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Elliptic KZ system, braid group of the torus and Vassiliev invariants

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Introduction

The purpose of this paper is to construct Vassiliev invariants for links in the product of the torus and the unit interval by means of the elliptic Knizhnik-Zamolodchikov (KZ) equation.

Let D be a chord diagram, which consists of oriented circles and chords marked on them. Let Σ be a closed oriented surface. We are going to consider chord diagrams on Σ. Namely, we consider the homotopy classes of continuous maps $\gamma : D \to \Sigma$ for any chord diagram D. The vector space spanned by all such chord diagrams on Σ modulo the 4-term relation is denoted by $\mathcal{A}(\Sigma)$. As was explained by Reshetikhin in [R], the vector space $\mathcal{A}(\Sigma)$ has a structure of a Poisson algebra.

Let G be a Lie group whose Lie algebra is equipped with a non-degenerate adjoint invariant symmetric bilinear form. Let ϕ be a flat G connection on Σ. To a chord diagram Γ on Σ and the flat connection ϕ we associate a scalar $\mathcal{T}_\phi(\Gamma)$ satisfying the 4 term relation. In other words $\mathcal{T}_\phi$, which is called the weight system associated with the flat connection ϕ, is considered to be an element of the dual space $\mathcal{A}(\Sigma)^*$ of $\mathcal{A}(\Sigma)$.

In [V], Vassiliev investigated the 0-th cohomology of the space of embeddings of a circle into $\mathbf{R}^3$, and defined the notion of the invariants of finite order for oriented knots. In this paper we adapt the formulation due to Birman and Lin [BL] to define the invariants of finite order for oriented links in a 3-manifold M. Let us consider the case when M is the product of the closed oriented surface Σ and the unit interval I. Let v be an invariant of finite order for oriented framed links in $\Sigma \times I$. It can be shown that v determines in a natural way an element of $\mathcal{A}(\Sigma)^*$.

Now the problem is to reconstruct an invariant of finite order for oriented framed links from a given element of $\mathcal{A}(\Sigma)^*$. In the case of knots in $\mathbf{R}^3$, this problem was solved by Kontsevich [K] using the iterated integral of the universal KZ connection.

Let us recall that the KZ connection is defined associated with the simplest rational solution of the classical Yang-Baxter equation of the form $r(u) = \Omega/u$, where Ω is the Casimir element. A systematic classification of non-degenerate solutions of the classical Yang-Baxter equation was established by Belavin and Drinfel'd ([BD]). In this paper we focus on the elliptic solutions to define the associated local system on the space of configuration of n distinct points on the elliptic curve $E = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\tau$, Im $\tau > 0$. We shall call such local system the elliptic KZ system. This system was studied extensively by Etingof [E] from the viewpoint of vertex operators. As the holonomy of the elliptic KZ system, we obtain projectively linear representations of the braid group of the torus.

The situation we are going to discuss in this paper is the case when M is the product of the elliptic curve E and the unit interval I. We consider a projective local system on E determined by a representation of the Heisenberg group, which is considered to be a central extension of $H_1(E, \mathbf{Z}_N)$. Associated with the Lie algebra $sl(N, \mathbf{C})$ and the above local system on E we can define a weight system for chord diagrams on the torus. The KZ connection in the case $M = \mathbf{R}^3$ is replaced by the elliptic KZ system in our case. We integrate the above weight system to construct invariants of finite order for oriented framed links in $E \times I$ by investigating the holonomy of the elliptic KZ system.

The paper is organized in the following way. In Section 1, we recall basic properties of the elliptic solution of the classical Yang-Baxter equation and the associated elliptic KZ system on the configuration space of the elliptic curve. In Section 2, we describe representations of the braid group of the torus obtained as the holonomy of the elliptic KZ system. Section 3 starts from an exposition of a general framework to define the weight system for chord diagrams on a closed oriented surface, associated with a flat G connection and representations of G. Then, we explain the case of the torus for representations of the Lie algebra $sl(N, \mathbf{C})$ together with the Heisenberg group action. Finally, in Section 4, we integrate the weight system for chord diagrams on the torus defined in Section 3 to construct invariants of finite order for oriented framed links in $E \times I$. A part of this work was presented at the meeting "Knotentheorie", Oberwolfach in September, 1995.

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1. Elliptic KZ system

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. First, we recall the definition of the classical Yang-Baxter equation following Belavin and Drinfel'd [BD]. We fix an associative algebra A with unit containing $\mathfrak{g}$. Let $r(u)$ be a meromorphic function with values in the tensor product $\mathfrak{g} \otimes \mathfrak{g}$. The functional equation for $r(u)$ in $A \otimes A \otimes A$ of the form

$$[r_{12}(u_1-u_2), r_{13}(u_1-u_3)]+[r_{12}(u_1-u_2), r_{23}(u_2-u_3)]+[r_{13}(u_1-u_3), r_{23}(u_2-u_3)] = 0 \quad (1.1)$$

is called the classical Yang-Baxter equation. Here the meaning of the suffix is as follows. We define the embedding $\varphi_{12} : \mathfrak{g} \otimes \mathfrak{g} \to A \otimes A \otimes A$ by $\varphi_{12}(a \otimes b) = a \otimes b \otimes 1$ and we put $\varphi_{12}(r(u)) = r_{12}(u)$. Analogously we define $r_{13}(u)$ and $r_{23}(u)$.

Non-degenerate solutions of the classical Yang-Baxter equation with the unitarity condition

$$r_{12}(-u) = -r_{21}(u) \tag{1.2}$$

have been classified by Belavin-Drinfel'd [BD] into three classes – rational solutions, trigonometric solutions and elliptic solutions. We denote by $\{I_\mu\}$ an orthonormal basis of g with respect to the Cartan-Killing form. We put

$$\Omega = \sum_\mu I_\mu \otimes I_\mu. \tag{1.3}$$

Then the function $r(u) = \Omega/u$ is a typical rational solution of the classical Yang-Baxter equation.

