

## A generalization of power series to real several dimensions

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**Introduction.** In this paper I would like to propose a generalization of power series to real several dimensions.

Weierstrass laid the foundation of the theory of analytic functions on power series and analytic continuations. Elementary functions like  $e^z$ ,  $\sin z$ ,  $\cos z$ , and  $\log(1+z)$  are all represented by convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with real coefficients  $a_n$ . These real analytic functions are however much better understood when we consider  $z$  as a complex variable rather than a real variable. One natural thought might be to consider  $z$  in the power series as a quaternion variable. This gives us a generalization of power series to real four dimension. However this simpleminded generalization is not satisfactory for a couple of reasons. First it gives a generalization only to four dimension. More importantly an essential difficulty arises when one tries analytic continuations; a translation of the quaternion variable in a power series results in a series which is not a power series any more since the quaternions are not commutative.

This difficulty of analytic continuations in a noncommutative situation lead me to consider the use of a special kind of Jordan algebras which I denote by  $J_n$ . These Jordan algebras  $J_n$  are explained in the first section.

In the second section I will give a short account of the Clifford algebras  $Cl_n$  and state the relationship between the Jordan algebra  $J_n$  and the Clifford algebra  $Cl_n$ . This relationship is in fact the reason why I started studying the Jordan algebras  $J_n$ . The generalization of power series is actually formulated in terms of the Jordan algebras  $J_n$  only. In this sense one can go without referring to the Clifford algebras at all. However without the perspective of the Clifford algebras the understanding of the generalization would be one-sided.

A theory of nonassociative polynomials over the Jordan algebra  $J_n$  will be developed in the third section, and the generalization of power series will be given in the forth section. The main features of the generalization given in this paper are the following:

- (1) Every complex power series canonically gives rise to a local analytic transformation of the euclidean space of any dimension  $n > 2$  which extends the complex analytic function defined by the power series on the embedded complex plane.
- (2) Analytic continuations can be performed.

One of my original motivations for this work was to try to define a class of analytic transformations of the  $n$ -dimensional euclidean space which is larger than the class of conformal maps but is not as large as the class of all analytic maps. From this point of view Theorem in the last section is a negative statement. It suggests that such a class cannot be defined directly by the generalization of power series given here.

**1. The Jordan algebra  $J_n$ .** Most, if not all, of the materials presented in this and the next sections are well-known to algebraists in more general forms. The purposes of the first two sections are to provide notations suitable for the generalization of power series given later in this paper and to make the subjects more accessible to the readers in other fields of mathematics.

A real vector space  $J$  equipped with a bilinear map  $\mu: J \times J \rightarrow J$ , which will be called the multiplication, is a *Jordan algebra* if the following two conditions are satisfied:

$$\{a, b\} = \{b, a\}, \quad (\forall a, b \in J), \quad (1.1)$$

$$\{\{a, a\}, \{a, b\}\} = \{a, \{\{a, a\}, b\}\}, \quad (\forall a, b \in J). \quad (1.2)$$

Here  $\{a, b\}$  denotes the image of  $(a, b) \in J \times J$  under the multiplication  $\mu$ . The notion of Jordan algebras was first introduced by P. Jordan in order to develop an algebraic theory of Hermitian operators (see [4]), and has later been used to construct certain Lie algebras. For a systematic treatment of Jordan algebras the reader is referred to [3].

The multiplication by an element  $a \in J$  defines a linear transformation  $T_a: J \rightarrow J$ . Namely

$$T_a(x) = \{a, x\}. \quad (1.3)$$

Using this notation the condition (1.2) can be written as  $T_a \circ T_{\{a, a\}} = T_{\{a, a\}} \circ T_a$ . Notice that the multiplication is not assumed to be associative. Therefore the equality  $T_a \circ T_b = T_{\{a, b\}}$  does not hold in general.

The Jordan algebra  $J_n$  is defined as follows. The underlying space of  $J_n$  is simply  $\mathbb{R}^{n+1}$  which we regard as the direct sum  $J_n^{(0)} \oplus J_n^{(1)}$  of the set of real numbers  $J_n^{(0)} = \mathbb{R}$  and the  $n$ -dimensional euclidean space  $J_n^{(1)} = \mathbb{R}^n$ . Every element  $a$  of  $J_n$  can be uniquely written as  $a = a^{(0)} + a^{(1)}$ , where  $a^{(0)} \in J_n^{(0)}$  and  $a^{(1)} \in J_n^{(1)}$ . We call the real number  $a^{(0)}$  the real part of  $a$ , and the vector  $a^{(1)}$  the imaginary part of  $a$ . For  $a = a^{(0)} + a^{(1)} \in J_n$  and  $b = b^{(0)} + b^{(1)} \in J_n$ , the multiplication of  $J_n$  is defined by

$$\{a, b\} = (a^{(0)}b^{(0)} - a^{(1)} \cdot b^{(1)}) + (a^{(0)}b^{(1)} + b^{(0)}a^{(1)}), \quad (1.4)$$

where  $a^{(1)} \cdot b^{(1)}$  denotes the inner product of the two vectors  $a^{(1)}$  and  $b^{(1)}$  in the euclidean space  $J_n^{(1)}$ . One can immediately see that the multiplication satisfies the commutativity condition (1.1). The condition (1.2) can also be verified directly by an easy computation, but the relationship between  $J_n$  and the Clifford algebra  $Cl_n$  explained in the next section will make it obvious. In any case  $J_n$  is therefore a Jordan algebra. Note that the multiplication by a real number  $r \in J_n^{(0)}$  in the Jordan algebra  $J_n$  is simply the scalar multiplication by  $r$ . In particular the real number 1 acts as the unit element of  $J_n$ .

One would probably notice that the multiplication of  $J_n$  is exactly the multiplication of the complex numbers when  $n = 1$ . In fact we can canonically identify  $J_0$  with the field of real numbers  $\mathbb{R}$ , and  $J_1$  with the field of complex numbers  $\mathbb{C}$ . The multiplication of  $J_n$  is therefore associative for  $n = 0$  and 1. However for  $n > 1$  the Jordan algebra  $J_n$  is not associative.

