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On the zero-maps and automorphism groups of a compact Riemann surface

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1. Introduction

The purpose of this paper is to classify the automorphism group $\mathrm{Aut}(\boldsymbol{P}^1)$ of the complex projective line $\boldsymbol{P}^1$ by the " r-signatures " independent of geometric properties of $\boldsymbol{P}^1$.

Let M be a compact Riemann surface of genus $g\,(\geq 0)$. Assume that G is a finite subgroup of the automorphism group $\mathrm{Aut}(M)$ of M. We are interested in the classification of the pairs (M, G). As for a criterion of the classification, we consider it natural to use the relation of topological equivalence. Our classification gives some information in relation with the problem of the moduli or Teichmüller space (cf. [8], [9]). We make the classification by virtue of the character theory of groups, in particular by the r-signatures r (see Definition 3.1), which is invariant up to topological equivalence. We characterize this classification as the relation between " r is representable " and " r is realizable. "

The Lefschetz trace $\sum(-1)^i \operatorname{Tr}(G|H^i(M, \Omega^{\otimes q}))$ of the natural action of G on the space of q-differentials on M can be expressed as the Chevalley-Weil formula (see Proposition 3.1). We shall abstract the notion concerning the branch points on M/G of the natural projection $\pi : M \to M/G$ which appears in this formula. Namely, given a finite (abstract) group G a priori, we introduce a quantity $r = [\, g_0\,;\, l_1, \ldots, l_h\,]$, which we call the virtual r-signature of G. Further we denote by $\chi_{[r]}^{(q)}$ the right-hand side of the Chevalley-Weil formula, abstractly. And we put $\chi_{[r]} = 1_G + \chi_{[r]}^{(1)}$, $g = \chi_{[r]}(1)$. Here 1_G is the principal character of G. Then we can study automorphism groups of compact Riemann surfaces by virtue of the character theory of groups with the class function $\chi_{[r]}$ on G (see Definition 4.1, 4.2). From this formula, we get our basic tools, the Eichler trace formula and the Riemann-Hurwitz relation (see Proposition 4.1, 4.2).

In this paper we state the following theorem :

Theorem. *Let G be a finite (abstract) group of order $n\,(\geq 2)$. Let $C_0, C_1, \ldots, C_h$ be the conjugacy classes of G and $s_0 = 1, s_1, \ldots, s_h$ their representatives, respectively. Let $r = [\,g_0\,; l_1, \ldots, l_h\,]$ be a virtual r-signature of G such that $g_0 \in \boldsymbol{Z}_{\geq 0}$, $l_i \in \boldsymbol{Z}_{\geq 0}\,(1 \leq i \leq h)$.*

(I) *The following two conditions are equivalent :*

(i) $\chi_{[r]} = 0$.

(ii) *r is realizable of genus zero.*

(II) *The automorphism group of genus zero is classified by the r-signatures as follows :*

(1) $\boldsymbol{Z}_n$:

(i) *In case* $n > 2$: $[\,0\,;\, \ldots, \underbrace{1}^{i}, \ldots, \underbrace{1}^{n-i}, \ldots]$ *with* $(i, n) = 1,\ 1 \leq i < n.$

(ii) *In case* $n = 2$: $[\,0\,;2\,]$.

(2) D_m :

(i) *In case* $m = 2k$: $[\,0\,;\, \ldots, \underbrace{1}^{i}, \ldots, \underbrace{1}^{k+1}, \underbrace{1}^{k+2}\,]$ *with* $(i, m) = 1,\ 1 \leq i \leq k.$

(ii) *In case* $m = 2k+1$: $[\,0\,;\, \ldots, \underbrace{1}^{i}, \ldots, \underbrace{2}^{k+1}\,]$ *with* $(i, m) = 1,\ 1 \leq i \leq k.$

(3) A_4 : $[\,0\,;\,1,1,1\,]$.

(4) S_4 : $[\,0\,;\,0,1,1,1\,]$.

(5) A_5 : $[\,0\,;\,1,1,1,0\,]$.

Here the symbol "$\ldots$" means that $l_j = 0$.

Here "$\chi_{[r]} = 0$" means the zero-map. The condition "$\chi_{[r]} = 0$" implies that r is representable. In general, we say that r is representable if $\chi_{[r]}$ is a linear combination of the irreducible characters of G with non-negative integer coefficients. Then $\chi_{[r]}$ is a character of some representation of G. In general, we say that r is realizable of genus g if there exist a compact Riemann surface M of genus g and an inclusion $\iota : G \to \mathrm{Aut}(M)$ such that g_0 is the genus of M/G and $l_i\ (1 \leq i \leq h)$ means the number of branch points on M/G of $\pi : M \to M/G$ (see Definition 4.3).

The result of this paper is used in the classification in case that M is hyperelliptic. In the same line, our method may be available to study the case of $g = 1, 2, \ldots$, which we shall consider in another place (cf. [5], [7], [8], [10]).