Let $\pi_j : \mathbf{g} \to End(V_j), 1 \le j \le n$, be finite dimensional representations of the Lie algebra g. Let $r(u)$ be an arbitrary solution of the classical Yang-Baxter equation. We denote by $r_{ij}(u) \in End(V_1 \otimes \cdots \otimes V_n)$, $1 \le i, j \le n$, the operation of $r(u)$ on the i-th and j-th components through the above representations. Let us consider the system of partial differential equation for a function $\varphi(z_1, \cdots, z_n)$ with values in $V_1 \otimes \cdots \otimes V_n$ of the form

$$\frac{\partial \varphi}{\partial z_i} = \sum_{j,\, j \ne i} r_{ij}(z_i - z_j)\ \varphi. \tag{1.4}$$

A solution of the above differential equation is considered to be a horizontal section of the meromorphic connection

$$\omega = \sum_{i<j} r_{ij}(z_i - z_j)\ (dz_i - dz_j) \tag{1.5}$$

for a trivial vector bundle over $\mathbf{C}^n$ with fiber $V_1 \otimes \cdots \otimes V_n$. The following lemma was observed by Cherednik [Ch].

Lemma 1.6. *If $r(u)$ is a solution of the classical Yang-Baxter equation, then the equation (1.4) is consistent. Namely, we have $\frac{\partial^2 \varphi}{\partial z_i \partial z_j} = \frac{\partial^2 \varphi}{\partial z_j \partial z_i}$ for any i, j.*

In the following of this section, we restrict ourselves to consider the Lie algebra $\mathbf{g} = sl(N, \mathbf{C})$. Let $\mathbf{e}_j$, $1 \le j \le N$, be the standard basis of the complex vector space $\mathbf{C}^N$. We put $\varepsilon = e^{2\pi\sqrt{-1}/N}$. Let A_1 and A_2 be the matrices defined by

$$\begin{aligned} &A_1 \mathbf{e}_j = \varepsilon^{j-1} \mathbf{e}_j, \quad 1 \le j \le N \\ &A_2 \mathbf{e}_j = \mathbf{e}_{j+1},\ 1 \le j \le N-1, \quad A_2 \mathbf{e}_N = \mathbf{e}_1. \end{aligned} \tag{1.7}$$

Then, A_1 and A_2 satisfy $A_1 A_2 = \varepsilon A_2 A_1$. Let a_1 and a_2 be the inner automorphisms of $sl(N, \mathbf{C})$ defined by

$$a_j(X) = A_j^{-1} X A_j, \quad j = 1, 2 \tag{1.8}$$

for $X \in sl(N, \mathbf{C})$. We see that a_1 and a_2 are commuting automorphisms of order N and that they do not have a common non zero fixed vector. For $l, m \in \mathbf{Z}$, we define $\Omega^{(l,m)}$ by

$$\Omega^{(l,m)} = (a_1^l a_2^m \otimes 1)(\Omega). \tag{1.9}$$

It can be checked that we have the relation

$$(a_1^l a_2^m \otimes 1)(\Omega) = (1 \otimes a_1^{-l} a_2^{-m})(\Omega). \tag{1.10}$$

Putting $\alpha = (l, m)$, and considering α as an element of the direct sum $\mathbf{Z}_N \oplus \mathbf{Z}_N$, we write Ω^α for $\Omega^{(l,m)}$. Here $\mathbf{Z}_N$ denotes the cyclic group of order N. We can easily check the following lemma.

Lemma 1.11. *The above $\Omega^\alpha, \alpha \in \mathbf{Z}_N \oplus \mathbf{Z}_N$, satisfies the following properties.*

(1) We have $P\Omega^\alpha = \Omega^{-\alpha}$, where P is the permutation operator defined by $P(x \otimes y) = y \otimes x$.
(2) The relation

$$[\Omega_{12}^\alpha + \Omega_{13}^\beta, \Omega_{23}^\gamma] = 0$$

holds if $\alpha - \beta + \gamma = 0$. Here the meaning of the suffix for Ω is the same as r_{ij} in the equation (1.1).
(3) We have

$$\sum_{\alpha \in \mathbf{Z}_N \oplus \mathbf{Z}_N} \Omega^\alpha = 0.$$

The elliptic solution which we are going to discuss appeared in the work of Belavin [B]. To describe the solution we first recall some basic properties of the Weierstrass ζ function. Let ω_1 and ω_2 be complex numbers with Im $\omega_2/\omega_1 > 0$ and L the lattice defined by $L = \{l\omega_1 + m\omega_2 \mid l, m \in \mathbf{Z}\}$. The Weierstrass ζ function is defined by the series

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Gamma, \omega \neq 0} \left[\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right], \tag{1.12}$$

which is a meromorphic function with simple poles at $\omega \in L$. We put $\omega_3 = \omega_1 + \omega_2$. The function $\zeta(z)$ is an odd function of z, with the properties

$$\zeta(z + \omega_j) = \zeta(z) + 2\zeta\left(\frac{\omega_j}{2}\right), \quad j = 1, 2, 3. \tag{1.13}$$

In particular, for $\omega_1 = 1$ and $\omega_2 = \tau$ with Im $\tau > 0$, the function $\zeta(z)$ is also denoted by $\zeta(z|\tau)$.

With the above notation, we put

$$\rho(z) = \Omega\zeta(z|N\tau) + \sum_{0\leq l,m\leq N-1,(l,m)\neq(0,0)} \Omega^{(l,m)}\left[\zeta(z-l-m\tau|N\tau)+\zeta(l+m\tau|N\tau)\right]. \quad (1.14)$$

The following proposition was shown in [BD](see also [E]).

Proposition 1.15. *The function $\rho(z)$ satisfies the following properties.*

(1) $\rho(z)$ is a meromorphic function which has only poles of order 1 at $l+m\tau$ with $l,m \in \mathbf{Z}$. The residue of $\rho(z)$ at $z = l+m\tau$ is $\Omega^{(l,m)}$.
(2)

$$\rho(z+1) = (a_1\otimes 1)\rho(z), \quad \rho(z+\tau) = (a_2\otimes 1)\rho(z)$$

where a_1 and a_2 are inner automorphisms of $sl(N,\mathbf{C})$ defined as in (1.8).
(3) $\rho(z)$ is a solution of the classical Yang-Baxter equation.
Moreover, $\rho(z)$ is characterized by the above properties (1) – (3).

