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# Unknotting tunnels and canonical decompositions of punctured torus bundles over a circle

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Let $M$ be a compact orientable 3-manifold whose boundary is a torus. An *unknotting tunnel* for $M$ is a properly embedded arc $\tau$ in $M$ such that $cl(M - N(\tau))$ is a handlebody (of genus 2). In [SW] it is observed through computer experiments that the unknotting tunnels for $M$ seem to have nice geometric features when $M$ is hyperbolic. In fact, it is conjectured that any unknotting tunnel for a hyperbolic knot complement is isotopic to an edge of its *canonical decomposition.* (See [A] for related theoretical study, and see [EP, W] for the definition of the canonical decomposition.)

One purpose of this note is to point out that the conjecture holds for punctured torus bundles over $S^1$ if we assume "a result of J$\phi$rgensen" [Jr]. In fact, we show that any unknotting tunnel of such a manifold is isotopic to an edge of the topological ideal decomposition of $M$ explained by [FH]. The term "a result of J$\phi$rgensen" means that the decomposition of $M$ in [FH] is the canonical decomposition. Further, we report that experiments using Snappea show that any unknotting tunnel of such a manifold is isotopic to the "shortest vertical arc". However, as is complained in [B2], there is no proof of the "result of J$\phi$rgensen" written down; there is no proof even of the "fact" that the decompositions are genuine geometric ideal decompositions.

The other purpose of this note is to record the current state of my understanding of J$\phi$rgensen's result (for more detailed record, please see [S]). However, I must confess that it is far from satisfactory, and I ask the readers to let me know any suggestions concerning the note.

After having obtained the topological results in Part 1, I learned that they are already obtained by Klaus Johannson [Jh] and Tsuyoshi Kobayashi [K]. I would like to thank them for teaching me their results. Concerning Part II of this note, though it is far from complete by the lack of my ability, I would like to thank many mathematicians for their kind support. Troels J$\phi$rgensen kindly explained his result to me in a day in 1992. Colin Adams and Alan Reid sent me copies of J$\phi$rgensen's unfinished paper on my request. Hyam Rubinstein and Iain Aitchison gave me a chance to stay at the University of Melbourne during the summer (= the winter in Australia) in 1995, where I devoted to the project to understand J$\phi$rgensen's paper. Iain Aitchison and Craig Hodgson kindly shared much time for the project and offered me valuable suggestions and warm encouragement. Sadayoshi Kojima, Ken'ichi Ohsika, Yoshihide Okumura, and Masahiko Taniguchi kindly listened to my unsatisfactory talk and gave me valuable suggestions and informations.

## I. Topological classification of the unknotting tunnels for punctured torus bundles over a circle

Let $T$ be a punctured torus. For an orientation-preserving self-homeomorphism $\phi$ on $T$, let $M_\phi$ be the bundle over $S^1$ with fiber $T$ and with monodromy $\phi$.

**Theorem 1.** **[Johannson, Kobayashi]** *Let $\alpha$ be a properly embedded arc in $M_\phi$. Then $\alpha$ is an unknotting tunnel for $M_\phi$ if and only if $\alpha$ lies on a fiber $T$ after an isotopy and satisfies $\alpha \cap \phi(\alpha) = \emptyset$.*

**Theorem 2.** **[Johannson]** *(1) Suppose $\phi$ is elliptic, i.e., $|tr(\phi)| < 2$. Then $M_\phi$ admits a unique unknotting tunnel up to isotopy.*

*(2) Suppose $\phi$ is parabolic, i.e., $|tr(\phi)| = 2$. Then $M_\phi$ admits an unknotting tunnel if and only if $\phi$ or $\phi^{-1}$ is conjugate to $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In this case, there is only one unknotting tunnel for $M_\phi$ up to isotopy.*

*(3) Assume that $\phi$ is hyperbolic, i.e., $|tr(\phi)| > 2$. Then $M_\phi$ admits an unknotting tunnel if and only if $\phi$ or $\phi^{-1}$ is conjugate to $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ for some positive integer $n$. In this case, the number of the unknotting tunnels for $M_\phi$ up to isotopy is equal to 2 or 1 according as $n = 1$ or $n > 1$.*

**Remark 3.** In Theorem 2 (3), we can easily see that each of the unknotting tunnel is an edge of the topological ideal decomposition of $M_\phi$ explained in [FH].

The characterization of punctured torus bundles follows from the result of Ochiai and Takahashi [OT] which determines the Heegaard genera of closed torus bundles over $S^1$. Since Theorem 1 and its generalization to arbitral punctured surface bundles are already obtained by K Johannson and T. Kobayashi, we give only the proof of Theorem 2 (3) by using Theorem 1.

To do this, we identify the set $V$ of the isotopy classes of essential arcs in $T$ with $\mathbb{Q} \cup \{\infty\}$. The latter set is embedded in the ideal boundary $\partial \mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2$. Let $D$ be the diagram of $SL(2, \mathbb{Z})$, i.e., the tessellation of $\mathbb{H}^2$ by ideal triangles with vertices in $V$ such that the ideal simplex spanned by three points in $V$ belongs to $D$ if and only if the corresponding three arcs on $T$ are mutually disjoint. Then $\phi$ induces an isometry of $\mathbb{H}^2$ respecting $D$, which we denote by $\phi_*$. An element $\alpha$ of $V$ satisfies $\alpha \cap \phi(\alpha) = \emptyset$ as arcs in $T$ if and only if the hyperbolic line $\mu$ spanned by the ideal vertices $\alpha$ and $\phi_*(\alpha)$ belongs to $D$. Suppose this condition is satisfied and suppose $\phi$ is hyperbolic. Then $\phi_*$ is a hyperbolic translation along a line, say $\lambda$. Let $f_-$ and $f_+$ be the repelling and the attractive fixed points of $\phi_*$. Then these points do not lie in $V$, and $\lambda$ is a line joining them. Let $I_1$ be the closure of component of $\partial \mathbb{H}^2 - \{f_-, f_+\}$ containing the point $\alpha$, and let $I_2$ be the closure of the remaining component. For two points $x$ and $y$ in $int(I_1)$, we denote $x < y$ if $f_-$, $x$, $y$, and $f_2$ lies in $I_1$ in this order. It should be noted that the relation $\alpha < \phi_*(\alpha) < \phi_*^2(\alpha)$ holds. Let $\sigma_i$ $(i = 1, 2)$ be the ideal simplices in $D$ having $\mu$ as an edge, and let $\beta_i$ $(\in V)$ be the remaining vertex of $\sigma_i$. We may assume $\beta_1$ lies "between" $\alpha$ and $\phi_*(\alpha)$. Let $\mu_2$ be the edge of $\sigma_2$ with endpoints $\alpha$ and $\beta_2$.