The Jordan algebra  $J_n$  is naturally equipped with a norm as the  $(n + 1)$ -dimensional euclidean space. Namely the norm of an element  $a = a^{(0)} + a^{(1)} \in J_n$  is given by

$$|a| = \sqrt{a^{(0)2} + |a^{(1)}|^2}. \quad (1.5)$$

The following inequality is crucial in our generalization of power series.

PROPOSITION 1.  $|\{a, b\}| \leq |a||b|$  for all  $a, b \in J_n$ .

PROOF: A straightforward computation by the definition (1.5) shows

$$\begin{aligned} & |a|^2|b|^2 - |\{a, b\}|^2 \\ &= (a^{(0)2} + |a^{(1)}|^2)(b^{(0)2} + |b^{(1)}|^2) - (a^{(0)}b^{(0)} - a^{(1)} \cdot b^{(1)})^2 - |a^{(0)}b^{(1)} + b^{(0)}a^{(1)}|^2 \\ &= |a^{(1)}|^2|b^{(1)}|^2 - |a^{(1)} \cdot b^{(1)}|^2 \\ &\geq 0, \end{aligned}$$

where the last is the Schwarz's inequality.

We define the conjugate of an element  $a$  of the Jordan algebra  $J_n$  to be  $\bar{a} = a^{(0)} - a^{(1)}$ . This is a direct generalization of the conjugation for complex numbers. We have  $\{a, \bar{a}\} = |a|^2 \in \mathbb{R}$  for any  $a \in J_n$ .

Let us investigate the linear transformation  $T_a: J_n \rightarrow J_n$  defined for  $a \in J_n$  by (1.3). First assume that  $|a| = 1$ . Then the element  $a$  is of the form

$$a = \cos \theta + (\sin \theta)v, \quad (0 \leq \theta \leq \pi, v \in J_n^{(1)}, |v| = 1).$$

Every element  $z \in J_n$  can be written as

$$z = z_0 + z_1v + w, \quad (z_0, z_1 \in \mathbb{R}, w \in J_n^{(1)}, w \perp v).$$

Then by an easy computation we obtain

$$T_a(z) = (z_0 \cos \theta - z_1 \sin \theta) + (z_0 \sin \theta + z_1 \cos \theta)v + (\cos \theta)w.$$

Geometrically this means the following. Let  $P$  denote the 2-dimensional subspace of  $J_n$  spanned by 1 and  $v$ , and  $P^\perp$  its orthogonal complement so that  $J_n = P \oplus P^\perp$ . Then the linear transformation  $T_a: J_n \rightarrow J_n$  is the direct sum of the rotation of angle  $\theta$  in  $P$  and the scalar multiplication by  $\cos \theta$  in  $P^\perp$ . In general every element of  $J_n$  can be written as  $ra$  for some nonnegative real number  $r$  and for some  $a \in J_n$  with  $|a| = 1$ . The linear transformation  $T_{ra}$  is the composition of  $T_a$  and the scalar multiplication by  $r$ .

As a consequence of the above argument we obtain:

PROPOSITION 2. The linear transformation  $T_a: J_n \rightarrow J_n$  is

- (1) non-singular if the real part  $a^{(0)}$  of  $a \in J_n$  is nonzero,
- (2) of rank at most two if  $a \in J_n^{(1)}$ .

**2. The Clifford algebra  $Cl_n$ .** The fields of real and complex numbers are both associative and commutative. In order to extend these fields to higher dimensional real algebras we must give up one of the two properties. Giving up the associativity naturally leads to the Jordan algebras  $J_n$  explained in the previous section. If we abandon the commutativity instead we arrive at the concept of Clifford algebras.

In [1] Ahlfors initiated the use of Clifford algebras in the study of higher dimensional Möbius transformations, and I partially demonstrated its usefulness in [6] and [7]. The general philosophy behind these works is to simplify computations involved in the  $n$ -dimensional transformations by the use of an algebraic operation. And this is in fact the origin of the current work.

The Clifford algebras have mainly been studied for the sake of the spin groups and their representations. The readers interested in this aspect of Clifford algebras are referred to [2] and [5].

Consider the  $n$ -dimensional euclidean space  $\mathbb{R}^n$  and the quadratic form  $q: \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $q(v) = -|v|^2$  for  $v \in \mathbb{R}^n$ , where  $|v|$  is the norm of the vector  $v$  in the euclidean space  $\mathbb{R}^n$ . We denote by  $Cl_n$  the Clifford algebra associated to  $(\mathbb{R}^n, q)$ . In terms of an orthonormal basis  $\{i_1, \dots, i_n\}$  for  $\mathbb{R}^n$ , the Clifford algebra  $Cl_n$  can be described as follows. The Clifford algebra  $Cl_n$  is the real associative algebra generated by the elements  $i_1, \dots, i_n$  satisfying the following fundamental relations.

$$\begin{aligned} i_j^2 &= -1, & (j = 1, \dots, n), \\ i_j i_k &= -i_k i_j, & (j, k = 1, \dots, n, j \neq k). \end{aligned}$$

For  $n = 0, 1$ , and  $2$ , the Clifford algebra  $Cl_n$  can be naturally identified with the field of real numbers  $\mathbb{R}$ , the field of complex numbers  $\mathbb{C}$ , and the skew field  $\mathbb{H}$  of quaternions respectively.

Every element  $a$  of the Clifford algebra  $Cl_n$  can be uniquely written as

$$a = \sum a_I I, \quad (a_I \in \mathbb{R}),$$

where the sum is taken over all the products  $I$  of generators of the form

$$I = i_{j_1} \cdots i_{j_r}, \quad (1 \leq j_1 < \cdots < j_r \leq n).$$

In particular the Clifford algebra  $Cl_n$  is  $2^n$ -dimensional as a real vector space. The number  $r$  of generators in the product  $I$  is called the degree of  $I$ .