Since a_1 and a_2 are automorphisms of order N, it follows from the above property (2) that we have

$$\rho(z+N) = \rho(z), \quad \rho(z+N\tau) = \rho(z). \quad (1.16)$$

This implies that $\rho(z)$ defines a meromorphic function on the elliptic curve $E_N = \mathbf{C}/L_N$, with the lattice

$$L_N = \{lN + mN\tau \mid l,n \in \mathbf{Z}\}.$$

On the elliptic curve $E = \mathbf{C}/\mathbf{Z}+\mathbf{Z}\tau$, $\rho(z)$ defines a multivalued meromorphic function with only one pole.

2. Representations of the braid group of the torus

Let H_N denote the Heisenberg group with generators x, y with relations

$$x^N = y^N = 1, \ [[x,y],x] = [[x,y],y] = 1. \quad (2.1)$$

The central element $[x,y]$ is denoted by c. We have the following exact sequence

$$0 \to \mathbf{Z}_N \to H_N \xrightarrow{p} H_1(E,\mathbf{Z}_N) \to 0 \quad (2.2)$$

where $H_1(E,\mathbf{Z}_N) \cong \mathbf{Z}_N \oplus \mathbf{Z}_N$ has as a basis the homology cycles corresponding to the deck transformations λ and μ defined by $\lambda(z) = z+1$ and $\mu(z) = z+\tau$ respectively.

The above map $p : H_N \to H_1(E, \mathbf{Z}_N)$ is given by $p(x) = \lambda$ and $p(y) = \mu$. We have the embedding $\iota : H_N \to GL(N, \mathbf{C})$ defined by $\iota(x) = A_1, \iota(y) = A_2$ and $j(c) = \varepsilon I$, where the matrices A_1 and A_2 are given as in (1.7). Let $s : H_1(T; \mathbf{Z}_N) \to H_N$ be the map defined by $s(l\lambda + m\mu) = A_1^l A_2^m$. In the following, we fix the above section s for the exact sequence (2.2).

For a manifold M, we denote by $Conf_n(M)$ the configuration space of ordered n points in M. Namely, we set

$$Conf_n(M) = \{(x_1, \cdots, x_n) \mid x_1, \cdots, x_n \in M, \quad x_i \neq x_j \text{ if } i \neq j\}.$$

As is the previous section, we denote by E_N the elliptic curve $\mathbf{C}/L_N$. We fix finite dimensional representations $\pi_j : sl(N, \mathbf{C}) \to End(V_j)$, $1 \leq j \leq n$. Let us consider the meromorphic 1-form on $\mathbf{C}^n$ with values in $End(V_1 \otimes \cdots \otimes V_n)$ defined by

$$\omega = \frac{1}{2\pi\sqrt{-1}\kappa} \sum_{1 \leq i < j \leq n} \rho_{ij}(z_i - z_j)\ (dz_i - dz_j), \tag{2.3}$$

where $\rho(z)$ is the elliptic solution of the classical Yang-Baxter equation as in Section 1 and κ is a non-zero complex parameter.

Let us denote by λ_j and μ_j, $1 \leq j \leq n$, the deck transformations on $\mathbf{C}^n$ defined by $\lambda_j(z_1, \cdots, z_j, \cdots, z_n) = (z_1, \cdots, z_j + 1, \cdots, z_n)$ and $\mu_j(z_1, \cdots, z_j, \cdots, z_n) = (z_1, \cdots, z_j + \tau, \cdots, z_n)$ respectively. It follows from (1.16) that the 1-form ω is invariant under the deck transformations λ_j^N and μ_j^N for $1 \leq j \leq n$. Thus it defines a meromorphic 1-form over the Cartesian product E_N^n. The 1-form ω on E_N^n has poles whenever $z_i - z_j \in \mathbf{Z} + \mathbf{Z}\tau$. We consider ω as a meromorphic connection for a trivial vector bundle with fiber $V_1 \otimes \cdots \otimes V_n$ over the base space E_N^n. This determines a local system $\mathcal{L}$ over E_N^n with singularities.

The Heisenberg group H_N acts on E_N by $x(z) = z + 1$, $y(z) = z + \tau$ and $c(z) = z$, which induces a natural action of the direct sum $H_N^{\oplus n} = H_N \oplus \cdots \oplus H_N$ on E_N^n. On the other hand, $H_N^{\oplus n}$ acts naturally on $V_1 \otimes \cdots \otimes V_n$ through $\iota : H_N \to GL(N, \mathbf{C})$. It follows from part (2) of Proposition 1.15 that the connection ω is compatible with the action of $H_N^{\oplus n}$. Considering the quotient by this action, we obtain a projective local system $\overline{\mathcal{L}}$ over E^n. The induced connection does not have poles on $Conf_n(E)$. We call the above local system $\overline{\mathcal{L}}$ the elliptic KZ system.

The 1-form ω defines a projectively flat connection on $\overline{\mathcal{L}}$. The holonomy of this connection gives a projectively linear representation of the pure braid group of the torus with n strings

$$\theta : \pi_1(Conf_n(E), *) \to GL(V_1 \otimes \cdots \otimes V_n). \tag{2.4}$$

Let us notice that the meromorphic 1-form ω defined on $\mathbf{C}^n$ is written as

$$\omega = \frac{1}{2\pi\sqrt{-1}\kappa} \sum_{1 \leq i < j \leq n} \sum_{l,m \in \mathbf{Z}} \frac{\Omega_{ij}^{(l,m)}}{z_i - z_j - l - m\tau}\ (dz_i - dz_j) + \varphi \tag{2.5}$$

with a holomorphic 1-form φ. We describe the relations satisfied by the matrices $\Omega_{ij}^{(l,m)}$. Since $\Omega_{ij}^{(l,m)} = \Omega_{ij}^{(l',m')}$ if $l \equiv l', m \equiv m'$ modulo N, we consider $\alpha = (l,m)$ as an element of $\mathbf{Z}_N \oplus \mathbf{Z}_N$. It follows immediately from Lemma 1.11 that the matrices Ω_{ij}^{α}, $1 \le i \ne j \le n$, satisfy the following relations:

(1) $\Omega_{ij}^{\alpha} = \Omega_{ji}^{-\alpha}$,

(2) $[\Omega_{ij}^{\alpha} + \Omega_{ik}^{\beta}, \Omega_{jk}^{\gamma}]$ for distinct i,j,k with $\alpha - \beta + \gamma = 0$,

(3) $[\Omega_{ij}^{\alpha}, \Omega_{kl}^{\beta}] = 0$ for distinct i,j,k,l,

(4) $\sum_{\alpha \in \mathbf{Z}_N \oplus \mathbf{Z}_N} \Omega_{ij}^{\alpha} = 0$.