**Lemm 3.** *$\beta_2$ lies in $I_2$ and hence $\mu_2$ intersects $\lambda$.*

*Proof.* Suppose $\beta_2$ lies in $I_1$. Then we have $\alpha < \beta_1 < \phi_*(\alpha) < \beta_2 < \phi_*(\beta_2)$, and hence $\phi_*(\mu_2)$ intersect the interior of $\sigma_2$ (see Figure 1 (a)), a contradiction.

Let $n$ be the number of simplices in $D$ between the edges $\mu_2$ and $\phi_*(\mu_2)$ except $\sigma_2$ (see

Figure 1 (b)). Then we can see that $\phi^{-1}$ is conjugate to $\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^n\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$. If $n > 1$, then for an element $\gamma$ of $V$ the line joining $\gamma$ and $\phi_*(\gamma)$ belongs to $D$ if and only if $\gamma$ is the image of $\alpha$ by a power of $\phi_*$. On the other hand, if $n = 1$, then the above condition is satisfied if and only if $\gamma$ is the image of $\alpha$ or $\beta_2$ by a power of $\phi_*$. Thus the number of the unknotting tunnels for $M_\phi$ is two or one according as $n = 1$ or $n > 1$.

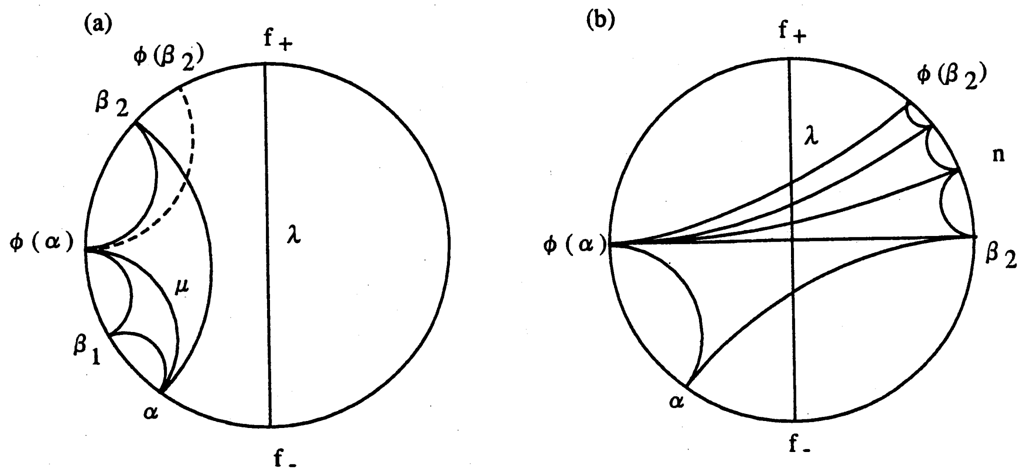



Figure 1

## II. Notes on J$\phi$rgensen's paper

### 1. Isometric circles and Ford domains

Let $A$ be a Möbius transformation $A(z) = (az+b)/(cz+d)$, where $ad-bc = 1$. Suppose $A(\infty) \neq \infty$, then the *isometric circle* $I(A)$ of $A$ is defined by

$$I(A) = \{z \in \mathbb{C} \,|\, |A'(z)| = 1\} = \{z \in \mathbb{C} \,|\, |cz+d| = 1\}.$$

$I(A)$ is a circle in $\mathbb{C}$ with center $-d/c = A^{-1}(\infty) = \text{pole}(A)$ and radius $1/|c|$. $A$ sends $I(A)$ to $I(A^{-1})$, which has the center $\text{pole}(A^{-1}) = a/c$ and the radius $1/|c|$. It should be noted that

$$\text{pole}(A^{-1}) - \text{pole}(A) = (a+d)/c = \tau(A)/c \tag{1.1}$$

where $\tau(A)$ is the trace of $A$. Let $\ell(A)$ be the line in $\mathbb{C}$ which passes through the center of $I(A)$ and forms the angle $\pi/2 - \arg(\tau(A))$ with the vector $\tau(A)/c$. (Thus the intersection of $I(A)$ and $\ell(A)$ is $\text{pole}(A) \pm i/\tau(A)$.) Then the Möbius transformation $A$ has the following expression (see [F]).

$$A = (\text{translation by } \tau(A)/c) \circ (\text{reflection in } \ell(A)) \circ (\text{inversion in } I(A)). \tag{1.2}$$

**Lemma 1.3.** *Let $A$ and $B$ be Möbius transformations. Then we have the following:*

*(1) If $z$ belongs to $I(A) \cap I(B)$, then $B(z)$ belongs to $I(AB^{-1}) \cap I(B^{-1})$.*

*(2) If $I(A) \cap I(B) \neq \emptyset$ then $I(AB^{-1}) \cap I(B^{-1}) \neq \emptyset$ and $I(BA^{-1}) \cap I(A^{-1}) \neq \emptyset$.*

*(3) If $I(A)$ is tangent to $I(B)$ from the outside, then $I(AB^{-1})$ [resp. $I(BA^{-1})$] is tangent to $I(A^{-1})$ [resp. $I(B^{-1})$] from the inside.*

*(4) $\theta(A,B) + \theta(B^{-1}, AB^{-1}) + \theta(BA^{-1}, A^{-1}) = 2\pi$, where $\theta(X,Y)$ denote the exterior angle between the isometric circles $I(X)$ and $I(Y)$.*

Let $G$ be a discrete subgroup of $PSL(2,\mathbb{C})$, and assume that the stabilizer $G_\infty$ of $\infty$ in $G$ consists of only parabolic elements. For an element $A$ of $G - G_\infty$, let $Ih(A)$ be the hyperplane in the upper half space $\mathbb{H}^3$ bounded by $I(A)$. Denote by $\tilde{P}h(G)$ [resp. $\tilde{P}(G)$] the subset of $\mathbb{H}^3$ [resp. $\mathbb{C}$] which consists of all points lying exterior to each of $Ih(A)$ [resp. $I(A)$] ($A \in G - G_\infty$). Let $Ph(G)$ [resp. $P(G)$] be the intersection of $\tilde{P}h(G)$ [resp. $\tilde{P}(G)$] with a "vertical" fundamental region of the action of $G_\infty$ on $\mathbb{H}^3$ [resp. $\mathbb{C}$]. Then $Ph(G)$ [resp. $P(G)$] is a fundamental region of the action of $G$ on $\mathbb{H}^3$ [resp. $\mathbb{C}$], and is called a *Ford domain* of $G$.