We denote by  $Cl_n^{(r)}$  the subspace of  $Cl_n$  spanned by the products  $I$  of generators of degree  $r$ . Then we have the direct sum decomposition

$$Cl_n = Cl_n^{(0)} \oplus Cl_n^{(1)} \oplus \cdots \oplus Cl_n^{(n)}.$$

Accordingly every element  $a$  of the Clifford algebra  $Cl_n$  is written as

$$a = a^{(0)} + a^{(1)} + \cdots + a^{(n)}, \quad (2.1)$$

where  $a^{(r)} \in Cl_n^{(r)}$ , ( $r = 0, \dots, n$ ). The subspace  $Cl_n^{(0)}$  is in fact a subalgebra, and can be naturally identified with the field of real numbers  $\mathbb{R}$ . The subspace  $Cl_n^{(1)}$  is nothing but the euclidean space  $\mathbb{R}^n$ . We can easily verify that

$$uv + vu = -2u \cdot v, \quad (2.2)$$

for any vectors  $u, v \in Cl_n^{(1)}$ , and in particular that  $v^2 = -|v|^2 = q(v)$ .

The norm of an element  $a = \sum a_I I$  of  $Cl_n$  is defined to be

$$|a| = \sqrt{\sum a_I^2}.$$

With respect to this norm the multiplication in the Clifford algebra  $Cl_n$  satisfies the following inequality:

$$|ab| \leq |a| |b|, \quad (\forall a, b \in Cl_n). \quad (2.3)$$

In the Clifford algebra  $Cl_n$  there are three standard involutions. The main involution  $a \mapsto a'$  is the automorphism of  $Cl_n$  characterized by  $v' = -v$  for  $v \in Cl_n^{(1)}$ . The reversion  $a \mapsto a^*$  is the anti-automorphism of  $Cl_n$  which acts identically on  $Cl_n^{(1)}$ . These two involutions of  $Cl_n$  commute with each other; their composition is called the conjugation and denoted by  $\bar{a} = a'^* = a^{*'} (a \in Cl_n)$ . In terms of the direct sum decomposition (2.1) we have

$$\begin{aligned} a' &= a^{(0)} - a^{(1)} + a^{(2)} - a^{(3)} + \dots + (-1)^n a^{(n)}, \\ a^* &= a^{(0)} + a^{(1)} - a^{(2)} - a^{(3)} + \dots + (-1)^{\frac{n(n-1)}{2}} a^{(n)}, \\ \bar{a} &= a^{(0)} - a^{(1)} - a^{(2)} + a^{(3)} + \dots + (-1)^{\frac{n(n+1)}{2}} a^{(n)}. \end{aligned}$$

For any element  $a \in Cl_n$  the product  $a\bar{a}$  is real and equal to  $|a|^2$  if  $n \leq 3$ . But  $a\bar{a}$  is not necessarily a real number in general; the real part  $(a\bar{a})^{(0)}$  is always equal to  $|a|^2$ .

Now let us consider the subspace  $Cl_n^{(0)} \oplus Cl_n^{(1)}$  of the Clifford algebra  $Cl_n$ . This subspace is not a subalgebra of  $Cl_n$  if  $n > 1$ . However the subspace is closed under the operation  $Cl_n \times Cl_n \rightarrow Cl_n$ ,  $(a, b) \mapsto \{a, b\}$  defined by

$$\{a, b\} = \frac{1}{2}(ab + ba). \quad (2.4)$$

The vector space  $Cl_n^{(0)} \oplus Cl_n^{(1)}$  equipped with this bilinear operation is essentially the same thing as the Jordan algebra  $J_n$ . In fact using (2.2) we see for  $r, s \in \mathbb{R}$  and for  $u, v \in Cl_n^{(1)}$  that

$$\begin{aligned} \{r + u, s + v\} &= \frac{1}{2}((r + u)(s + v) + (s + v)(r + u)) \\ &= (rs - u \cdot v) + (rv + su). \end{aligned}$$

Compare this with (1.4). It is therefore quite natural to identify  $J_n^{(0)}$  with  $Cl_n^{(0)}$  and  $J_n^{(1)}$  with  $Cl_n^{(1)}$ , and regard the Jordan algebra  $J_n$  as a subspace of the Clifford algebra  $Cl_n$ . We

can then think that the multiplication of the Jordan algebra  $J_n = Cl_n^{(0)} \oplus Cl_n^{(1)}$  is defined by (2.4) in terms of the associative multiplication of the Clifford algebra  $Cl_n$ . Notice that the property (1.2) of the Jordan algebra  $J_n$  is an obvious consequence of this observation. The two notions of the norms coincide. Therefore Proposition 1 is in fact an immediate corollary of (2.3) and the triangle inequality. Also the conjugation in the Jordan algebra  $J_n$  is nothing but the conjugation in the Clifford algebra  $Cl_n$  restricted to  $J_n$ .

If two elements  $a$  and  $b$  of the Jordan algebra  $J_n$  commute with each other as elements of the Clifford algebra  $Cl_n$  then by (2.4) we have  $\{a, b\} = ab$ . It follows that the  $k$ -th power of an element  $a \in J_n$  is the same whether we use the multiplication of the Jordan algebra  $J_n$  or the one of the Clifford algebra  $Cl_n$ . We denote the  $k$ -th power of an element  $a \in J_n$  simply by  $a^k$ .

Finally let us consider some low-dimensional cases. For  $n = 0$  and  $1$  we have  $J_0 = Cl_0 = \mathbb{R}$  and  $J_1 = Cl_1 = \mathbb{C}$  respectively. The Clifford algebra  $Cl_2$  is isomorphic to the skew field  $\mathbb{H}$  of quaternions by the identification of  $i_1$  with  $i$ ,  $i_2$  with  $j$ , and  $i_1 i_2$  with  $k$ . Therefore in terms of the quaternions the Jordan algebra  $J_2$  may be thought of as the 3-dimensional vector space

$$J_2 = \{a = a_0 + a_1 i + a_2 j \in \mathbb{H} \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

equipped with the multiplication defined by (2.4).