Remark 1.1. The symbols $\boldsymbol{Z}_n$, D_m, A_4, S_4 and A_5 denote, respectively, the cyclic group of order n, the dihedral group of order $2m$, the alternating group of degree 4, the symmetric group of degree 4 and the alternating group of degree 5.

Remark 1.2. It is well-known as the classical results that the finite subgroups of $\mathrm{Aut}(\boldsymbol{P}^1)$ are classified as cyclic $\boldsymbol{Z}_n$, dihedral D_m, tetrahedral A_4, octahedral S_4 and icosahedral A_5.

2. Notation

We denote by $\boldsymbol{Q}$ the field of rational numbers, and by $\boldsymbol{C}$ the field of complex numbers. We denote by $\boldsymbol{Z}_{\geq 0}$ the set of non-negative rational integers, and by $\boldsymbol{C}^{\times}$ the multiplicative group $\boldsymbol{C} \setminus \{0\}$. Put $\zeta_n = \exp\left(2\pi\sqrt{-1}/n\right)$ $(n = 1, 2, \ldots)$. We denote by $\sharp S$ the cardinality of a set S.

Throughout this paper, G is a finite group. As for the group theory and the representation theory, we use the general notations (e.g. [11]). For example, $\langle s \rangle$, $C_G(H)$, $[\,G : H\,]$, $\langle \chi, \varphi \rangle$, $\mathrm{Ind}_H^G(\varphi)$, etc. We denote by $\sharp s$ the cardinality $\sharp\langle s \rangle$. We denote by reg_G the regular character of G. For $a, b \in G$, we denote by $a \overset{G}{\sim} b$ that a is G-conjugate to b.

We use the following situation :

Situation 2.1. Let G be a finite(abstract) group. Let $C_0, C_1, \ldots, C_h$ be the conjugacy classes of G and $s_0 = 1, s_1, \ldots, s_h$ their representatives, respectively.

3. r-signatures and r-datum

We shall introduce the notion concerning the branch point on M/G of the natural projection $\pi : M \longrightarrow M/G$.

Let G be a finite (abstract) group. For an inclusion $\iota : G \longrightarrow \mathrm{Aut}(M)$, we say that G is an automorphism group of M. In this case, we identify G with its image via ι and denote that $G \subseteq \mathrm{Aut}(M)$. We shall specify ι, if necessary.

Assume that $G \subseteq \mathrm{Aut}(M)$. For a point P on M, we denote by

$$G_P = \{\, \sigma \in G \,;\, \sigma(P) = P \,\}$$

the stabilizer of P in G. We define an injective homomorphism $\theta_P : G_P \longrightarrow \boldsymbol{C}^{\times}$ by the equation

$$\theta_P(\sigma) = \zeta \quad (\sigma \in G_P)$$

where ζ is a $\sharp\sigma$-th root of unity satisfying the relation

$$\sigma^*(\tau) \equiv \zeta \cdot \tau \pmod{\tau^2 \mathcal{O}_P}$$

for some local parameter τ of the valuation ring $\mathcal{O}_P$ at P (in the function field of M).

Definition 3.1. In Situation 2.1, we assume that $G \subseteq \mathrm{Aut}(M)$. Denote by g_0 the genus of M/G. We put, for $s\,(\neq 1) \in G$,

$$l(s) = \sharp\,(\pi\,\{\, P \in M \,;\, G_P = \langle s \rangle \text{ and } \theta_P(s) = \zeta_{\sharp s} \})$$

where the natural projection $\pi : M \longrightarrow M/G$. Then we call the quantity

$$r = [\, g_0 \,;\, l(s_1), \ldots, l(s_h)\,]$$

the *r-signature* of G with respect to $C_1, \ldots, C_h$.

Remark 3.1. $l(s) = l(s')$ in case $s \overset{G}{\sim} s'$.

Now we introduce the notion concerning the fixed points on M.

Definition 3.2. Let G be a finite group. Assume that $G \subseteq \mathrm{Aut}(M)$. We put, for $s\,(\neq 1) \in G$,

$$r(s) = \sharp\,\{\, P \in M \,;\, G_P \supseteq \langle s \rangle \text{ and } \theta_P(s) = \zeta_{\sharp s} \},$$
$$r_*(s) = \sharp\,\{\, P \in M \,;\, G_P = \langle s \rangle \text{ and } \theta_P(s) = \zeta_{\sharp s} \},$$

and put

$$r(1) = 1 - g$$

where g denotes the genus of M. Then we get a class function $r : G \longrightarrow \boldsymbol{Q}$, which we call the *r-datum* of G.