## 2. A WARM-UP EXAMPLE

Let $A$ and $B$ be hyperbolic Möbius transformations preserving the real axes. Assume that $I(A)$, $I(A^{-1})$, $I(B)$, and $I(B^{-1})$ are mutually disjoint and lie as illustrated in Figure 2.1 (1). Then $A$ and $B$ generate a Schottky group of genus 2. In fact, the region exterior to $Ih(A^{\pm 1})$ and $Ih(B^{\pm 1})$ is the Ford domain of $< A, B >$.

Push $I(A)$ and $I(B^{-1})$ together until they touch, then the region exterior to $Ih(A^{\pm 1})$ and $Ih(B^{\pm 1})$ remains to be a fundamental region of $< A, B >$. By Lemma 1.3 (3), $I(AB)$ [resp. $I(B^{-1}A^{-1})$] is tangent to $I(B)$ [resp. $I(A^{-1})$] and is ready to become visible (see Figure 2.1 (2)).

Push further $I(A)$ and $I(B^{-1})$ together so that they intersects in two points. Then, by Lemma 1.3, $I(AB)$ and $I(B^{-1}A^{-1})$ become visible, and the region exterior to $Ih(A^{\pm 1})$, $Ih(B^{\pm 1})$, and $Ih((AB)^{\pm 1})$ is a fundamental region of $< A, B >$ by Lemma 1.3 (4) and by Poincaré's theorem on fundamental polyhedra [M] (see Figure 2.1 (3)).

Push $I(AB) \cup I(B)$ and $I(A) \cup I(B^{-1}A^{-1})$ together so that $I(B)$ and $I(A)$ intersects. Then, as in the above, $I((BA)^{\pm 1})$ become visible and the region exterior to $Ih(A^{\pm 1})$, $Ih(B^{\pm 1})$, $Ih((AB)^{\pm 1})$, and $Ih((BA)^{\pm 1})$ is a fundamental region of $< A, B >$ (see Figure 2.1 (4), (5)) The simplest arrangement satisfying the above conditions is the one illustrated in Figure 2.1 (6).

Push the two blocks in Figure 2.1 (6) together until they meet. Then $I(K^{\pm 1})$ with $K = [A, B] = ABA^{-1}B^{-1}$ break out and the region exterior to $Ih(A^{\pm 1})$, $Ih(B^{\pm 1})$, $Ih((AB)^{\pm 1})$, $Ih((BA)^{\pm 1})$, and $Ih(K^{\pm 1})$ is a fundamental region of $< A, B >$ (see Figure 2.1 (7), (8)).

As $I(AB)$ approaches $I(BA)$, $I((K^{\pm 1}))$ become bigger (see Figure 2.1 (9)). Finally, when $I(AB)$ and $I(BA)$ coincide, $K$ becomes parabolic with $\infty$ as its fixed points. (see Figure 2.1 (10)). Thus $G_0 =< A, B >$ is a Fuchsian group representing a punctured torus.

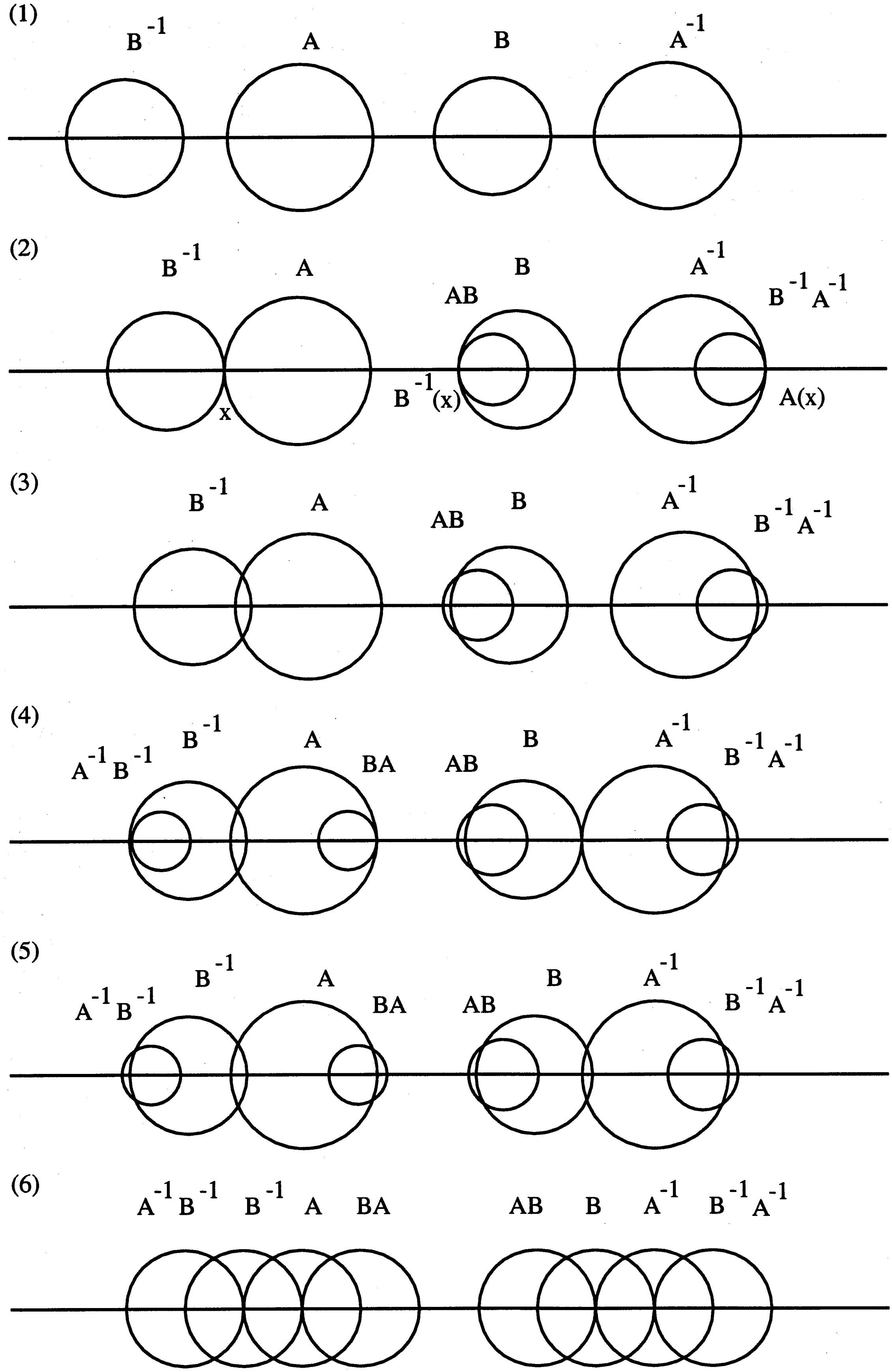



Figure 2.1 (Part I)