The Jordan algebra  $J_3$  is naturally identified with the 4-dimensional subspace  $Cl_3^{(0)} \oplus Cl_3^{(1)}$  of  $Cl_3$ . Actually one can express the Jordan algebra  $J_3$  also in terms of the quaternions. There is a homomorphism  $\Phi: Cl_3 \rightarrow \mathbb{H}$  of real associative algebras determined by  $\Phi(i_1) = i$ ,  $\Phi(i_2) = j$ , and  $\Phi(i_3) = k$ . This homomorphism  $\Phi$  maps the subspace  $Cl_3^{(0)} \oplus Cl_3^{(1)}$  of  $Cl_3$  isomorphically as a vector space onto the quaternion algebra  $\mathbb{H}$ . The restriction of the linear map  $\Phi$  to the subspace  $Cl_3^{(0)} \oplus Cl_3^{(1)}$  preserves the Jordan multiplication defined by (2.4). Therefore we can identify the Jordan algebra  $J_3$  with the set  $\mathbb{H}$  of quaternions equipped with the multiplication defined by (2.4).

**3.  $J_n$ -polynomials.** A  $J_n$ -algebra is a commutative, but not necessarily associative, real algebra  $A$  containing the unit element  $1$ , together with a homomorphism  $\epsilon_A: J_n \rightarrow A$ . By a homomorphism I mean a linear map preserving the multiplication and the unit element. Let  $A$  be a  $J_n$ -algebra. Then we can multiply an element  $a \in A$  by an element  $s \in J_n$ ; we define

$$\{s, a\} = \{a, s\} = \{\epsilon_A(s), a\}$$

for  $s \in J_n$  and  $a \in A$ . In particular the multiplication by a real number  $r$  is the same as the scalar multiplication by  $r$  in  $A$ ; i.e.,  $\{r, a\} = ra$  for  $r \in \mathbb{R}$  and  $a \in A$ .

Let  $A$  and  $B$  be two  $J_n$ -algebras, and  $\epsilon_A, \epsilon_B$  their associated homomorphisms. A homomorphism  $\varphi: A \rightarrow B$  is called a  $J_n$ -homomorphism if it commutes with the associated homomorphisms; i.e.,  $\varphi \circ \epsilon_A = \epsilon_B$ . This condition can also be written as

$$\varphi(\{s, a\}) = \{s, \varphi(a)\}, \quad (\forall s \in J_n, \forall a \in A).$$

The above definitions are just a nonassociative analog of the definitions of commutative algebras over a commutative ring with the unit element  $1$ , and their homomorphisms.

Now let us imitate the construction of polynomials over a commutative ring, and define a nonassociative version of polynomials.

Let  $X$  be an arbitrary set; we are actually interested in the case where the set  $X$  consists of one element  $z$ . We denote by  $J_n[X]$  the free  $J_n$ -algebra generated by  $X$ . Namely  $J_n[X]$  is a  $J_n$ -algebra containing the set  $X$  which satisfies the following universal property:

- (P) Every map  $\varphi: X \rightarrow A$  of the set  $X$  to any  $J_n$ -algebra  $A$  uniquely extends to a  $J_n$ -homomorphism  $\tilde{\varphi}: J_n[X] \rightarrow A$ .

Such a  $J_n$ -algebra is easily proved to be unique up to  $J_n$ -isomorphisms. An element of the free  $J_n$ -algebra  $J_n[X]$  is called a  $J_n$ -polynomial.

The  $J_n$ -polynomials can also be defined in a constructive manner as follows:

- (1) Every element of the Jordan algebra  $J_n$  is a  $J_n$ -polynomial.
- (2) Every element of the set  $X$  is a  $J_n$ -polynomial.
- (3) If  $f$  and  $g$  are  $J_n$ -polynomials then the product  $\{f, g\}$  is also a  $J_n$ -polynomial.
- (4) If  $f$  and  $g$  are  $J_n$ -polynomials then the sum  $f + g$  is also a  $J_n$ -polynomial.

Only those explicitly shown to be  $J_n$ -polynomials by (1)–(4) above are called  $J_n$ -polynomials. The product  $\{f, g\}$  in (3) and the sum  $f + g$  in (4) are formal expressions, unless both  $f$  and  $g$  are elements of  $J_n$  in which case they are respectively the product and the sum of  $f$  and  $g$  in the Jordan algebra  $J_n$ . For any real number  $r$  and for any  $J_n$ -polynomial  $f$ , we identify the product  $\{r, f\}$  with the scalar multiple  $rf$ . Also the product  $\{f, g\}$  is formally considered to be symmetric and bilinear. Namely we identify  $\{f, g\}$  with  $\{g, f\}$ ,  $\{rf, g\}$  with  $r\{f, g\}$ , and  $\{f + g, h\}$  with  $\{f, h\} + \{g, h\}$  for any  $J_n$ -polynomials  $f, g, h$ , and for any real number  $r$ .

The set  $J_n[X]$  of  $J_n$ -polynomials constructed this way is in fact a free  $J_n$ -algebra. First of all  $J_n[X]$  is a commutative, nonassociative, real algebra in an obvious fashion. The unit element  $1 \in J_n$  considered as a  $J_n$ -polynomial is the unit element of  $J_n[X]$ , and the inclusion map  $\epsilon: J_n \rightarrow J_n[X]$  given by (1) is obviously a homomorphism. Therefore  $J_n[X]$  is a  $J_n$ -algebra. Let  $A$  be any  $J_n$ -algebra with the associated homomorphism  $\epsilon_A: J_n \rightarrow A$ , and  $\varphi: X \rightarrow A$  an arbitrary map of the set  $X$  to the  $J_n$ -algebra  $A$ . We can extend the map  $\varphi$  to a  $J_n$ -homomorphism  $\tilde{\varphi}: J_n[X] \rightarrow A$  recursively by the following:

- (1) If  $s \in J_n$  then  $\tilde{\varphi}(s) = \epsilon_A(s)$ .
- (2) If  $x \in X$  then  $\tilde{\varphi}(x) = \varphi(x)$ .
- (3) If  $f$  and  $g$  are  $J_n$ -polynomials then  $\tilde{\varphi}(\{f, g\}) = \{\tilde{\varphi}(f), \tilde{\varphi}(g)\}$ .
- (4) If  $f$  and  $g$  are  $J_n$ -polynomials then  $\tilde{\varphi}(f + g) = \tilde{\varphi}(f) + \tilde{\varphi}(g)$ .

