$ is a slowly decaying solution.

(iii) TypeM $\underset{\text{def}}{\Leftrightarrow}$ There exists a unique positive number α_* such that $u(r;\alpha_*)$ is a rapidly decaying solution. Moreover, $u(r;\alpha)$ is a crossing solution for every $\alpha \in (\alpha_*,\infty)$ and $u(r;\alpha)$ is a slowly decaying solution for every $\alpha \in (0,\alpha_*)$.

Now we will summarize the known results on (4.1) as follows:

(I) If $n \geq 1$, $p > 1$ and $\lambda \geq n/2$, then the structure of solutions to (4.1) is TypeC. (Weissler [W])

(II) If $n > 2$, $p \geq (n+2)/(n-2)$ and $0 < \lambda \leq \max\{1, n/4\}$, then the structure of solutions to

(4.1) is TypeS. (Atkinson and Peletier [AP])

(III) If $n \geq 3$ ($n \in \mathrm{N}$), $p = (n+2)/(n-2)$ and $\max\{1, n/4\} < \lambda < n/2$, then there exists a rapidly decaying solution. (Escobedo and Kavian [EK])

(IV) If $n \geq 1$, $1 < p < (n+2)/(n-2)^+$ and $1/2(p-1) < \lambda < n/2$, then

$$\alpha_* := \inf\{\alpha > 0\ ;\ u(r;\alpha) \text{ is a crossing solution}\}$$

exists and is finite. Moreover, $u(r;\alpha_*)$ is a rapidly decaying solution and for sufficiently small $\alpha > 0$ $u(r;\alpha)$ is a slowly decaying solution. (Haraux and Weissler [HaW])

(V) If $n = 1$, $p > 1$ and $0 < \lambda < 1/2$, then the structure of solutions to (4.1) is TypeM. (Weissler [W])

(VI) If $n \geq 3$, $1 < p < (n+2)/(n-2)$ and $\lambda = 1$, then the structure of solutions to (4.1) is TypeM. (Hirose [Hi])

In view of above results, the existence and nonexistence of rapidly decaying solutions for subcritical p (i.e., $1 < p < (n+2)/(n-2)^+$) is well understood. Although the uniqueness has remained open, Claus Dohmen and I succeed in getting a result analogous to (V) for higher space dimension and λ ranging between 0 and $(n-2)/2$:

<u>THEOREM C.</u> ([DHi]) If $n \geq 3$, $1 < p < (n+2)/(n-2)$ and $0 < \lambda \leq (n-2)/2$, then the structure of solutions to (4.1) is TypeM.

This theorem can be also proved by using Theorem 2.1.

<u>REMARK.</u> On the following range of n, p and λ, the structure of solutions to (4.1) still remains open:

(i) $n \in (1,3)$, $p \in \left(1, \dfrac{(n+2)}{(n-2)^+}\right)$, $\lambda \in \left(0, \dfrac{n}{2}\right)$.

(ii) $n \in [3,4)$, $p \in \left(1, \dfrac{(n+2)}{(n-2)}\right)$, $\lambda \in (0,1) \cup \left(1, \dfrac{n}{2}\right)$.

(iii) $n \in [4,\infty)$, $p \in \left(1, \frac{(n+2)}{(n-2)}\right)$, $\lambda \in \left(\frac{n-2}{2}, \frac{n}{2}\right)$.

(iv) $n \in (2,\infty)$, $p \in \left[\frac{(n+2)}{(n-2)}, \infty\right)$, $\lambda \in \left(\max\left\{1, \frac{n}{4}\right\}, \frac{n}{2}\right)$.

Finally, we will show domains of Types C, S and M in (λ, p)-plane for $n \geq 4$.

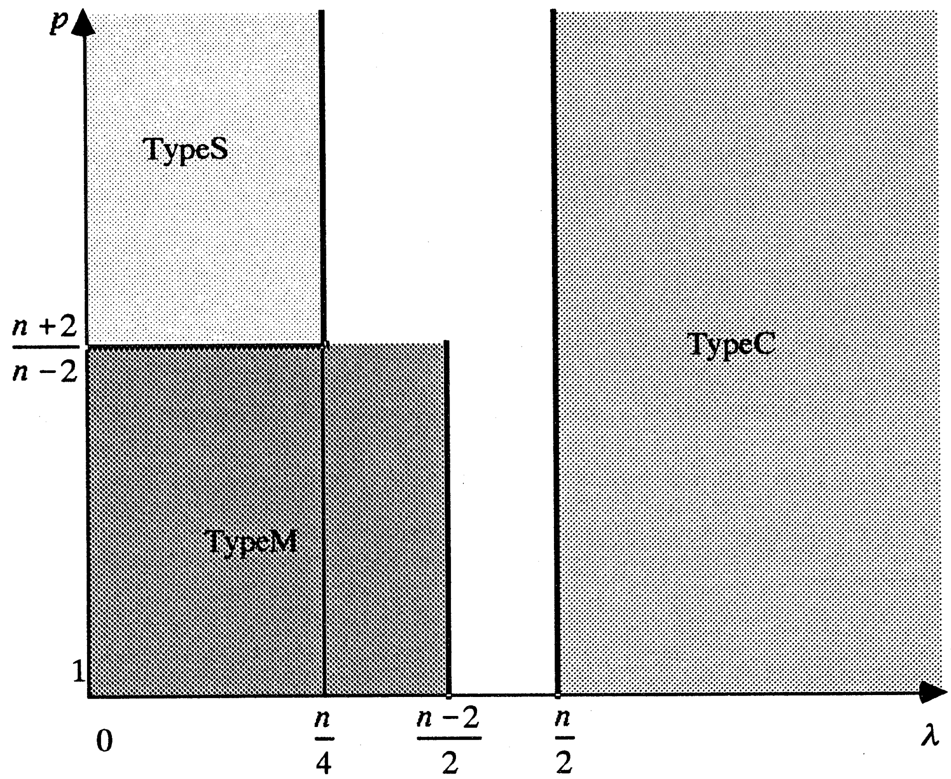

Fig.4

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