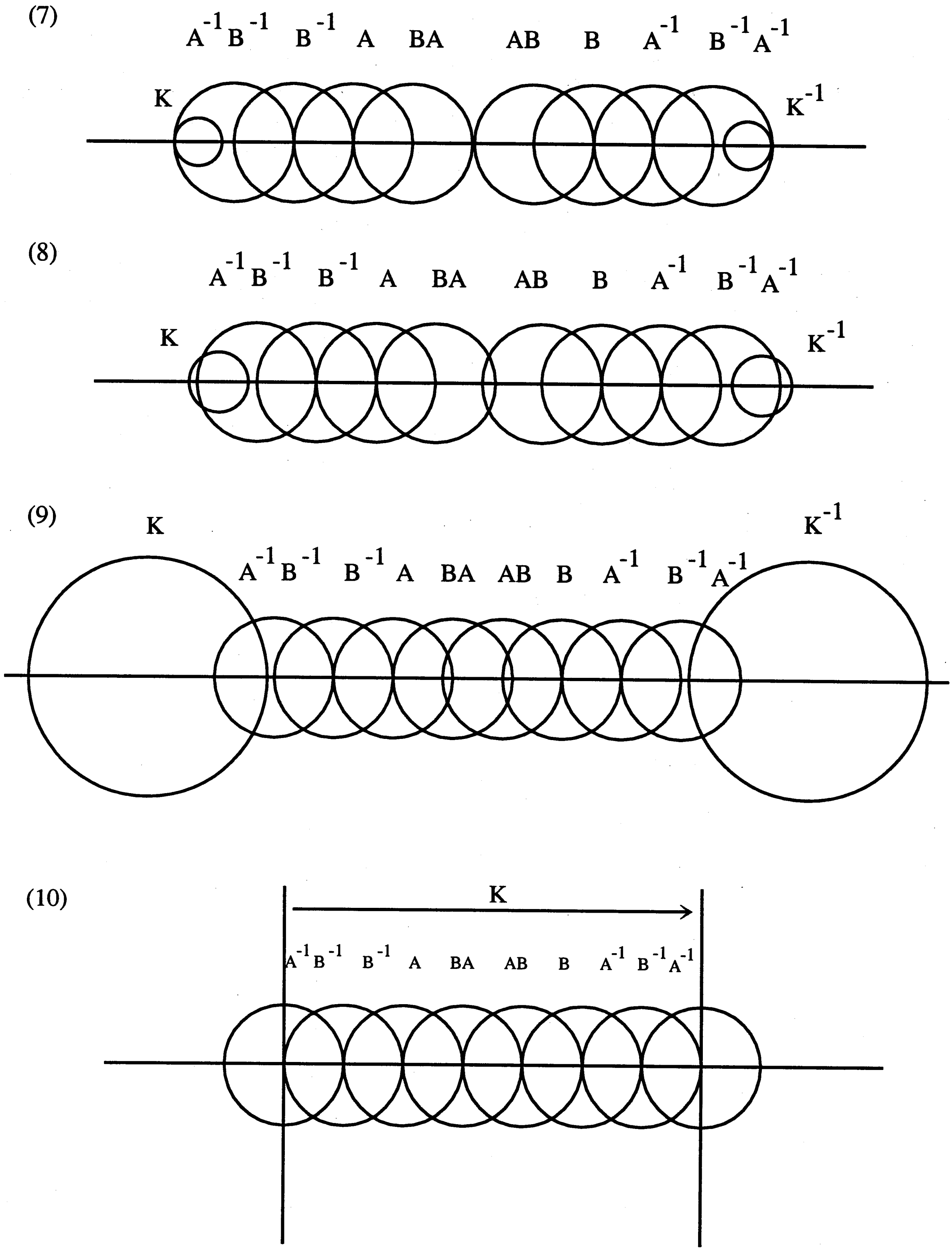



Figure 2.1 (Part II)

## 3. Two generator subgroups of $PSL(2,\mathbb{C})$

Let $G$ be the group generated by two loxodromic elements $A$ and $B$ of $PSL(2,\mathbb{C})$, and let $K$ be the commutator $[A,B]$. If $\tau(K)=2$, then we can see that either $G$ is elementary or indiscrete. So, we assume $\tau(K)\neq 2$. Then the axes of $A$ and $B$ do not share a common endpoint and hence we can find an element $R$ of $PSL(2,\mathbb{C})$ which satisfies the following conditions (see [Th, Prop 5.4.1]).

$$R^2=1,\quad RAR=A^{-1},\quad RBR=B^{-1} \tag{3.1}$$

(In fact, $R$ is given by $(2-\tau(K))^{-1/2}(AB-BA)$.) Put $\tilde{G}=<A,B,R>$. Then the index $[\tilde{G},G]$ is at most 2, and $\tilde{G}$ is discrete if and only if $G$ is discrete. Put $P=AR$ and $Q=BR$. Then we see $P^2=ARAR=AA^{-1}=1$, $Q^2=1$, and $K=(PR)(QR)(RP)(RQ)=(PRQ)^2$. Thus $K$ has a square root $\sqrt{K}=PRQ$, and we see

$$A=\sqrt{K}Q,\quad B=\sqrt{K}^{-1}P,\quad AB=\sqrt{K}R \tag{3.2}$$

Suppose further that $K$ is parabolic. After conjugation, we may assume $K=\begin{pmatrix}-1 & -2\\ 0 & -1\end{pmatrix}$. Then $\sqrt{K}$ acts on $\mathbb{C}$ as the Euclidean translation $\sqrt{K}(z)=z+1$. So the following holds for any Möbius transformation $X$.

$$I(\sqrt{K}^{m}X\sqrt{K}^{n})=\sqrt{K}^{-n}(I(X)) \tag{3.3}$$

By using (3.1)-(3.3), we obtain the following:

$$\begin{aligned} I(A^{-1})&=\sqrt{K}(I(A)),\quad I(B^{-1})=\sqrt{K}^{-1}(I(B)),\\ I((AB)^{-1})&=\sqrt{K}(I(AB)),\quad I(AB)=I(BA). \end{aligned} \tag{3.4}$$

An ordered pair $\{X,Y\}$ of elements of $G$ is called a *generator pair* of $G$ if $X$ and $Y$ generate $G$ and $[X,Y]=K$ $(=[A,B])$. If $\{X,Y\}$ is a generator pair, the ordered triple $\{X,XY,Y\}$ is called a *generator triple* of $G$. Note that (3.4) holds not only for the generator triple $\{A,AB,B\}$ but also for any generator triple $\{X,XY,Y\}$. By identifying $G$ with the image of the fundamental group of a punctured torus, there is a natural correspondence between the set of the generator triples of $G$ up to conjugation and the set of the ideal triangles of the diagram of $SL(2,\mathbb{Z})$.