The identifications of  $J_n$ -polynomials we make cause no difficulty since the multiplication by a real number  $r$  is also the scalar multiplication by  $r$  in  $A$ , and the multiplication in  $A$  is symmetric and bilinear. The map  $\tilde{\varphi}: J_n[X] \rightarrow A$  thus obtained is linear and preserves the multiplication by (3) and (4). The map  $\tilde{\varphi}$  also satisfies  $\tilde{\varphi} \circ \epsilon = \epsilon_A$  by (1), therefore is a  $J_n$ -homomorphism. On the other hand any  $J_n$ -homomorphism which extends the map  $\varphi: X \rightarrow A$  necessarily satisfies the above conditions (1)–(4). Therefore it must coincide with the  $J_n$ -homomorphism  $\tilde{\varphi}$ . This shows the uniqueness of the  $J_n$ -homomorphism  $\tilde{\varphi}$ .

A  $J_n$ -polynomial is called a  $J_n$ -monomial if it can be obtained from (1) and (2) by multiplication (3) only. Every  $J_n$ -polynomial can be “expanded” into a finite sum of  $J_n$ -monomials. The number of times elements of the set  $X$  appear in a  $J_n$ -monomial  $f$  is

called the degree of  $f$ , and is denoted by  $\deg f$ . Formally the degree of a  $J_n$ -monomial  $f$  is recursively defined by (1)  $\deg s = 0$  for  $s \in J_n$  ( $s \neq 0$ ), (2)  $\deg x = 1$  for  $x \in X$ , and (3)  $\deg \{f, g\} = \deg f + \deg g$ . We do not define the degree of the  $J_n$ -monomial  $0$ ; namely  $\deg 0$  is indefinite.

We note that if  $n \leq m$  then  $J_n$ -algebras and  $J_n$ -homomorphisms can be naturally considered as  $J_m$ -algebras and  $J_m$ -homomorphisms respectively. And a  $J_n$ -polynomial can be canonically thought of as a  $J_m$ -polynomial.

Now let us restrict ourselves to the case where the set  $X$  consists of one element  $z$ . In this case the set  $J_n[X]$  of  $J_n$ -polynomials is simply denoted by  $J_n[z]$ . Elements of  $J_n[z]$  are called  $J_n$ -polynomials of  $z$ . We use a symbol like  $f(z)$ , as well as  $f$ , to denote a  $J_n$ -polynomial of  $z$ .

Let  $A$  be any  $J_n$ -algebra. Then a  $J_n$ -polynomial  $f \in J_n[z]$  canonically determines a transformation of  $A$  which we denote by the same symbol  $f: A \rightarrow A$ . The image  $f(a)$  of an element  $a \in A$  by  $f$  is given by "substituting" the variable  $z$  by  $a$ . Formally  $f(a)$  is defined as follows. Define the map  $\varphi: \{z\} \rightarrow A$  by  $\varphi(z) = a$ . By the universal property (P) this map  $\varphi$  uniquely extend to a  $J_n$ -homomorphism  $\tilde{\varphi}: J_n[z] \rightarrow A$ . Then we define  $f(a) = \tilde{\varphi}(f)$ . If  $f$  is a  $J_n$ -monomial of degree  $d$  then we have  $f(ra) = r^d f(a)$  for all  $r \in \mathbb{R}$  and for all  $a \in J_n$ .

The composition of two  $J_n$ -polynomials  $f$  and  $g$  is defined similarly. Define the map  $\psi: \{z\} \rightarrow J_n[z]$  by  $\psi(z) = g$ . Using the  $J_n$ -homomorphism extension  $\tilde{\psi}: J_n[z] \rightarrow J_n[z]$ , we define the composition  $f \circ g \in J_n[z]$  to be  $\tilde{\psi}(f)$ . It can be easily verified that  $f \circ g(a) = f(g(a))$  for all  $a \in A$ .

As a special case of the above every  $J_n$ -polynomial  $f(z) \in J_n[z]$  gives rise to a map  $f: J_n \rightarrow J_n$ . It should be noted that different  $J_n$ -polynomials may give the same map. For instance the two  $J_1$ -monomials  $z$  and  $\{\{z, i_1\}, -i_1\}$  both give the identity map of  $J_1$  since the Jordan algebra  $J_1$  is associative. However these  $J_1$ -monomials give different maps of  $J_n$  to itself for  $n > 1$ ; while the  $J_n$ -monomial  $z$  gives the identity map of  $J_n$  the other  $J_n$ -monomial defines a linear transformation of  $J_n$  of rank at most two by Proposition 2. A more suggestive example is the following. The  $J_2$ -monomial  $\{\{\{z, i_1\}, i_2\}, i_1\}$  as considered as a map of  $J_n$  to itself is constantly zero for all  $n \geq 2$ .

These examples make it clear that we have to distinguish between form and matter; the  $J_n$ -polynomials and the transformations represented by the  $J_n$ -polynomials. Our standpoint is to deal with the "outer shells" rather than the "guts."