Let G be as above. We denote by $\chi^{(q)}$ ($q = 1, 2, \ldots$) the Lefschetz trace (cf. [2])

$$\sum_{i\geq 0}(-1)^i \operatorname{Tr}(G|H^i(M, \Omega^{\otimes q}))$$

of the natural action of G on the space of q-differentials on M. Then we have

$$\chi^{(1)} = \operatorname{Tr}(G|H^0(M, \Omega)) - 1_G,$$
$$\chi^{(q)} = \operatorname{Tr}(G|H^0(M, \Omega^{\otimes q})) \text{ for } q \geq 2.$$

Here 1_G is the principal character of G which is given by $1_G(s) = 1$ for all $s \in G$.

Proposition 3.1 (**The Chevalley-Weil formula**). *In Situation* 2.1, *we assume that* $G \subseteq \operatorname{Aut}(M)$. *Let* $r = [\, g_0\, ;\, l(s_1), \ldots, l(s_h)\,]$ *be an* r*-signature of* G. *Then we have, for* $q = 1, 2, \ldots$,

$$\chi^{(q)} = \left\{(2q-1)(g_0-1) + q\sum_{i=1}^{h} l(s_i)\cdot\left(1 - \frac{1}{\sharp s_i}\right)\right\}\cdot \operatorname{reg}_G - \sum_{i=1}^{h} l(s_i)\cdot \mu'^{(q)}_{s_i}$$

where

$$\mu'^{(q)}_s = \frac{1}{\sharp s}\sum_{d=0}^{\sharp s-1} d\cdot \operatorname{Ind}_{\langle s\rangle}^{G}\left(\theta_s^{d+q}\right)$$

for $s \in G$.

Here reg_G is the regular character of G which is given by

$$\operatorname{reg}_G(s) = 0 \text{ for } s\,(\neq 1) \in G,$$
$$\operatorname{reg}_G(1) = \sharp G.$$

Further $\operatorname{Ind}_{\langle s\rangle}^{G}$ is the induced character which is given by

$$\operatorname{Ind}_{\langle s\rangle}^{G}(\theta_s^\alpha)\,(t) = \frac{1}{\sharp s}\sum_{\substack{\sigma\in G\\ \sigma^{-1}t\sigma\in\langle s\rangle}} \theta_s^\alpha(\sigma^{-1}t\sigma) \quad \text{for } t \in G$$

where $\theta_s^\alpha : \langle s\rangle \to \boldsymbol{C}^\times$ by the equation $\theta_s^\alpha(s) = \zeta_{\sharp s}^\alpha$.

4. Virtual r-signatures and virtual r-datum

Given a finite (abstract) group G a priori, we shall abstract the notion in **3**.

Let G be a finite group. In Situation 2.1, for an $h+1$-tuple $[\, g_0\, ;\, l_1, \ldots, l_h\,]$ of rational numbers, we put, for $s\,(\neq 1) \in G$,

$$r(s) = \sum_{s'\in G}[\, C_G(\langle s'\rangle) : \langle s'\rangle\,]\cdot l(s')$$

where $s = s'^{[\langle s'\rangle:\langle s\rangle]}$, $l(s') = l_i$ in case $s' \overset{G}{\sim} s_i$. Here $[\ :\ ]$ is the index and $C_G(H)$ is the centralizer of H in G. Further we put

$$r(1) = \sharp G\left\{1 - g_0 - \frac{1}{2}\sum_{i=1}^{h} l_i\cdot\left(1 - \frac{1}{\sharp s_i}\right)\right\}.$$

Then we get a class function $r : G \to \boldsymbol{Q}$.

On the other hand, for a class function $r : G \to \boldsymbol{Q}$, we define an $h+1$-tuple $[g_0; l_1, \ldots, l_h]$ of rational numbers by the following relations :

$$\text{(i)} \quad r_*(s) = r(s) - \sum_{s' \in G} r_*(s') \quad \text{where} \quad s = s'^{[\langle s' \rangle : \langle s \rangle]}$$
for $s\,(\neq 1) \in G$ (defined by descending condition),

$$\text{(ii)} \quad l_i = \frac{r_*(s_i)}{[\, C_G(\langle s_i \rangle) : \langle s_i \rangle \,]} \quad (i \neq 0),$$

$$\text{(iii)} \quad g_0 = 1 - \frac{1}{\sharp G} r(1) - \frac{1}{2} \sum_{i=1}^{h} l_i \cdot \left(1 - \frac{1}{\sharp s_i}\right).$$

Hence we see that the tuple $[\, g_0 \,; l_1, \ldots, l_h \,]$ and the class function $r : G \to \boldsymbol{Q}$ are the same notion. So we use the same notation.

Definition 4.1. In Situation 2.1, we call the tuple

$$r = [\, g_0 \,;\, l_1, \ldots, l_h \,]$$

the *virtual r-signature* of G with respect to $C_1, \ldots, C_h$ and the class function $r : G \to \boldsymbol{Q}$ the *virtual r-datum* of G.

Now we shall denote by $\chi_{[r]}^{(q)}$ the right-hand side of the Chevalley-Weil formula in Proposition 3.1, abstractly. From this formula, we get our basic tools, the Eichler trace formula and the Riemann-Hurwitz relation.