Next, we give a parametrization of the groups $G$ up to conjugacy. The following fact is well-known (cf. [B1]).

**Lemma 3.5.** *(1) Let $X,Y,X'$, and $Y'$ be matrices in $SL(2,\mathbb{C})$. Suppose $(\tau(X),\tau(Y),\tau(XY))=(\tau(X'),\tau(Y'),\tau(X'Y'))$. Then, either $\tau([X,Y])=\tau([X',Y'])=2$ or there is a matrix $P$ in $SL(2,\mathbb{C})$ such that $X'=PXP^{-1}$ and $Y'=PYP^{-1}$.*

*(2) For any complex numbers $x$, $y$, and $z$, there are matrices $X$ and $Y$ in $SL(2,\mathbb{C})$ such that $(\tau(X),\tau(Y),\tau(XY))=(x,y,z)$.*

The following trace identities are well-known, where $X$ and $Y$ are matrices in $SL(2,\mathbb{C})$.

$$\tau(X)\tau(Y)=\tau(XY)+\tau(XY^{-1}) \tag{3.6}$$

$$\tau(X)^2+\tau(Y)^2+\tau(XY)^2-\tau(X)\tau(Y)\tau(XY)=2+\tau([X,Y]) \tag{3.7}$$

Let $\tilde{\mathcal{R}}$ [resp. $\mathcal{R}$] be the set of the pairs $(A,B)$ of matrices in $SL(2,\mathbb{C})$ [resp. $PSL(2,\mathbb{C})$] with $\tau([A,B]) = -2$ up to conjugacy. Then by the above result, there is a bijection between $\tilde{\mathcal{R}}$ and the set of the solutions of the *Markoff equations* $x^2+y^2+z^2 = xyz$ over $\mathbb{C}$, where $(A,B)$ corresponds to $(\tau(A),\tau(B),\tau(AB))$. For a solution $(x,y,z)$ of the Markoff equation with $z \neq 0$, the following matrices give the cross section for the correspondence.

$$A = \begin{pmatrix} x-y/z & x/z^2 \\ x & y/z \end{pmatrix}, B = \begin{pmatrix} y-x/z & -y/z^2 \\ -y & x/z \end{pmatrix}, AB = \begin{pmatrix} z & -1/z \\ -z & 0 \end{pmatrix}.$$

For this group, we have $K = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}$. This cross section is constructed by using the following facts. Put $A = (a_{ij})$, $B = (b_{ij})$, and $AB = (c_{ij})$.

(1) By (1.2) and the identity $A = \sqrt{K}Q$, we have $Q =$ (reflection in $\ell(A)$)$\circ$(inversion in $I(A)$), and $\tau(A)/a_{21} = 1$. Thus $a_{21} = \tau(A) = x$. Similarly, we have $b_{21} = -\tau(B) = y$ and $c_{21} = \tau(AB) = z$.

(2) $\mathrm{pole}(A) - \mathrm{pole}(B^{-1}) = -a_{22}/a_{21} - b_{11}/b_{21} = -c_{21}/a_{21}b_{21} = -z/(xy)$. Similarly, $\mathrm{pole}(AB) - \mathrm{pole}(A) = y/(zx)$ and $\mathrm{pole}(B) - \mathrm{pole}(AB) = x/(yz)$.

(3) Normalize the group so that $\mathrm{pole}(AB) = 0$. Then $c_{22} = 0$.

The above observation (2) leads us to another parameter of $\mathcal{R}$, namely, the triple $(a_1,a_2,a_3)$ determined by

$$a_1 = x/(yz), \quad a_2 = y/(zx), \quad a_3 = z/(xy).$$

This parameter satisfies $a_1+a_2+a_3 = 1$ and is called the *complex probabilities*. Since

$$x^2 = 1/(a_2a_3), \quad y^2 = 1/(a_3a_1), \quad z^2 = 1/(a_1a_2),$$

$(a_1,a_2,a_3)$ with $a_i \neq 0$ determines the trace parameter $(x,y,z)$ modulo signs and hence it determines the conjugacy class of a group $G$ in $PSL(2,\mathbb{C})$. The Fuchsian group constructed in the last section corresponds to $(a_1,a_2,a_3) = (1/3,1/3,1/3)$.

The identity (3.6) enables us to know the effect of a base change to the parameters. If we change the generator triple $\{A, AB, B\}$ to $\{AB^{-1}, B, A\}$, then the corresponding trace parameter $(x',y',z')$ and the complex probabilities $(a_1',a_2',a_3')$ are calculated as follows:

$$(3.8) \quad (x',y',z') = (xy-z,y,x), \quad (a_1',a_2',a_3') = (1-a_3, a_2a_3/(1-a_3), a_1a_3/(1-a_3)).$$

## 4. A WAY TO THE FIGURE-EIGHT KNOT

Consider the subspace of $\mathcal{R}$ consisting of those pairs $(A,B)$ which generate quasi-Fuchsian groups, and let $\mathcal{T}$ be the connected component of the subspace which contains the Fuchsian group $G_0$ in Section 2. We start from the Fuchsian group $G_0$ and deform it in $\mathcal{T}$. The domain $\tilde{P}(G_0)$ in the complex plane has two components, the *upper polygon* $\tilde{P}_+(G_0)$ and the *lower polygon* $\tilde{P}_-(G_0)$. The quotient of $\Omega(G_0)$, the domain of discontinuity, by $G_0$ consists of a pair of punctured tori $\tilde{P}_+(G_0)/G_0$ and $\tilde{P}_-(G_0)/G_0$. Each of the *upper boundary* $\partial\tilde{P}_+(G_0)$ and the *lower boundary* $\partial\tilde{P}_-(G_0)$ consists of arcs of the isometric circles of

$A$, $AB$, and $B$ (in this order) and their images by the powers of the Euclidean translation $\sqrt{K}(z) = z + 1$. In this sense, we say that each of the upper and lower boundaries is of type $\{A, AB, B\}$ (see Figure 4.1 (1)).