**4.  $J_n$ -series.** As stated in the previous section every  $J_n$ -polynomial is a finite sum of  $J_n$ -monomials. In this section we consider an infinite sum

$$f(z) = \sum_{k=0}^{\infty} f_k(z) \quad (4.1)$$

of  $J_n$ -monomials  $f_k(z)$  ( $k = 0, 1, \dots$ ). Such an infinite sum is called a  $J_n$ -series of  $z$ , and the  $J_n$ -monomial  $f_k(z)$  is called the  $k$ -th term. The set of  $J_n$ -series of  $z$  is denoted by  $J_n[[z]]$ . Note that we make no assumptions about the degree of the terms  $f_k(z)$ . In fact we allow infinitely many terms of the same degree to appear in a  $J_n$ -series; this generality is necessary for analytic continuations discussed later.



In order to talk about convergence of a  $J_n$ -series we first define the norm  $\|f\|$  of a  $J_n$ -monomial  $f$  of  $z$  by

- (1)  $\|a\| = |a|$  for  $a \in J_n$ ,
- (2)  $\|z\| = 1$ , and
- (3)  $\|\{f, g\}\| = \|f\| \|g\|$  for any  $J_n$ -monomials  $f$  and  $g$ .

Then we have:

PROPOSITION 3. *If  $f$  is a  $J_n$ -monomial of  $z$  then*

$$|f(a)| \leq \|f\| |a|^{\deg f} \quad (4.2)$$

for all  $a \in J_n$ .

PROOF: (1) If  $f \equiv b \in J_n$  then  $\|f\| = |b|$ , and  $|f(a)| = |b| = \|f\| |a|^0$ . (2) If  $f \equiv z$  then  $\|f\| = 1$ , and  $|f(a)| = |a| = \|f\| |a|^1$ . (3) Suppose that the  $J_n$ -monomial  $f$  is written as a product  $\{g, h\}$  for some  $J_n$ -monomials  $g$  and  $h$  of  $z$  which satisfy the condition (4.2). Then by Proposition 1 we obtain

$$\begin{aligned} |f(a)| &= |\{g(a), h(a)\}| \\ &\leq |g(a)| |h(a)| \\ &\leq \|g\| |a|^{\deg g} \|h\| |a|^{\deg h} \\ &= \|\{g, h\}\| |a|^{\deg g + \deg h} \\ &= \|f\| |a|^{\deg f}. \end{aligned}$$

This completes the proof of Proposition 3.

We define the *radius of norm convergence* of the  $J_n$ -series  $f = \sum_{k=0}^{\infty} f_k$  to be

$$R(f) = \sup \left\{ r \in \mathbb{R}_+ \mid \sum_{k=0}^{\infty} \|f_k\| r^{\deg f_k} < +\infty \right\}.$$

The following is an immediate corollary of Proposition 3.

COROLLARY 4. *Suppose that the radius of norm convergence  $R(f)$  of a  $J_n$ -series  $f(z) = \sum_{k=0}^{\infty} f_k(z) \in J_n[[z]]$  is positive. Then the series  $\sum_{k=0}^{\infty} f_k(a)$  converges absolutely for all  $a \in J_n$  with  $|a| < R(f)$ .*

A  $J_n$ -series with positive radius of norm convergence is said to be *norm convergent*. The norm convergence is a strong notion of convergence. For instance the  $J_n$ -series

$$f(z) = \sum_{k=0}^{\infty} \{\{\{z, i_1\}, i_2\}, i_1\}$$

is easily seen to have the radius of norm convergence 0, hence is not norm convergent. However this series considered as a series of maps of  $J_n$  is convergent for all  $z \in J_n$  in the strongest possible sense, since each term is constantly zero. This example also shows that

the notion of norm convergence explicitly depends on the formal expression of a  $J_n$ -series rather than the transformation represented by the  $J_n$ -series.

Analytic continuations are formulated as follows. Let

$$f(z) = \sum_{k=0}^{\infty} f_k(z) \quad (4.3)$$

be a norm convergent  $J_n$ -series and  $R(f)$  its radius of norm convergence. Suppose that  $a$  is a point in  $J_n$  such that  $|a| < R(f)$ . Now let  $z = a + w$  in the  $J_n$ -series (4.3) and expand the terms  $f_k(a + w)$  with respect to the variable  $w$  so that

$$f_k(a + w) = \sum_{l=1}^{m_k} f'_{kl}(w),$$

where each  $f'_{kl}(w)$  is a  $J_n$ -monomial of  $w$ . Thus we obtain a new  $J_n$ -series

$$f'(w) = \sum_{k=0}^{\infty} \sum_{l=1}^{m_k} f'_{kl}(w). \quad (4.4)$$

PROPOSITION 5. *The radius of norm convergence of the  $J_n$ -series (4.4) is at least  $R - |a|$ .*

We first prove a lemma.

LEMMA 6. *Let  $f(z)$  be a  $J_n$ -monomial of  $z$  and  $a$  an element of  $J_n$ . Suppose that the  $J_n$ -polynomial  $f(a + w)$  of  $w$  is expanded as*

$$f(a + w) = \sum_{l=1}^m f'_l(w)$$

into the sum of  $J_n$ -monomials  $f'_l(w)$  ( $l = 1, \dots, m$ ). Then

$$\|f\| (|a| + r)^{\deg f} = \sum_{l=1}^m \|f'_l(w)\| r^{\deg f'_l} \quad (4.5)$$

for any positive real number  $r$ .