Let us notice that in the case $N = 1$ the above relations (2) and (3) were called infinitesimal pure braid relations in [Ko1] and [Ko2].

Let us describe the monodromy representation θ in terms of the iterated integral of the 1-form ω. We take an element of $\pi_1(Conf_n(E), *)$, which is lifted to a path $\gamma(t)$, $0 \le t \le 1$, in $\mathbf{C}^n$ with a basepoint $\gamma(0) = (x_1^0, x_2^0, \cdots, x_n^0)$. We suppose that the basepoint satisfies $x_1^0, \cdots, x_n^0 \in \mathbf{R}$ and $0 < x_1^0 < \cdots < x_n^0 < 1$. For each j, $1 \le j \le n$, we denote by $\xi_j \in L$ the deck transformation sending $\gamma_j(0)$ to $\gamma_j(1)$. Identifying the lattice L with $H_1(E, \mathbf{Z}_N)$, we obtain an element of the Heisenberg group $s(\xi_j) \in H_N$ by means of the section s defined in the previous section. Let us recall that the Heisenberg group H_N acts naturally on the representation space V_j. We denote by X_j the linear transformation on $V_1 \otimes \cdots \otimes V_n$ obtained as the action of $s(\xi_j)$ on the j-th component of $V_1 \otimes \cdots \otimes V_n$.

Pulling back ω by $\gamma : [0,1] \to \mathbf{C}^n$, we set $\gamma^*\omega = \alpha(t)\, dt$. We consider the iterated integral

$$\int_\gamma \underbrace{\omega\omega\cdots\omega}_{m} = \int_{0<t_1<t_2<\cdots<t_m<1} \alpha(t_1)\alpha(t_2)\cdots\alpha(t_m)\, dt_1 dt_2 \cdots dt_m. \tag{2.6}$$

We have the following Proposition.

Proposition 2.7. *The holonomy of the local system $\overline{\mathcal{L}}$ over $Conf_n(E)$ for the horizontal section is expressed as the sum of the iterated integrals*

$$\theta(\gamma) = X_1 X_2 \cdots X_n \left(I + \int_\gamma \omega + \int_\gamma \omega\omega + \cdots + \int_\gamma \underbrace{\omega\omega\cdots\omega}_{m} + \cdots \right)$$

with the linear transformations $X_1, \cdots, X_n$ defined above. This determines a projectively linear representation

$$\theta : \pi_1(Conf_n(E), *) \to GL(V_1 \otimes \cdots \otimes V_n).$$

Here the associated 2-cocycle c determined by $\theta(x_1x_2) = c(x_1,x_2)\theta(x_1)\theta(x_2)$, $x_1,x_2 \in \pi_1(Conf_n(E),*)$, satisfies $c(x_1,x_2)^N = 1$.

Proof: We start with the trivial vector bundle over $\mathbf{C}^n$ with fiber $V_1\otimes\cdots\otimes V_n$ equipped with the connection ω. Let us consider the parallel transport for the connection ω along the lifted path $\gamma(t)$, $0 \le t \le 1$, in $\mathbf{C}^n$. We may suppose that $\gamma(t)$ does not pass through the poles of ω. The parallel transport is expressed as the iterated integral

$$\tilde\theta(\gamma) = I + \int_\gamma \omega + \int_\gamma \omega\omega + \cdots + \int_\gamma \underbrace{\omega\omega\cdots\omega}_{m} + \cdots .$$

The parallel transport for the induced connection of the trivial vector bundle over E_N is also expressed as the same integral. Now we consider the quotient by the action of $H_N^{\oplus n}$. The holonomy along the path γ on $Conf_n(E)$ is written as $s(\xi_1)\otimes\cdots\otimes s(\xi_n)\tilde\theta(\gamma)$, which shows the first part of the proposition. Let us suppose that x_1 and x_2 in $\pi_1(Conf_n(E),*)$ are represented by the paths γ_1 and γ_2 respectively. For the composition of paths, we have $\tilde\theta(\gamma_1\gamma_2) = \tilde\theta(\gamma_1)\tilde\theta(\gamma_2)$. We denote by $(\xi_{11},\cdots,\xi_{1n})$ and $(\xi_{21},\cdots,\xi_{2n})$ the elements of $H_N^{\oplus n}$ corresponding to γ_1 and γ_2 respectively. The 2-cocycle in the statement of the proposition is expressed as $c(x_1,x_2) = c_1\cdots c_n$ with c_j, $1 \le j \le n$, determined by $s(\xi_{1j}\xi_{2j}) = c_j s(\xi_{1j})s(\xi_{2j})$. Thus we have $c(x_1,x_2)^N = 1$. This completes the proof.

For a fixed elliptic curve $E = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\tau$, the above construction gives projective representations of the pure braid group of the torus with parameter κ. The term with the iterated integral of length m contains κ^{-m}. If $V_1 = \cdots = V_n$, then we get projective representations of the braid group of the torus. The explicit form of such representations was computed by Etingof [E]. We refer the reader to [CFW] for a different approach to representations of the braid group of the torus based on quantized universal enveloping algebras. In [Ko3], Vassiliev invariants for pure braids were discussed in terms of the representation of the pure braid group into the algebra defined by the infinitesimal pure braid relations. An elliptic analogue of such construction will be discussed in a separate publication.