If we push $I(AB)$ (and its images by the powers of $\sqrt{K}$) upward, then the part of the lower boundary contained in $I(AB)$ decreases and finally it vanishes. At this moment, the lower boundary consists of arcs of the isometric circles of $A$ and $B$ (in this order) and their images by the powers of $\sqrt{K}$. In this sense, we say that the lower boundary is of type $\{A, B\}$. $\partial \tilde{P}h(G)$ consists of parts of the isometric hemispheres of $A$, $AB$, and $B$, and their images by the powers of $\sqrt{K}$. Though these are the only *visible* isometric hemispheres, we can see by Lemma 1.3 that the isometric hemisphere of $AB^{-1}$ (and its images by the powers of $\sqrt{K}$) touch the lower boundary at the vertex on $I(A) \cap I(B^{-1})$ (and its images by the powers of $\sqrt{K}$) and are ready to appear (see Figure 4.1 (2)).

If we deform further, then the isometric circle of $AB^{-1}$ breaks out from the lower boundary, and the lower boundary becomes of type $\{AB^{-1}, A, B\}$ (see Figure 4.1 (3)).

We would be able to continue this process as illustrated in Figure 4.1 (4), (5) and would obtain a 1-parameter family of quasi Fuchsian groups, and finally as the limit of these groups we would obtain the group as illustrated in Figure 4.2 (see [JM]). This is not a quasi-Fuchsian group any more but is the fiber group of the once punctured torus bundle over a circle with monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

## 5. The side parameter

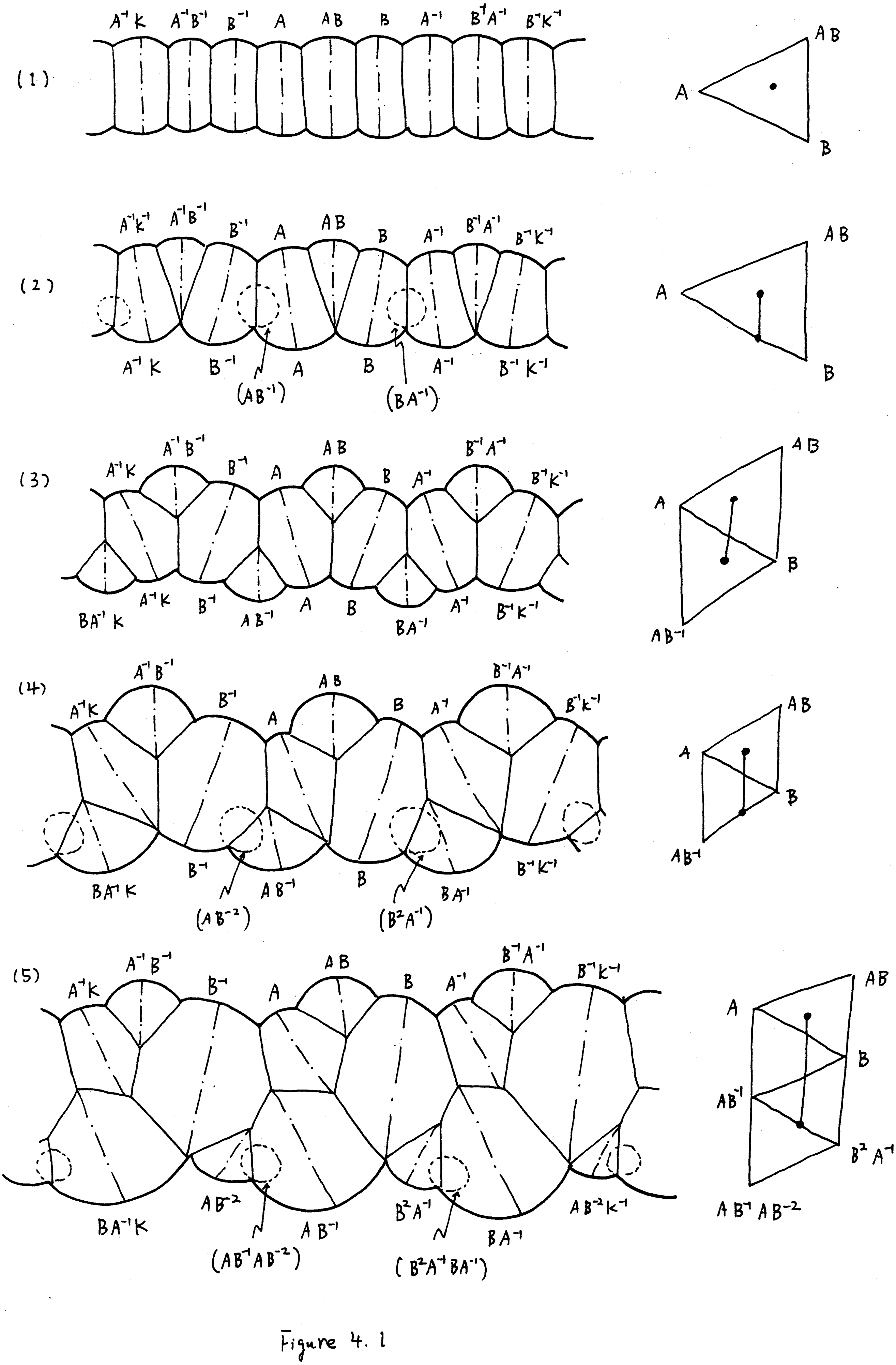
Each group $G$ in $\mathcal{T}$ has the upper polygon $\tilde{P}_+(G)$ and the lower polygon $\tilde{P}_-(G)$. As in the previous section, each of these polygons would be represenred by a generator triple (or by a generating pair in degenerate case). Let $\{A_*, A_*B_*, B_*\}$ be the generator triple corresponding to $\tilde{P}_*(G)$, where $* = +$ or $-$. Let $\theta(A_*)$, $\theta(A_*B_*)$, and $\theta(B_*)$ be the angles $\in [0, \pi/2]$ characterized by the following identities:

$$A_*^{-1}(\infty) + (*i/\tau(A_*))\exp(*i\theta(A_*)) = B_*^{-1}(\infty) + (*i/\tau(B_*))\exp(- * i\theta(B_*))$$
$$(A_*B_*)^{-1}(\infty) + (*i/\tau(A_*B_*))\exp(*i\theta(A_*B_*)) = A_*^{-1}(\infty) + (*i/\tau(A_*))\exp(- * i\theta(A_*))$$
$$B_*^{-1}(\infty) + (*i/\tau(B_*))\exp(*i\theta(B_*)) = (A_*B_*)^{-1}(\infty) + (*i/\tau(A_*B_*))\exp(- * i\theta(A_*B_*))$$