Definition 4.2. In Situation 2.1, let $r = [\, g_0 \,;\, l_1, \ldots, l_h \,]$ be a virtual r-signature of G. We put, for $q = 1, 2, \ldots,$

$$\chi_{[r]}^{(q)} = \left\{ (2q-1)(g_0 - 1) + q \sum_{i=1}^{h} l_i \cdot \left(1 - \frac{1}{\sharp s_i}\right) \right\} \cdot \mathrm{reg}_G - \sum_{i=1}^{h} l_i \cdot \mu_{s_i}^{\prime (q)}$$

where

$$\mu_s^{\prime (q)} = \frac{1}{\sharp s} \sum_{d=0}^{\sharp s - 1} d \cdot \mathrm{Ind}_{\langle s \rangle}^{G} \left(\theta_s^{d+q} \right)$$

for $s \in G$, and put

$$\chi_{[r]} = 1_G + \chi_{[r]}^{(1)}, \quad g = \chi_{[r]}(1).$$

Proposition 4.1 (**The Eichler trace formula**). *Let G be a finite group. Let $r : G \to \boldsymbol{Q}$ be a virtual r-datum of G. Then we have, for $s\,(\neq 1) \in G$,*

$$\chi_{[r]}^{(q)}(s) = \sum_{\beta} r(s^{\beta^*}) \frac{\zeta_{\sharp s}^{\beta q}}{1 - \zeta_{\sharp s}^{\beta}}$$

where $(\beta, \sharp s) = 1$, $\beta\, \beta^* \equiv 1 \pmod{\sharp s}$, *and*

$$\chi_{[r]}^{(q)}(1) = (1 - 2q)\, r(1)$$

$(q = 1, 2, \ldots)$.

Proposition 4.2 (**The Riemann-Hurwitz relation**). *In Situation* 2.1, *let* $r = [\, g_0\, ;\, l_1, \ldots, l_h\,]$ *be a virtual r-signature of G. Then we have the following relation :*

$$2g - 2 = \sharp G \left\{ 2g_0 - 2 + \sum_{i=1}^{h} l_i \cdot \left(1 - \frac{1}{\sharp s_i} \right) \right\}.$$

Now we are in a position to give the definition of " r is realizable. "

Definition 4.3. In Situation 2.1, let $r = [\, g_0\, ;\, l_1, \ldots, l_h\,]$ be a virtual r-signature of G. We say that r is *realizable* of genus g if there exist a compact Riemann surface M of genus g and an inclusion $\iota : G \to \mathrm{Aut}(M)$ such that

(i) g_0 is the genus of M/G and

(ii) $l_i = \sharp\,(\pi\{\, P \in M\, ;\, G_P = \langle s_i \rangle$ and $\theta_P(s_i) = \zeta_{\sharp s_i}\})$ for $s_i \in C_i\, (1 \le i \le h)$, where the natural projection $\pi : M \to M/G$ (cf. Definition 3.1).

Proposition 4.3 (**The Riemann existence theorem**). *In Situation* 2.1, *let* $r = [\, g_0\, ;\, l_1, \ldots, l_h\,]$ *be a virtual r-signature of G. Then r is realizable of genus g if and only if*

(i) $g_0 \in \boldsymbol{Z}_{\ge 0},\ l_i \in \boldsymbol{Z}_{\ge 0}\, (1 \le i \le h)$ *and*

(ii) *there exist elements* $s_{ij} \in C_i\, (1 \le i \le h,\ 1 \le j \le l_i)$, *and* $\alpha_k, \beta_k \in G$ $(1 \le k \le g_0)$ *such that*

$$G = \langle\, \alpha_1, \beta_1, \ldots, \alpha_{g_0}, \beta_{g_0}, s_{1,1}, \ldots, s_{1,l_1}, \ldots, s_{h,1}, \ldots, s_{h,l_h}\, \rangle$$

with the relation

$$\prod_{k=1}^{g_0} [\, \alpha_k,\ \beta_k\,] \prod_{i,j} s_{ij} = 1.$$

5. Proof of Theorem

Proof of Theorem (I). The implication : (ii) $\Rightarrow$ (i) is trivial. To prove the converse, we introduce the notion for the virtual r-signature r of G. We put

$$(\, g_0\, ;\, \sharp s_1, \ldots, \sharp s_1, \ldots, \sharp s_h, \ldots, \sharp s_h\,).$$

Here $\sharp s_i$ appears l_i-times $(1 \le i \le h)$.