3. Chord diagrams on surfaces and their weight systems

First, we describe some basic facts on chord diagrams on surfaces. Let G be a Lie group whose Lie algebra $\mathbf{g}$ is equipped with an adjoint invariant symmetric non-degenerate bilinear form $B : \mathbf{g}\times\mathbf{g} \to \mathbf{R}$. Let Σ be a closed oriented surface of genus g, and consider the moduli space $\mathcal{M}_\Sigma(G)$ of flat G connections on Σ. The moduli space $\mathcal{M}_\Sigma(G)$ is identified with the set of conjugacy classes of representations of the fundamental group $\pi_1(\Sigma)$ into G. The variety $\mathcal{M}_\Sigma(G)$ contains an open set $\mathcal{M}_\Sigma(G)^o$ corresponding to the conjugacy classes of irreducible representations of $\pi_1(\Sigma)$, which has a structure of a symplectic manifold.

A chord diagram is a collection of finitely many oriented circles with finitely many chords attached on them, regarded up to orientation preserving diffeomorphisms of the circles. Here we assume that the endpoints of the chords are distinct and lie on the circles. The chords are depicted by dashed lines as in Figure 1.

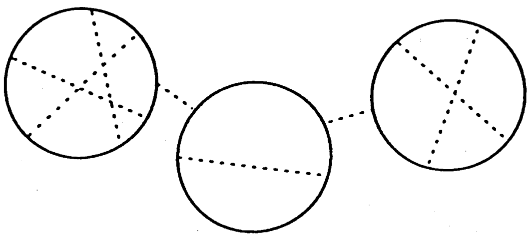


Figure 1: a chord diagram

Let D be a chord diagram. We consider a continuous map $\gamma : D \to \Sigma$ and we denote by $[\gamma]$ its free homotopy class. We call such $[\gamma]$ a chord diagram on Σ. Up to homotopy we shrink the chords on Σ as shown in Figure 2 to get loops with transversal intersections. We represent $[\gamma]$ by loops with specified vertices. Here the vertices correspond to the shrunk chords as depicted in Figure 2.

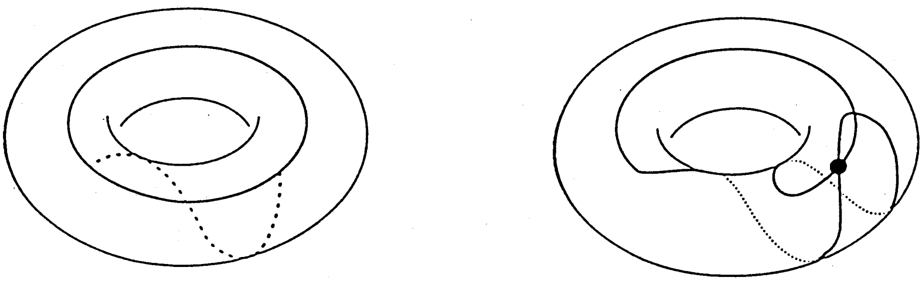


Figure 2: shrinking chords on the torus

We denote by $\mathcal{D}_\Sigma$ the complex vector space spanned by all chord diagrams on Σ and by $\mathcal{A}(\Sigma)$ its quotient space modulo 4 term relations as depicted in Figure 3.

As was explained by Reshetikhin in [R], $\mathcal{A}(\Sigma)$ has a structure of a Poisson algebra in the following way. Let Γ_1 and Γ_2 be chord diagrams on Σ where the chords are shrunk and are represented by the specific vertices as explained above. We suppose that Γ_1 and Γ_2 intersect transversely on Σ. Let p be one of the intersections of Γ_1 and Γ_2. We denote by $\Gamma_1 \cup_p \Gamma_2$ the chord diagram on Σ which is the union of Γ_1 and Γ_2, with p considered to be the specific vertex corresponding to a shrunk chord. For a chord diagram Γ we denote by $[\Gamma]$ its equivalence class in $\mathcal{A}(\Sigma)$. We can check the following.

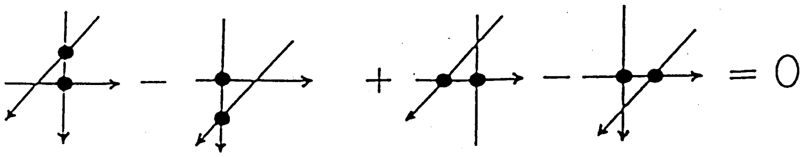


Figure 3: 4 term relation

Proposition 3.1. *We define the bracket by*

$$\{[\Gamma_1],[\Gamma_2]\} = \sum_{p\in\Gamma_1\cap\Gamma_2} \varepsilon_{12}(p)[\Gamma_1 \cup_p \Gamma_2]$$

where $\varepsilon_{12}(p)$ *is set to be* 1 *or* -1 *according as the way of intersection as shown in Figure 4a. Then the above bracket is anti-symmetric and satisfies the Jacobi identity.*

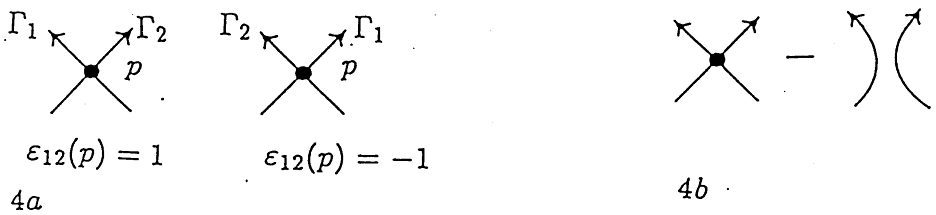


Figure 4

Let $\mathcal{A}^n(\Sigma)$ denote the subspace of $\mathcal{A}(\Sigma)$ spanned by the chord diagrams on Σ with n circles. Then $\mathcal{A}^n(\Sigma)$ is a graded vector space $\oplus_{k\geq 0}\mathcal{A}^n_k(\Sigma)$ where $\mathcal{A}^n_k(\Sigma)$ is the subspace spanned by the chord diagrams on Σ with n circles and k chords.

We observe that the quotient space of $\mathcal{A}^1(\Sigma)$ by the ideal spanned by the diagrams which look locally as depicted in Figure 4b is the Poisson algebra structure on the free homotopy classes of loops on Σ introduced by Goldman [G].