Each of these identities arises from expressing the common vertex of two adjoining sides by its position on each of the two circles, relatively to the mid-points of the sides. It follows that $\theta(A_*) + \theta(A_*B_*) + \theta(B_*) = \pi/2$. Jφrgensen asserts that the following holds:

**Theorem 5.1.** *For each pair of ideal triangles in the diagram of $SL(2, \mathbb{Z})$ with associated side parameters $\{\theta(A_*), \theta(A_*B_*), \theta(B_*)$ satsifying $\theta(A_*) + \theta(A_*B_*) + \theta(B_*) = \pi/2$, where $* = \pm$ and $0 \leq \theta < \pi/2$, there is a unique group in $\mathcal{T}$ realizing the parametr.*

It is not difficult to prove the theorem in case $\{A_+, A_+B_+, B_+\} = \{A_-, A_-B_-, B_-\}$ (cf. [S]) and we can confirm that the first step of the series of deformations in Section 4 is actually possible. But, I do not know how to prove the theorem for the general case and how to show that the sequence of the groups in Section 4 actually has the limit.


(1)
$A^{-1}K$ $A^{-1}B^{-1}$ $B^{-1}$ $A$ $AB$ $B$ $A^{-1}$ $B^{-1}A^{-1}$ $B^{-1}K^{-1}$
$A$ $AB$ $B$
(2)
$A^{-1}K^{-1}$ $A^{-1}B^{-1}$ $B^{-1}$ $A$ $AB$ $B$ $A^{-1}$ $B^{-1}A^{-1}$ $B^{-1}K^{-1}$
$A^{-1}K$ $B^{-1}$ $A$ $B$ $A^{-1}$ $B^{-1}K^{-1}$
$(AB^{-1})$ $(BA^{-1})$
$AB$ $A$ $B$
(3)
$A^{-1}K$ $A^{-1}B^{-1}$ $B^{-1}$ $A$ $AB$ $B$ $A^{-1}$ $B^{-1}A^{-1}$ $B^{-1}K^{-1}$
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$AB$ $A$ $B$ $AB^{-1}$
(4)
$A^{-1}K$ $A^{-1}B^{-1}$ $B^{-1}$ $A$ $AB$ $B$ $A^{-1}$ $B^{-1}A^{-1}$ $B^{-1}K^{-1}$
$BA^{-1}K$ $B^{-1}$ $AB^{-1}$ $B$ $BA^{-1}$ $B^{-1}K^{-1}$
$(AB^{-2})$ $(B^{2}A^{-1})$
$AB$ $A$ $B$ $AB^{-1}$
(5)
$A^{-1}K$ $A^{-1}B^{-1}$ $B^{-1}$ $A$ $AB$ $B$ $A^{-1}$ $B^{-1}A^{-1}$ $B^{-1}K^{-1}$
$BA^{-1}K$ $AB^{-2}$ $AB^{-1}$ $B^{2}A^{-1}$ $BA^{-1}$ $AB^{-2}K^{-1}$
$(AB^{-1}AB^{-2})$ $(B^{2}A^{-1}BA^{-1})$
$A$ $AB$ $B$ $AB^{-1}$ $B^{2}A^{-1}$ $AB^{-1}AB^{-2}$