PROOF: (1) If  $f$  is a constant  $b \in J_n$  then  $m = 1$ ,  $f'_1 = b$ , and the formula (4.5) obviously holds. (2) If  $f \equiv z$  then both sides of (4.5) are equal to  $|a| + r$ . (3) Suppose that the  $J_n$ -monomial  $f$  is written as a product  $\{g, h\}$  of two  $J_n$ -monomials  $g$  and  $h$  for which the lemma has been proved. Namely assume that there are expansions

$$g(a + w) = \sum_{k=1}^p g'_k(w),$$

$$h(a + w) = \sum_{l=1}^q h'_l(w),$$

and that we have

$$\|g\| (|a| + r)^{\deg g} = \sum_{k=1}^p \|g'_k\| r^{\deg g'_k},$$

$$\|h\| (|a| + r)^{\deg h} = \sum_{l=1}^q \|h'_l\| r^{\deg h'_l}.$$

The expansion of  $f(a + w) = \{g(a + w), h(a + w)\}$  into the sum of  $J_n$ -monomials of  $w$  is given by

$$f(a + w) = \sum_{k=1}^p \sum_{l=1}^q \{g'_k(w), h'_l(w)\}.$$

Then we have

$$\begin{aligned} \|f\| (|a| + r)^{\deg f} &= \|\{g, h\}\| (|a| + r)^{\deg g + \deg h} \\ &= \|g\| (|a| + r)^{\deg g} \|h\| (|a| + r)^{\deg h} \\ &= \left( \sum_{k=1}^p \|g'_k\| r^{\deg g'_k} \right) \left( \sum_{l=1}^q \|h'_l\| r^{\deg h'_l} \right) \\ &= \sum_{k=1}^p \sum_{l=1}^q \|g'_k\| \|h'_l\| r^{\deg g'_k + \deg h'_l} \\ &= \sum_{k=1}^p \sum_{l=1}^q \|\{g'_k, h'_l\}\| r^{\deg \{g'_k, h'_l\}}. \end{aligned}$$

This completes the proof of Lemma 6.

Now it is easy to prove Proposition 5. Let  $r$  be an arbitrary real number such that  $|a| + r < R(f)$ . By the definition of the radius of norm convergence of  $f(z)$  we have

$$\sum_{k=0}^{\infty} \|f_k\| (|a| + r)^{\deg f_k} < +\infty.$$

By Lemma 6 this is equivalent to

$$\sum_{k=0}^{\infty} \sum_{l=1}^{m_k} \|f'_{kl}\| r^{\deg f'_{kl}} < +\infty.$$

Therefore the radius of norm convergence of the  $J_n$ -series (4.4) is at least  $r$ . Since  $r$  was an arbitrary real number such that  $r < R(f) - |a|$  this means that the radius of norm convergence of the  $J_n$ -series (4.4) is at least  $R(f) - |a|$ . This completes the proof of Proposition 5.

Let us look at some examples. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{4.6}$$

be a convergent power series in the usual sense. First assume that the coefficients  $a_k$  are all real numbers. The standard power series expansions of functions like  $e^z$ ,  $\sin z$ ,  $\cos z$ ,  $\log(1+z)$  are all of this type. Such a power series is automatically a  $J_n$ -series for all  $n = 0, 1, \dots$ . The radius of norm convergence  $R(f)$  of the  $J_n$ -series is simply the radius of convergence of the power series (4.6) in the usual sense. Therefore the  $J_n$ -series canonically gives a family of maps

$$f_n: J_n \rightarrow J_n, \quad (n = 0, 1, \dots)$$

defined at least in the inside of the ball of radius  $R(f)$ . The map  $f_0: J_0 \rightarrow J_0$  is simply the real analytic function given by the power series (4.6), and the map  $f_1: J_1 \rightarrow J_1$  is its extension to the complex plane. For higher dimensions the map  $f_n$  is obtained by "rotating" the map  $f_1$  around "the real axis."

Suppose next that the coefficients  $a_k$  of the power series (4.6) are complex numbers. While the power  $z^k$  naturally makes sense in  $J_n$ , multiplying  $z^k$  by anything other than a real number using the Clifford multiplication does not make a good sense in the Jordan algebra  $J_n$ . The natural thing to do here is to replace the usual multiplication of  $a_k$  and  $z^k$  by the Jordan multiplication  $\{a_k, z^k\}$ . Namely we consider the  $J_n$ -series

$$\sum_{k=0}^{\infty} \{a_k, z^k\} \quad (4.7)$$

as the generalization of the power series (4.6). Since the Jordan multiplications in  $J_0$  and  $J_1$  coincide with the usual multiplications of  $\mathbb{R}$  and  $\mathbb{C}$  respectively the  $J_n$ -series (4.7) is in fact a direct extension of (4.6).

**5. Polynomial maps.** As we saw in the sections 3 and 4  $J_n$ -polynomials and norm convergent  $J_n$ -series are not in one to one correspondence with the transformations of the space  $J_n$  represented by them. Therefore the following problem is of fundamental importance in the study of the subject. Let  $n$  be a non-negative integer.

**PROBLEM 1.** *Characterize the  $J_n$ -polynomials (norm convergent  $J_n$ -series) which define the zero map of  $J_n$  to itself.*

Or more strongly:

**PROBLEM 2.** *Give the normal forms of  $J_n$ -polynomials ( $J_n$ -series).*

Problem 1 for  $J_n$ -polynomials can be given an answer as follows. Suppose that  $f(z)$  is a  $J_n$ -polynomial. Consider  $z$  and  $w$  specifically as variables in  $J_n$  and let

$$w = f(z). \quad (5.1)$$

Write the variables  $z$  and  $w$  as

$$\begin{aligned} z &= z_0 + z_1 i_1 + \cdots + z_n i_n, \\ w &= w_0 + w_1 i_1 + \cdots + w_n i_n, \end{aligned}$$

using the real variables  $z_0, z_1, \dots, z_n$  and  $w_0, w_1, \dots, w_n$ . Then the relation (5.1) gives a family of polynomials  $f_0, f_1, \dots, f_n \in \mathbb{R}[z_0, z_1, \dots, z_n]$  in the usual sense so that

$$w_j = f_j(z_0, z_1, \dots, z_n), \quad (j = 0, 1, \dots, n).$$

Then the answer to Problem 1 is the following: The  $J_n$ -polynomial  $f(z) \in J_n[z]$  defines the zero map  $f: J_n \rightarrow J_n$  if and only if these polynomials  $f_0, f_1, \dots, f_n \in \mathbb{R}[z_0, z_1, \dots, z_n]$  are all zero.