Let D be a chord diagram with n oriented circles $C_1, C_2, \cdots, C_n$ and $\gamma : D \to \Sigma$ a chord diagram on Σ, considered up to free homotopy. As in the previous paragraph we shrink the chords on Σ and represent $\Gamma = [\gamma]$ by n loops on Σ with transversal intersections and with the specified vertices corresponding to the shrunk chords. We assign finite dimensional representations $R_j : G \to Aut(V_j)$, $1 \leq j \leq n$, and the associated representations of the Lie algebra are denoted by $r_j : \mathrm{g} \to End(V_j)$, $1 \leq j \leq n$.

Let ϕ be a flat G connection on Σ. Associated with ϕ and the above representations $r_j : \mathrm{g} \to End(V_j), 1 \leq j \leq n$, we define a function $\mathcal{T}_\phi : \mathcal{A}^n(\Sigma) \to \mathbf{C}$ in the following way. The representation r_j, $1 \leq j \leq n$, is considered to be an element of $\mathrm{g}^* \otimes V_j^* \otimes V_j$.

We show it graphically as in Figure 5a. The invariant bilinear form $B : \mathrm{g} \times \mathrm{g} \to \mathbf{R}$ defines $\Omega \in \mathrm{g} \otimes \mathrm{g}$ by identifying g with its dual, which is shown graphically as in Figure 5b associated to each chord. By the endpoints of the chords, each oriented circle C_j, $1 \leq j \leq n$, is divided into several arcs C_{jk}, $k = 1, 2, \cdots$. Considering the holonomy along the path $\gamma(C_{jk})$ on Σ we obtain a linear map $Hol_{\gamma(C_{jk})} : V_j \to V_j$, which is considered to be an element of $V_j^* \otimes V_j$ and is shown graphically as in Figure 5c.

Our way of defining $\mathcal{T}_\phi(\Gamma)$ is quite similar to the method to define the weight system in [BN]. Contracting the above three kind of tensors according to the chord diagram on Σ we obtain a scalar which is denoted by $\mathcal{T}_\phi(\Gamma)$. We call $\mathcal{T}_\phi(\Gamma)$ the weight system associated with the holonomy ϕ and the representations $r_j : \mathrm{g} \to End(V_j), 1 \leq j \leq n$. We have the following proposition.

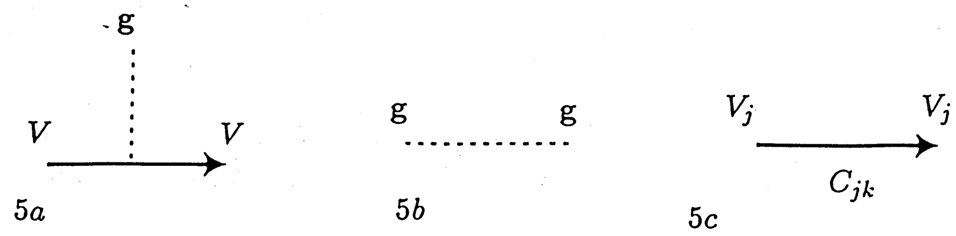


Figure 5: graphical representation of the tensors

Proposition 3.2. *The above $\mathcal{T}_\phi(\Gamma)$ is compatible with the 4 term relation and defines a map*

$$\mathcal{T}_\phi : \mathcal{A}^n(\Sigma) \to \mathbf{C}.$$

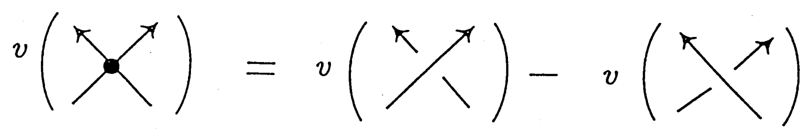
Let us go back to the case of the torus. We are going to discuss a slightly modified version of the above general framework in the case of the torus. Let $E = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\tau$ be an elliptic curve with the basis λ, μ of $H_1(E, \mathbf{Z})$ as in the previous section. Let g be the Lie algebra $sl(N, \mathbf{C})$ and we fix representations $\pi_j : \mathrm{g} \to End(V_j), 1 \leq j \leq n$. The Heisenberg group H_N acts naturally on V_j by means of the embedding $\iota : H_N \to GL(N, \mathbf{C})$ defined by the matrices A_1 and A_2 as in Section 2. We consider the projectively linear representation $\alpha : H_1(E, \mathbf{Z}) \to Aut(V_j)$, $1 \leq j \leq n$ defined by $\lambda \mapsto A_1$ and $\mu \mapsto A_2$. More precisely, we consider the representation of the Heisenberg group $\tilde{\alpha} : H_N \to Aut(V_j)$, by corresponding to each element x of $H_1(E, \mathbf{Z}_N)$ the linear transformation $\rho(s(x))$, where s is as defined in Section 2.

Let Γ be a chord diagram on the torus with n circles. Then, by means of the process of the contraction of the tensors using the above representations π_j, $1 \leq j \leq n$, and the projectively linear representation α of $H_1(E, \mathbf{Z}_N)$, we obtain a scalar, which is denoted by $\mathcal{T}(\Gamma)$. Let us notice that since our representation of the fundamental group of the torus is projectively linear, $\mathcal{T}(\Gamma)$ is only well-defined up to a multiplication of a N-th root of unity.

So far, we have represented the chord diagram on the torus by shrinking the chords. To compute $\mathcal{T}(\Gamma)$ using the chord diagram with chords not necessarily shrunk, we notice that for $v \in V$, $w \in V^*$, $g \in H_N$ and $X \in \mathrm{g}$ we have $< w, X(gv) >=< gw, Ad(g)X\ (v) >$. The above adjoint action of the Heisenberg group is exactly the same as in (1.8).