On the other hand, we put

$$(\, g_0\, ;\, m_1, \ldots, m_\nu\,)$$

where $2 \le m_1 \le \cdots \le m_\nu \le n,\ m_j | n$, which we call the *virtual branching data* of r. For the sake of brevity, we shall call, for example, the data (1). These are equal except their order. Then we have the relation between them as follows :

$$\sharp\{\, j\, ;\, m_j = m\,\} = \sum_{\substack{i \\ \sharp s_i = m}} l_i \quad \text{with} \quad m | n,\ m \ne 1.$$

To determine r corresponding to the data, we use this relation. Further we can reform the Riemann-Hurwitz relation in Proposition 4.2 as follows :

$$2g-2=n\left\{2g_0-2+\sum_{j=1}^{\nu}\left(1-\frac{1}{m_j}\right)\right\}.$$

Now we assume that (i) $\chi_{[r]}=0$. Hence $g=0$. Then there exist the following five possibilities :

(1) $(\,0\,;\,n,\,n\,)\;:\;n\geq 2$.

(2) $(\,0\,;\,2,\,2,\,m\,)\;:\;n=2m\;(m\geq 2)$.

(3) $(\,0\,;\,2,\,3,\,3\,)\;:\;n=12$.

(4) $(\,0\,;\,2,\,3,\,4\,)\;:\;n=24$.

(5) $(\,0\,;\,2,\,3,\,5\,)\;:\;n=60$.

In particular we shall determine r corresponding to the date (2).

The case of the data (2) : Let $m\neq 2$. We claim that $G=D_m$. In fact, first, we shall show that

$$G=\langle\, a,b\,;\,a^m=b^2=1,\;bab^{-1}=a^u\,\rangle.$$

For the data (2), we have r as follows :

(i) $g_0=0,\;l_i=l_{i'}=l_{i''}=1,$ other $l_j=0$ where $\sharp s_i=\sharp s_{i'}=2,\;\sharp s_{i''}=m.$

(ii) $g_0=0,\;l_i=2,\;l_{i'}=1,$ other $l_j=0$ where $\sharp s_i=2,\;\sharp s_{i'}=m.$

From the data (2), we see that there exists an element $a\in G$ with $\sharp a=m$. In case m : odd, there exist no elements of order 2 in $\langle a\rangle$. From the data (2), we see that there exists an element $b\notin\langle a\rangle$ with $\sharp b=2$. In case m : even, there exists a unique element $a^{\frac{m}{2}}\in\langle a\rangle$ with $a^{\frac{m}{2}}=2$. Now we assume that $l(a^{\frac{m}{2}})=2$.

Remark 5.1. $l(a^{\frac{m}{2}})=2$ means that $l_i=2$ in case $a^{\frac{m}{2}}\overset{G}{\sim}s_i$.

Then $\chi_{[r]}\neq 0$. In fact, by the Eichler trace formula in Proposition 4.1,

$$\chi_{[r]}(a^{\frac{m}{2}})=1-\frac{1}{2}r(a^{\frac{m}{2}}).$$

Here, by Definition 4.1,

$$r(a^{\frac{m}{2}})=\sum_{s'\in G}[\,C_G(\langle s'\rangle):\langle s'\rangle\,]\cdot l(s')$$

where $a^{\frac{m}{2}}=s'^{\frac{\sharp s'}{2}}$. Consider the elements s' satisfying the condition $a^{\frac{m}{2}}=s'^{\frac{\sharp s'}{2}}$. From the virtual r-signature r, it is sufficient to consider elements order 2 and m. If s' is satisfied with this condition, so is elements which is G-cojugate to s'. The element of order 2 satisfying this condition is exactly one $a^{\frac{m}{2}}$. As for the elements of order m satisfying this condition, we can consider the following cases :

Case (1) : $l(s') = 0$ for every s'.

Case (2) : There exists an element s' such that $l(s') = 1$.

Hence we have

$$\begin{aligned} r(a^{\frac{m}{2}}) &= [\, C_G(\langle a^{\frac{m}{2}} \rangle) : \langle a^{\frac{m}{2}} \rangle \,] \cdot l(a^{\frac{m}{2}}) + [\, G : C_G(\langle s' \rangle) \,] \cdot [\, C_G(\langle s' \rangle) : \langle s' \rangle \,] \cdot l(s') \\ &= 2m + 2\,l(s'). \end{aligned}$$

Therefore

$$\chi_{[r]}(a^{\frac{m}{2}}) = 1 - m - l(s') = \begin{cases} 1-m & \text{in case (1),} \\ -m & \text{in case (2).} \end{cases}$$

Hence $\chi_{[r]} \neq 0$. This is absurd to the assumption. Thus we see that $l(a^{\frac{m}{2}}) \neq 2$. From the virtual r-signature r, we see that there exists an element $b \notin \langle a \rangle$ with $\sharp b = 2$. Since $[\, G : \langle a \rangle \,] = 2$, we have

$$G = \langle a \rangle + \langle a \rangle b \quad (\text{the right coset decomposition})$$

and $\langle a \rangle$ is a normal subgroup of G. Hence

$$G = \langle\, a, b\,;\; a^m = b^2 = 1,\; bab^{-1} = a^u \,\rangle.$$

Further we have $u^2 \equiv 1 \pmod{m}$.

Remark 5.2. $l(a^{\frac{m}{2}}) = 0$.