Figure 4. 1

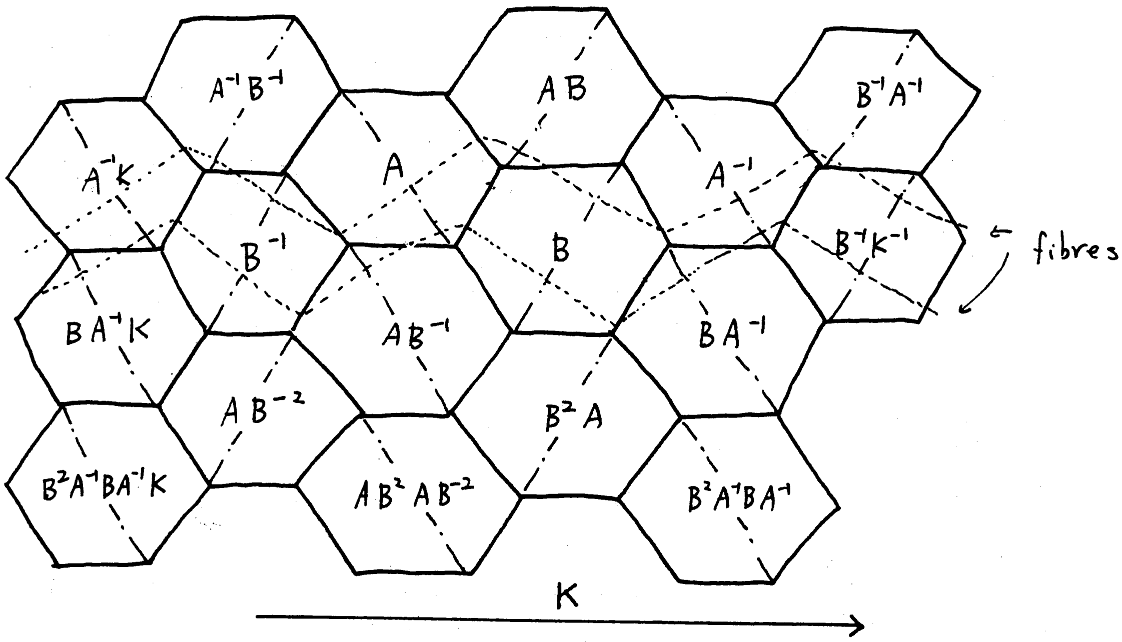


Figure 4.2

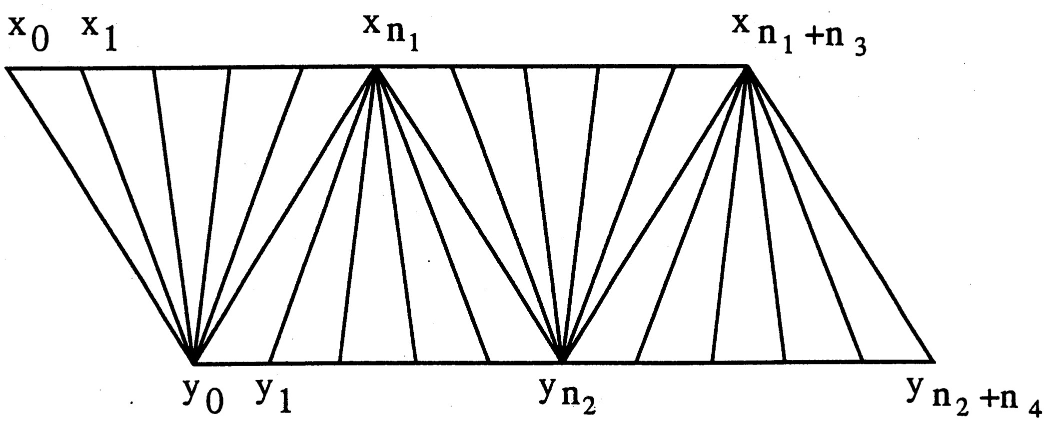


Figure 6.1

## 6. The canonical decompositions of punctured torus bundles

Put $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then any hyperbolic matrix $\phi$ of $SL(2,\mathbb{Z})$ is conjugate to $\epsilon L^{n_1}R^{n_2}\cdots L^{n_{2k-1}}R^{n_{2k}}$ where $\epsilon = \pm 1$, $k \geq 1$, and $n_i > 0$ ($1 \leq i \leq 2k$). To construct the complete hyperbolic structure on the punctured torus bundle $M_\phi$, we construct a complete hyperbolic structure of the infinite cyclic cover $\tilde{M}_\phi$ ($\cong T \times \mathbb{R}$) such that the covering transformation $(x,t) \to (\phi(x),t)$ acts as an isometry. Such a structure would be obtained as the limit of quasi-Fuchsian groups as in the previous section. To be precise, consider the axis $\lambda$ of the action $\phi_*$ on $\mathbb{H}^2$. Then the infinite strip consisting of the ideal triangles of the diagram $D$ of $SL(2,\mathbb{Z})$ intersecting $\lambda$ is periodic with period $L^{n_1}R^{n_2}\cdots L^{n_{2k-1}}R^{n_{2k}}$" (see Figure 6.1). Consider an increasing sequence of subarcs $\lambda_1 \subset \lambda_2 \subset \cdots$ of $\lambda$ such that $\cup_i \lambda_i = \lambda$. Then for each $i$, there is a quasi-Fuchsian group $G_i$ in $\mathcal{T}$ such that the "types" of the upper polygon $\tilde{P}_+(G_i)$ and the lower polygon $\tilde{P}_-(G_i)$ are represented by the two points $\partial\lambda_i$ in $D$. Then the sequence $\{G_i\}$ would converge to a group $G$, which gives the desired hyperbolic structure of $T \times \mathbb{R}$. Further there would be a natural bijection between the set of the faces of the extended Ford domain of $G$ and the set of the vertices of the infinite strip about $\lambda$. Recall that the canonical decomposition is the geometric dual of the Ford domain, in particular, each edge of the canonical decomposition is obtained as the image of the vertical line through the center of the isometric hemisphere supporting a face of the Ford domain. Thus there would be a natural bijection between the set of the edges of the canonical decomposition of $M_\phi$ and the set of the vertices of the infinite strip modulo the action of $\phi_*$. The induced triangulation of the cusp cross section would be described as follows (see [B2, FH]). Identify the infinite strip with $[0,1] \times \mathbb{R}$ in $\mathbb{R}^2$, so that the transformation $\phi_*$ is conjugated to the map $(x,y) \to (x, y+1)$. We extend this to a triangulation of $\mathbb{R}^2$ by a process of repeated reflection in the pair of vertical lines which form the boundary of this strip. The triangulation of $\partial M_\phi$ is given by the quotient by the group generated by $(x,y) \to (x,y+1)$ and $(x,y) \to (x+4,y)$ if $\epsilon = +1$. The "length" of the edges of the canonical decompositions are calculated from the traces of the generators of the fibre group $G$.

The explicit form of the group $G$ is obtained from a solution of the following equations. Put $s_1 = \sum_{i=odd} n_i$ and $s_2 = \sum_{i=even} n_i$. Let $x_i$ ($0 \leq i \leq s_1$) and $y_i$ ($0 \leq i \leq s_2$) be the traces of the elements of $SL(2,\mathbb{Z})$ corresponding to the vertices of the infinite strip as illustrated in Figure 5.1. Then the following holds:

(1) If $x$, $y$, $z$, and $w$ are the traces of two adjacent "triangles" $xyz$ and $xyw$, then $z + w = xy$ (see (3.6)). This enables us to express $\{x_i\}$ and $\{y_i\}$ in terms of $x_0$, $x_1$, and $y_0$.

(2) $x_0^2 + x_1^2 + y_0^2 = x_0 x_1 y_0$ by (3.7).

(3) $x_0 = x_{s_1}$ and $y_0 = y_{s_2}$ by the periodicity.

The above equations are reduced to a single polynomial equation in one variable, and the geometric solution satisfies the triangle inequality $\sqrt{|a_1|} - \sqrt{|a_2|} < \sqrt{|a_3|} < \sqrt{|a_1|} + \sqrt{|a_2|}$, where $a_1 = x_0/(x_1 y_0)$, $a_2 = x_1/(y_0 x_0)$, and $a_3 = y_0/(x_1 x_0)$.

Suppose $\phi$ is as in Theorem 3 in Part I, i.e., $k = 1$, $n_1 = 1$, and $n_2 = n$. Then the traces are obtained as follows. For each natural number $n$, let $[n]$ be the polynomial in $x$ defined by

$$[0]=0, \quad [1]=1, \quad [n+1]+[n-1]=x[n].$$

Then the trace $x_1$ is a solution of

$$[n+1]^2+[n]^2-[n+1][n-1]-3[n+1]+[n-1]+2=0.$$

The traces $\{y_i\}$ are obtained by

$$y_{-1}=x_0, \quad y_0=[n]x_0/([n+1]-1), \quad y_{i+1}+y_{i-1}=x_0 y_i.$$

**Example 6.1.** (1) If $n=1$, then $x_0=y_0=(3+\sqrt{3}i)/2$.

(2) If $n=2$, then $x_0=(8)^{1/4}$, $y_0=(16)^{1/4}$, and $y_1=(32)^{1/4}$. Since the unknotting tunnel corresponds to $x_0$ and since $|x_0|<|y_0|<|y_1|$, it is the shortest vertical edge.

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