4. Vassiliev invatiants

Let us recall the definition of an invariant of finite order for oriented links in an oriented 3-manifold M following [BL]. Any $\mathbf{C}$ valued invariant v of oriented links in M can be extended to be an invariant of immersed circles in M, which are allowed to have transversal intersections, using the rule:

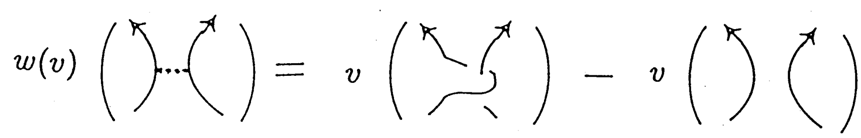


Here we think of the above graphs as parts of bigger graphs which are identical outside a small sphere. Let k be a non-negative integer. An invariant v of oriented links in M is called an invariant of order k, if v vanishes on singular links with more than k intersections. An invariant v of oriented links in M is called a Vassiliev invariant, or an invariant of finite order, if it is of order k for some non-negative integer k. We denote by $\mathcal{V}_k$ the vector space of Vassiliev invariants of order k for oriented links in M. The space of all Vassiliev invariants $\mathcal{V} = \cup_{k\geq 0}\mathcal{V}_k$ is a vector space with the increasing filtration

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \mathcal{V}_k \subset \cdots. \tag{4.1}$$

Let us now consider the case when M is the product of a closed oriented surface Σ and the unit interval $I = [0,1]$. We have a natural projection map $p : M \to \Sigma$. Let L be an oriented link with n components in $M = \Sigma \times I$. Projecting L onto Σ by p, we obtain a link diagram drawn on Σ. The notion of the framing is well-defined for links in $\Sigma \times I$.

Let $\mathcal{V}_k^n(\Sigma)$ denote the space of all $\mathbf{C}$ valued invariants of order k for oriented framed links in $\Sigma \times I$. For a chord diagram Γ on Σ we define $w(v)(\Gamma) \in \mathbf{C}$ by the rule:



It can be checked that the above $w(v)$ is compatible with the 4 term relation and is called the weight system associated with the invariant v. This induces a map

$$w : \mathcal{V}_k^n(\Sigma)/\mathcal{V}_{k-1}^n(\Sigma) \to Hom_{\mathbf{C}}(\mathcal{A}_k^n(\Sigma), \mathbf{C}). \tag{4.2}$$

In the previous section we have shown that associated with representations of a Lie group $R_j : G \to Aut(V_j), 1 \leq j \leq n$, and a flat G connection ϕ on Σ, we can define $\mathcal{T}_\phi \in Hom_{\mathbf{C}}(\mathcal{A}_k^n(\Sigma), \mathbf{C})$. From the viewpoint of the Chern-Simons perturbative theory it would be natural to ask if one can integrate $\mathcal{T}_\phi$ to construct a Vassiliev invariant v such that $w(v) = \mathcal{T}_\phi$.

We now go back to the case of the torus with the projective local system defined in the previous section. For a chord diagram Γ on the elliptic curve $E = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\tau$ we have defined the weight $\mathcal{T}(\Gamma)$ satisfying the 4 term relation. We are going to construct a Vassiliev invariant v of an oriented framed link in $E \times I$ satisfying $w(v) = \mathcal{T}$. Our method is based on the holonomy of the elliptic KZ system. We refer the reader to [Go] and [S] for a different approach in the case of links in a solid torus.

Before explaining our construction of Vassiliev invariants for links, let us first describe tangles in $E \times I$. We set $E_t = E \times \{t\} \subset E \times I, 0 \leq t \leq 1$. A tangle T in $E \times I$ is a one-dimensional submanifold with boundary of $E \times I$ such that the boundary ∂T is contained in $E_0 \cup E_1$. Let J denote the segment in E defined as the image of the open interval $(0,1)$ by the covering map $\pi : \mathbf{C} \to E$. We suppose that $\partial T \cap E_0$ and $\partial T \cap E_1$ consist of distinct points in the segment J. In the following we consider a tangle in $E \times I$ such that each connected component is oriented and framed.

Let T be an oriented framed tangle in $E \times I$. To each connected component T_j of T, we assign a finite dimensional representation of $sl(N, \mathbf{C})$. We consider the parameter t for the unit interval $[0,1]$ as a height function. Deforming the tangle up to regular isotopy, we may suppose that there exists a partition of the unit interval $0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < t_p = 1$ satisfying the following conditions:

(1) For each i, $1 \leq i \leq p$, E_{t_i} intersects transversely with the tangle T, and $T \cap E_{t_i}$ consists of distinct points in the segment J.
(2) The restricted tangle $T \cap E \times [t_i, t_{i+1}]$ is one of the following three types.

(i) a tangle with only one minimal point,
(ii) a tangle with only one maximal point,
(iii) a braid of E.

We denote by $n(i)$ the number of points in $T \cap E_{t_i}$ and we put $T \cap E_{t_i} = \{z_1, z_2, \cdots, z_{n(i)}\}$ with $z_1, \cdots, z_{n(i)} \in J$ and $z_1 < z_2 < \cdots < z_{n(i)}$. Let V_{ij}, $1 \leq j \leq n(i)$, be the representation of $sl(N, \mathbf{C})$ assigned to the component of T passing through $z_j \in T \cap E_{t_i}$. We

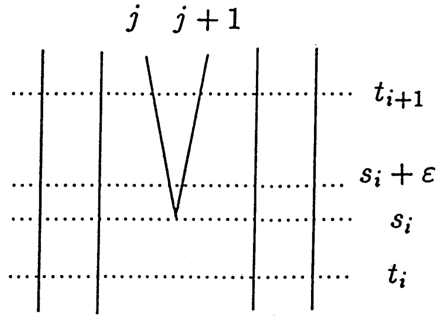


Figure 6: tangle with one minimal point

correspond to $z_j \in T \cap E_{t_i}$ the representation $V_{ij}^{\epsilon(i)}$, where $\epsilon(i)$ is 1 or -1 according as T passes through E_{t_i} downward or upward. The notation V_{ij}^{ϵ} stands for V_{ij} if $\epsilon = 1$ and for the dual representation of V_{ij} if $\epsilon = -1$.

For each t_i, $1 \le i \le p$, we consider the tensor product

$$V(t_i) = V_{i1}^{\epsilon(1)} \otimes V_{i2}^{\epsilon(2)} \otimes \cdots \otimes V_{i,n(i)}^{\epsilon(n(i))}. \tag{4.3}$$

Let us denote by $T_{i,i+1}$ the tangle T restricted to the interval $[t_i, t_{i+1}]$. We are going to construct a map

$$Z_i^{i+1} : V(t_i) \to V(t_{i+1}),\ 0 \le i \le p-1. \tag{4.4}$$

Our construction is quite similar to the well-known one due to Reshetikhin and Turaev [RT] and others, except that we are considering braids of the torus.