However this answer to Problem 1 is somehow against the basic idea of "simplifying computations involved in the  $n$ -dimensional transformations by the use of an algebraic operation." What we really want is a characterization which does not refer to the coordinates.

In any case the transformation  $f: J_n \rightarrow J_n$  given by a  $J_n$ -polynomial  $f(z) \in J_n[z]$  is after all a polynomial map in the usual sense. If  $n \neq 1$  then the converse is also true.

**PROPOSITION 7.** *If  $n \neq 1$  then every polynomial transformation  $f: J_n \rightarrow J_n$  can be represented by a  $J_n$ -polynomial  $f(z) \in J_n[z]$ .*

**PROOF:** If  $n = 0$  then the statement is obvious; let us assume that  $n > 1$ . Every element  $z \in J_n$  can be uniquely written as

$$z = z_0 + z_1 i_1 + \dots + z_n i_n, \quad (z_0, z_1, \dots, z_n \in \mathbb{R}).$$

Therefore the real part  $z_0$  and the coefficients  $z_j$  of  $i_j$  ( $j = 1, \dots, n$ ) can be considered as real valued functions on  $J_n$ . First we show that these real valued functions are representable by  $J_n$ -polynomials. Let us consider  $z_j$  ( $j = 1, \dots, n$ ). We can take an index  $k$  different from  $j$  since  $n > 1$ . Then by an easy computation we get

$$\begin{aligned} \{z, i_j\} &= -z_j + z_0 i_j, \\ \{\{z, i_j\}, i_k\} &= -z_j i_k, \end{aligned}$$

therefore

$$\{\{\{z, i_j\}, i_k\}, i_k\} = z_j.$$

Similarly we obtain

$$\{\{\{\{z, i_1\}, i_1\}, i_2\}, i_2\} = z_0.$$

Since the Jordan multiplication coincides with the usual multiplication of the real numbers, we can multiply these  $J_n$ -polynomials together with a real number to get a  $J_n$ -monomial which represents any monomial of  $z_0, z_1, \dots, z_n$  in the usual sense. We then add some of these  $J_n$ -monomials to get  $J_n$ -polynomials  $f_0, f_1, \dots, f_n \in J_n[z]$  which represent arbitrary polynomial functions. Finally let

$$f(z) = f_0 + \{f_1, i_1\} + \dots + \{f_n, i_n\}.$$

This way we can find a  $J_n$ -polynomial which represents any desired polynomial map.

For  $n = 1$  the Jordan algebra  $J_1$  is the field of complex numbers and the transformations  $f: J_1 \rightarrow J_1$  given by  $J_1$ -polynomials are just complex polynomial functions. Since such maps are always conformal, not all polynomial maps on  $J_1$  are representable by  $J_1$ -polynomials. Contrary to the similarity of outside appearances, the structures of the  $J_n$ -polynomials are quite different for  $n = 1$  and for  $n > 1$ .

Analytic maps are just the limits of polynomial maps. As a corollary of the above we have the following:

**THEOREM.** *If  $n \neq 1$  then every real analytic map  $f: J_n \rightarrow J_n$  defined on a neighborhood of the origin can be represented by a norm convergent  $J_n$ -series  $f(z) \in J_n[[z]]$ .*

**PROOF:** Let  $f: J_n \rightarrow J_n$  be any real analytic map defined on a neighborhood of the origin. Write  $w = f(z)$  in terms of the coordinates as

$$w_j = f_j(z_0, \dots, z_n), \quad (j = 0, \dots, n).$$

Then expand each function  $f_j$  as a power series in  $n + 1$  variables  $z_0, \dots, z_n$ ;

$$f_j(z_0, \dots, z_n) = \sum_{k_0=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} c_{j k_0 \dots k_n} z_0^{k_0} \cdots z_n^{k_n},$$

where the coefficients  $c_{j k_0 \dots k_n}$  are real numbers. Since the power series is convergent in a neighborhood of  $0 \in J_n$  there is a positive real number  $r$  such that

$$\sum_{k_0=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} |c_{j k_0 \dots k_n}| r^{k_0 + \dots + k_n} < +\infty, \quad (j = 0, \dots, n). \quad (5.2)$$

As in the proof of Proposition 7 we can express each term  $z_0^{k_0} \cdots z_n^{k_n}$  as a  $J_n$ -monomial  $g_{k_0 \dots k_n}(z)$  of degree  $k_0 + \dots + k_n$  satisfying  $\|g_{k_0 \dots k_n}\| = 1$ . The  $J_n$ -series

$$f(z) = \sum_{k_0, \dots, k_n=0}^{\infty} c_{0 k_0 \dots k_n} g_{k_0 \dots k_n} + \sum_{j=1}^n \sum_{k_0, \dots, k_n=0}^{\infty} c_{j k_0 \dots k_n} \{g_{k_0 \dots k_n}, i_j\}$$

is norm convergent because

$$\begin{aligned} & \sum_{k_0, \dots, k_n=0}^{\infty} \|c_{0 k_0 \dots k_n} g_{k_0 \dots k_n}\| r^{\deg g_{k_0 \dots k_n}} \\ & + \sum_{j=1}^n \sum_{k_0, \dots, k_n=0}^{\infty} \|c_{j k_0 \dots k_n} \{g_{k_0 \dots k_n}, i_j\}\| r^{\deg \{g_{k_0 \dots k_n}, i_j\}} \\ & = \sum_{k_0, \dots, k_n=0}^{\infty} |c_{0 k_0 \dots k_n}| r^{k_0 + \dots + k_n} + \sum_{j=1}^n \sum_{k_0, \dots, k_n=0}^{\infty} |c_{j k_0 \dots k_n}| r^{k_0 + \dots + k_n} \\ & < +\infty \end{aligned}$$

by (5.2). The norm convergent  $J_n$ -series  $f(z) \in J_n[[z]]$  thus obtained obviously represents the given real analytic map  $f$ . This completes the proof of Theorem.

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