In case $u = 1$. We note that G is abelian. Then $\chi_{[r]} \neq 0$. In fact we have

$$\chi_{[r]}(b) = 1 - \frac{1}{2} r(b).$$

Here

$$r(b) = \sum_{s' \in G} [\, C_G(\langle s' \rangle) : \langle s' \rangle \,] \cdot l(s')$$

where $b = s'^{\frac{\sharp s'}{2}}$. Consider the elements s' satisfying the condition $b = s'^{\frac{\sharp s'}{2}}$. From the virtual r-signature r, it is sufficient to consider elements order 2 and m. The element of order 2 satisfying this condition is exactly one b. There exist no elements of order m satisfying this condition. Hence we have

$$r(b) = [\, C_G(\langle b \rangle) : \langle b \rangle \,] \cdot l(b) = m\,l(b).$$

Therefore

$$\chi_{[r]}(b) = 1 - \frac{1}{2} m\,l(b) = \begin{cases} 1 & \text{if } l(b) = 0, \\ 1 - \frac{1}{2} m & \text{if } l(b) = 1, \\ 1 - m & \text{if } l(b) = 2. \end{cases}$$

Hence $\chi_{[r]} \neq 0$. This is absurd to the assumption.

Let $u \neq 1$. We claim that $u = -1$. In fact we consider that

$$\chi_{[r]}(a) = 1 + \sum_{\beta^*} r(a^\beta) \frac{\zeta^{\beta^*}}{1 - \zeta^{\beta^*}}$$

where $(\beta^*, m) = 1,\ \beta\,\beta^* \equiv 1 \pmod m,\ \zeta = \zeta_m$. Here we note that $(\beta, m) = 1$. Further

$$r(a^\beta) = \sum_{s' \in G} [\, C_G(\langle s' \rangle) : \langle s' \rangle \,] \cdot l(s')$$

where $a^\beta = s'^{\frac{\sharp s'}{m}}$. Consider the elements s' satisfying the condition $a^\beta = s'^{\frac{\sharp s'}{m}}$. From the virtual r-signature r, it is sufficient to consider elements order 2 and m. The element of order m satisfying this condition is exactly one a^β.

Remark 5.3. In case m : even and $\beta = 1$, we have $a^{\frac{m}{2}}$ with $\sharp a^{\frac{m}{2}} = 2$ satisfying this condition.

By Remark 5.2, we can consider that there exist no elements of order 2 satisfying this condition. Hence we have

$$r(a^\beta) = [\, C_G(\langle a^\beta \rangle) : \langle a^\beta \rangle \,] \cdot l(a^\beta) = l(a^\beta).$$

Therefore

$$\chi_{[r]}(a) = 1 + \sum_{\beta^*} l(a^\beta) \frac{\zeta^{\beta^*}}{1 - \zeta^{\beta^*}}.$$

As for the quantity $l(a^\beta)$, we can consider the following cases :

Case (1) : $l(a^\beta) = 0$ for every β.

Case (2) : There exists β such that $l(a^\beta) = 1$.

In case (1), we have $\chi_{[r]}(a) = 1$. This is absurd to the assumption. In case (2), since $a^\beta \overset{G}{\sim} a^{u\beta}$, we have $l(a^\beta) = l(a^{u\beta}) = 1$. Hence

$$\chi_{[r]}(a) = 1 + \frac{\zeta^{\beta^*}}{1 - \zeta^{\beta^*}} + \frac{\zeta^{(u\beta)^*}}{1 - \zeta^{(u\beta)^*}}.$$

By the assumption, it must be $\chi_{[r]}(a) = 0$. Then

$$\frac{\zeta^{\beta^*}}{1 - \zeta^{\beta^*}} + \frac{\zeta^{(u\beta)^*}}{1 - \zeta^{(u\beta)^*}} = -1.$$

Hence we have $\zeta^{\beta^* + (u\beta)^*} = 1$ by simple calculation. Therefore we have $u^* \equiv -1 \pmod m$. Hence $u = -1$. Thus we see that $G = D_m$, i.e.,

$$G = \langle\, a, b\, ;\ a^m = b^2 = 1,\ bab^{-1} = a^{-1} \,\rangle.$$

Then we have r in this case as follows :

In case $m = 2k$: $\sharp s_i = m,\ \sharp s_{k+1} = \sharp s_{k+2} = 2,\ (i, m) = 1,\ 1 \leq i \leq k - 1$.