In the case when the tangle $T_{i,i+1}$ is a braid of the torus, we assign the linear map $Z_i^{i+1} : V(t_i) \to V(t_{i+1})$ obtained as the holonomy of the elliptic KZ system.

Let us now consider the case when the tangle $T_{i,i+1}$ contains only one minimal point at $t = s_i$, $t_i < s_i < t_{i+1}$ as shown in Figure 6. We set $V(s_i) = V(t_i)$ and to the tangle restricted to $[t_i, s_i]$ we assign the identity map. We denote by $\cup$ the tangle T restricted to $[s_i, t_{i+1}]$. Let ε be a sufficiently small positive number and we decompose the tangle $\cup$ into 2 parts, $[s_i, s_i+\varepsilon]$ and $[s_i+\varepsilon, t_{i+1}]$. We denote by $\cup_\varepsilon$ the tangle restricted to $[s_i+\varepsilon, t_{i+1}]$. To the tangle $\cup_\varepsilon$ we assign the linear map $f_\varepsilon : V(s_i+\varepsilon) \to V(t_{i+1})$ obtained as the holonomy of the elliptic KZ system. We have a natural injection $e : V(s_i) \to V(s_i+\varepsilon)$ determined by the canonical embedding $\mathbf{C} \to V_{ij}^{\varepsilon(j)} \otimes V_{i,j+1}^{\varepsilon(j+1)}$. Here $V_{i,j+1}^{\varepsilon(j+1)}$ is the dual representation of $V_{ij}^{\varepsilon(j)}$. We define $Z(\cup_\varepsilon) : V(s_i) \to V(t_{i+1})$ to be the composition $f_\varepsilon \circ e$. We set

$$Z(\cup) = \lim_{\varepsilon \to 0} Z(\cup_\varepsilon)\ \exp\left(-\frac{\Omega_{j,j+1}}{2\pi\sqrt{-1}\kappa}\log\varepsilon\right). \tag{4.5}$$

Investigating the local behaviour of the solution of the elliptic KZ system, it can been shown in a similar way as in [LM] that the above limit is convergent. This construction defines a linear map

$$Z(\cup) : V(s_i) \to V(t_{i+1}). \tag{4.6}$$

By composing the identity map $V(t_i) \to V(s_i)$, we obtain the map $Z_i^{i+1} : V(t_i) \to V(t_{i+1})$.

In the case when the tangle $T_{i,i+1}$ contains only one maximal point, we define Z_i^{i+1} in a similar way using

$$Z(\cap) = \lim_{\varepsilon \to 0} \exp\left(\frac{\Omega_{j,j+1}}{2\pi\sqrt{-1}\kappa} \log \varepsilon\right) Z(\cap_\varepsilon). \tag{4.7}$$

Now we define $Z(T)$ by the composition $Z_{p-1}^p \cdots Z_1^2 Z_0^1$. Using the integrability of the elliptic KZ system, we can show in a similar way as in [BN] the following proposition.

Proposition 4.8. *For an oriented framed tangle T in $E \times I$, the map $Z(T) : V(0) \to V(1)$ is invariant by a horizontal move preserving the framing, up to a multiplication of a N-th root of unity.*

Let L be an oriented framed link in $E \times I$ with n components. To each component L_j we assign V_j, a finite dimensional representation of $sl(N, \mathbf{C})$ and we regard L as a colored oriented framed tangle. The above construction gives a linear map $Z(L) : \mathbf{C} \to \mathbf{C}$. We denote by the same symbol $Z(L)$, or $Z(L; V_1, \cdots, V_n)$ the complex number $Z(L)(1)$.

Let C be a trivial knot with 0-framing possessing 2 minimal points and 2 maximal points. We put $\gamma_j = Z(C; V_j)$. As in [K] (see also [BN] and [LM]), we normalize $Z(L)$ as

$$\widehat{Z}(L) = \gamma_1^{-m_1} \cdots \gamma_n^{-m_n} Z(L), \tag{4.9}$$

where m_j, $1 \leq j \leq n$, is the number of maximal points on the j-th component of L. The above $\widehat{Z}(L)$ has an expansion with respect to $h = \kappa^{-1}$ of the form

$$\widehat{Z}(L) = \widehat{Z}_0(L) + \sum_{k>0} \widehat{Z}_k(L) h^k. \tag{4.10}$$

Theorem 4.11. *Let L be an oriented framed link in the product of an elliptic curve E and the unit interval I. Then, up to a multiplication of a N-th root of unity, $\widehat{Z}(L)$ satisfies the following properties.*

(1) $\widehat{Z}(L)$ is a regular isotopy invariant of L.
(2) $\widehat{Z}_k(L)$ is a Vassiliev invariant of order k.
(3) For Γ a chord diagram on E with k chords, we have $w(\widehat{Z}_k)(\Gamma) = \mathcal{T}(\Gamma)$.

Proof: To show the assertion (1), it is enough to verify that $\widehat{Z}(L)$ is invariant under vertical moves preserving the framing. Let L' be the link obtained by the move on the j-th component, creating a kink with one extra minimal point and one extra maximal point. Then we have $Z(L') = \gamma_j Z(L)$. Thus $\widehat{Z}(L)$ is invariant under the above move. Using this, in a similar way as in [BN], one can conclude that $\widehat{Z}(L)$ is a regular isotopy invariant. The assertion (2) follows directly from the definition of invariants of finite order and the expression of the holonomy of the elliptic KZ system given as in Proposition 2.7. Finally, to evaluate $w(Z_k)$ on a chord diagram Γ with k chords, we notice that the 1-form ω is written in the form as in (2.5) and that the local monodromy along $z_i = z_j$ is given by Ω_{ij}/κ. Comparing with the definition of $\mathcal{T}(\Gamma)$, we obtain the assertion (3). This completes the proof.

Remark: If the link L is contained in a 3-ball, then it is clear that our invariants coincide with usual Vassiliev invariants of oriented framed links in $\mathbf{R}^3$ for $sl(N, \mathbf{C})$.

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