(i) $g_0 = 0,\ l_i = 1,\ l_{k+1} = 2,$ other $l_j = 0.$

(ii) $g_0 = 0,\ l_i = 1,\ l_{k+2} = 2,$ other $l_j = 0.$

(iii) $g_0 = 0,\ l_i = l_{k+1} = l_{k+2} = 1,$ other $l_j = 0.$

In case $m = 2k+1 :\ \sharp s_i = m,\ \sharp s_{k+2} = 2,\ (i, m) = 1,\ 1 \leq i \leq k.$

(i) $g_0 = 0,\ l_i = 1,\ l_{k+1} = 2,$ other $l_j = 0.$

Let $m = 2$. Then we have the data (2) $(\,0\,;\ 2,\ 2,\ 2\,)$. The finite groups of order 4 are

$\boldsymbol{Z}_2 \times \boldsymbol{Z}_2 = \langle\, a, b\,;\ a^2 = b^2 = 1,\ bab^{-1} = a^{-1}\,\rangle,$

$\boldsymbol{Z}_4.$

For these groups, we have r as follows :

$\boldsymbol{Z}_2 \times \boldsymbol{Z}_2 :\ \sharp s_1 = \sharp s_2 = \sharp s_3 = 2.$

(1) $g_0 = 0,\ l_1 = l_2 = l_3 = 1.$ (2) $g_0 = 0,\ l_1 = 3,$ other $l_i = 0.$

(3) $g_0 = 0,\ l_2 = 3,$ other $l_i = 0.$ (4) $g_0 = 0,\ l_3 = 3,$ other $l_i = 0.$

(5) $g_0 = 0,\ l_1 = 2,\ l_2 = 1,$ other $l_i = 0.$ (6) $g_0 = 0,\ l_1 = 2,\ l_3 = 1,$ other $l_i = 0.$

(7) $g_0 = 0,\ l_1 = 1,\ l_2 = 2,$ other $l_i = 0.$ (8) $g_0 = 0,\ l_2 = 2,\ l_3 = 1,$ other $l_i = 0.$

(9) $g_0 = 0,\ l_1 = 1,\ l_3 = 2,$ other $l_i = 0.$ (10) $g_0 = 0,\ l_2 = 1,\ l_3 = 2,$ other $l_i = 0.$

$\boldsymbol{Z}_4 :\ \sharp s_1 = 2.$

(1) $g_0 = 0,\ l_1 = 3,$ other $l_i = 0.$

Thus we have r for the data (2). By the assumption, however, we must exclude r such that $\chi_{[r]} \neq 0$. So we shall check such r. Let

$$\chi_{[r]} = n_0\chi_0 + n_1\chi_1 + \cdots + n_h\chi_h$$

be the decomposition of $\chi_{[r]}$ into the irreducible characters of G. Since

$$\chi_{[r]} = 0 \ \Leftrightarrow\ n_0 = n_1 = \cdots = n_h = 0,$$

it is sufficient to check an irreducible character χ_i such that $n_i = \langle\, \chi_{[r]},\ \chi_i\,\rangle \neq 0$. Here $\langle\, \chi,\ \varphi\,\rangle$ is a hermitian inner product which is given by

$$\langle\, \chi,\ \varphi\,\rangle = \frac{1}{\sharp G}\sum_{\sigma\in G} \chi(\sigma)\cdot\varphi(\sigma^{-1})$$

for characters χ and φ. Recall that

$$\chi_{[r]} = 1_G + \left\{ g_0 - 1 + \sum_{i=1}^{h} l_i \cdot \left(1 - \frac{1}{\sharp s_i}\right)\right\}\cdot \mathrm{reg}_G - \sum_{i=1}^{h} l_i \cdot \mu_{s_i}'^{(1)}$$

where

$$\mu_s'^{(1)} = \frac{1}{\sharp s}\sum_{d=0}^{\sharp s - 1} d \cdot \mathrm{Ind}_{\langle s\rangle}^{G}\left(\theta_s^{d+1}\right)$$

for $s \in G$. For an irreducible character χ of G, we have

$$\langle \chi_{[r]}, \chi \rangle = \langle 1_G, \chi \rangle + \left\{ g_0 - 1 + \sum_{i=1}^{h} l_i \cdot \left(1 - \frac{1}{\sharp s_i}\right)\right\} \cdot \langle \mathrm{reg}_G, \chi \rangle$$
$$- \sum_{i=1}^{h} l_i \cdot \langle \mu_{s_i}^{\prime(1)}, \chi \rangle.$$

Here

$$\langle 1_G, \chi \rangle = \begin{cases} 1 & \text{if } \chi \text{ is the principal character of } G, \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle \mathrm{reg}_G, \chi \rangle = \chi(1),$$

$$\begin{aligned} \langle \mu_s^{\prime(1)}, \chi \rangle &= \frac{1}{\sharp s} \sum_{d=0}^{\sharp s-1} d \cdot \langle \mathrm{Ind}_{\langle s \rangle}^{G}\left(\theta_s^{d+1}\right), \chi \rangle \\ &= \frac{1}{\sharp s} \sum_{d=0}^{\sharp s-1} d \cdot \langle \theta_s^{d+1}, \chi|_{\langle s \rangle} \rangle \quad (\text{by the Frobenius reciprocity law}) \\ &= \frac{1}{\sharp s} \sum_{d=0}^{\sharp s-1} \sum_{k=0}^{\sharp s-1} d x_k \cdot \langle \theta_s^{d+1}, \theta_s^k \rangle \\ &= \frac{1}{\sharp s} \sum_{d=0}^{\sharp s-1} d x_{d+1} \end{aligned}$$

where

$$\chi|_{\langle s \rangle} = x_0 \theta_s^0 + x_1 \theta_s^1 + \cdots + x_{\sharp s-1} \theta_s^{\sharp s-1}, \ x_{\sharp s} = x_0.$$

We can determine the coefficients of $\chi|_{\langle s \rangle}$, since we obtain

$$\chi|_{\langle s \rangle}(1), \ \chi|_{\langle s \rangle}(s), \ldots, \chi|_{\langle s \rangle}(s^{\sharp s-1})$$

by the character table of G (see **6**).

Remark 5.4. In case $\chi = 1_G$, we have $\langle \chi_{[r]}, 1_G \rangle = g_0$.

Thus we can determine r under the assumption as follows :

(2) D_m :

(i) In case $m = 2k$: $g_0 = 0, \ l_i = l_{k+1} = l_{k+2} = 1$, other $l_j = 0$ with $(i, m) = 1, \ 1 \leq i \leq k$.

(ii) In case $m = 2k+1$: $g_0 = 0, \ l_i = 1, \ l_{k+1} = 2$, other $l_j = 0$ with $(i, m) = 1, \ 1 \leq i \leq k$.

Remark 5.5. In case (2) $m = 2$, we have

$$\boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \ : \ g_0 = 0, \ l_1 = l_2 = l_3 = 1.$$

This belongs to (i).

Remark 5.6. In the same way, we can verify $\chi_{[r]} = 0$ for r as above.

Applying Proposition 4.3, we shall show that r is realizable of genus zero.

(2) D_m :

(i) We can take $a^i \in C_i,\ b \in C_{k+1},\ a^ib \in C_{k+2}$ such that $D_m = \langle\, a^i,\ b,\ a^ib\,\rangle$ with $a^i \cdot b \cdot a^ib = 1$.

(ii) We can take $a^{-i} \in C_i,\ a^ib \in C_{k+1},\ b \in C_{k+1}$ such that $D_m = \langle\, a^{-i},\ a^ib,\ b\,\rangle$ with $a^{-i} \cdot a^ib \cdot b = 1$.

Thus we see that r is realizable of genus zero.

Further by considering the other cases it is easy to see that we have a one-to-one correspondence between the data (2) and D_m. *q.e.d.*

6. Appendix

For an irreducible character χ such that $\langle\, \chi_{[r]},\ \chi\,\rangle \neq 0$, the character table is as follows (cf. [13]) :

$D_m\,(m = 2k)$:

	1	m	2	2
	C_0	C_i	C_{k+1}	C_{k+2}
χ_1	1	1	-1	-1
χ_2	1	$(-1)^i$	1	-1
χ_3	1	$(-1)^i$	-1	1

$1 \leq i \leq k$.

Here the first row gives the order of elements of each conjugacy class.

$$\chi_1|_{\langle s_i\rangle} = \theta^0_{s_i},\ \chi_1|_{\langle s_{k+1}\rangle} = \theta^1_{s_{k+1}},\ \chi_1|_{\langle s_{k+2}\rangle} = \theta^1_{s_{k+2}},$$
$$\chi_2|_{\langle s_i\rangle} = \theta^k_{s_i},\ \chi_2|_{\langle s_{k+1}\rangle} = \theta^0_{s_{k+1}},\ \chi_2|_{\langle s_{k+2}\rangle} = \theta^1_{s_{k+2}},$$
$$\chi_3|_{\langle s_i\rangle} = \theta^k_{s_i},\ \chi_3|_{\langle s_{k+1}\rangle} = \theta^1_{s_{k+1}},\ \chi_3|_{\langle s_{k+2}\rangle} = \theta^0_{s_{k+2}}.$$

(i) : $\langle\, \chi_{[r]},\ \chi_3\,\rangle = \dfrac{1}{2}$. (ii) : $\langle\, \chi_{[r]},\ \chi_2\,\rangle = \dfrac{1}{2}$.

$\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$:

See the character table of $D_m\,(m = 2)$.

(2), (6), (9) : $\langle\, \chi_{[r]},\ \chi_2\,\rangle = \dfrac{1}{2}$. (3), (5), (7) : $\langle\, \chi_{[r]},\ \chi_3\,\rangle = \dfrac{1}{2}$.

(4), (8), (10) : $\langle\, \chi_{[r]},\ \chi_1\,\rangle = \dfrac{1}{2}$.

$\boldsymbol{Z}_4$:

	1	2
	C_0	C_1
χ	1	1

$\chi|_{\langle s_1\rangle} = \theta^0_{s_1}$. (1) : $\langle\, \chi_{[r]},\ \chi\,\rangle = -1$.

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