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Recent developments in asymptotics obtained via integral representations of Stieltjes-transform type

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Abstract I discuss the recent developments in asymptotics, particularly my own work, which have been prompted by Berry's discovery in 1989 of how the remainder of an asymptotic expansion at optimal truncation can be universally described by an error function. In particular, it is now appreciated that integral representations of Stieltjes-transform type afford direct means by which asymptotic behaviour at high orders in an asymptotic expansion (often called "exponential asymptotics") can be discussed. I suggest that a direction for future research is the integration of the fields of exponential asymptotics and exact asymptotics.

1 Introduction

My purpose in this article is to review recent developments in an old subject. The old subject is that of asymptotic expansions, in which one seeks to describe, in terms of a series, the local behaviour of a function $f(z)$ in the neighbourhood of some special point $z = z_0$. We suppose that $f(z)$ is an analytic function of the complex variable z. If z_0 is a regular point of the function $f(z)$, the resulting series is, of course, simply the well-known Taylor expansion, which is a convergent power series. Even if z_0 is a singular point, the local series expansion will often be convergent. However, the term *asymptotic expansion* is usually used to denote a series expansion when the point z_0 is an essential singularity, and the expansion in question diverges, and it is this with which we shall be concerned.

Various long-standing, and much used, methods are available for finding asymptotic expansions. These may be found in the standard textbooks, such as de Bruijn (1958), Jeffreys (1962), Copson (1965), Olver (1974), Bleistein & Handelsman (1975), Bender & Orszag (1978), or Wong (1989). Two of the most common methods for obtaining the asymptotic expansion of a function $f(z)$ are by finding successive approximate solutions to a differential equation which $f(z)$ satisfies, or via stationary point methods applied to an integral representation of $f(z)$. To illustrate my discussion, I consider the behaviour of the modified Bessel function $K_0(z)$ as $z \to \infty$. I consider in turn the differential equation, and then the integral representation, approaches.

1.1 Functions defined by differential equations

To within an arbitrary multiplicative constant, the function $K_0(z)$ may be defined as that solution of the modified Bessel differential equation

$$z^2\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+z\frac{\mathrm{d}y}{\mathrm{d}z}-z^2y=0 \tag{1}$$

which tends to zero as $z\to\infty$ for real, positive z. (Any other linearly independent solution is unbounded in this limit: we say that the solution $K_0(z)$ is *recessive* in this limit—see Olver (1974), p. 155.) The change of dependent variable

$$w(z)=z^{\frac{1}{2}}\,y(z) \tag{2}$$

transforms the differential equation to

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-\left(1-\frac{1}{4z^2}\right)w=0\,. \tag{3}$$

For large z, one therefore expects that, to leading order, the $1/4z^2$ term on the right hand side of (3) may be neglected, and the solutions of (3) should, in some sense, be close to those of the differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-w=0\,, \tag{4}$$

a fundamental set for which is $\{\mathrm{e}^z,\mathrm{e}^{-z}\}$. Thus one would expect that for large positive z,

$$K_0(z)\approx Cz^{-\frac{1}{2}}\mathrm{e}^{-z} \tag{5}$$

for some constant C.

This approach may be refined further. Suppose we choose one of the solutions of the "approximate" differential equation (4), say $w_0(z)$. We can improve on this solution by adding successive correction terms, say $w_n(z)$, so that the successive partial sums

$$w_0(z),\quad w_0(z)+w_1(z),\quad w_0(z)+w_1(z)+w_2(z),\ \ldots\ldots\ldots\ldots \tag{6}$$

should be increasingly accurate approximations to a solution of (3). Here, $w_n(z)$ satisfies the inhomogeneous differential equation

$$\frac{\mathrm{d}^2w_n}{\mathrm{d}z^2}-w_n=-\frac{1}{4z^2}\,w_{n-1} \tag{7}$$

If we choose $w_0(z)$ to be a multiple of e^{-z}, and ask that each $w_n(z)$ is bounded as $z\to+\infty$, this procedure is well-defined.

Unfortunately, the procedure described above suffers from two defects. First, it is found that the sequence of partial sums in (6) diverges, no matter how large z is; that is, the expansion (in fact the *asymptotic expansion* for $w(z)$ as $z\to\infty$)

$$w(z)\sim w_0(z)+w_1(z)+w_2(z)+w_3(z)+\ldots\ldots\ldots \tag{8}$$

diverges. Nevertheless, for large z, taking the initial terms of the expansion does indeed give good approximations to a solution of (3). This suggests that we should truncate the expansion (8) at the point at which it is about to start diverging—usually referred to as *optimal truncation.* This overcomes the problem of divergence, albeit for the price of limited accuracy.

However, it is found that there is a second defect: the approximations are only appropriate for restricted values of ph(z). (Because I shall need often to refer to it, I shall denote ph(z) by θ from now on in this article without further comment.) Actually, this is only what one should expect. After all, the modified Bessel differential equation (1) has a single singular point in the finite z-plane, namely a regular singularity at $z = 0$, which gives rise to a logarithmic branch point of $K_0(z)$ at $z = 0$; in consequence, one should expect for large positive z, the asymptotic behaviour of $K_0(ze^{2\pi i})$ will differ radically from that of $K_0(z)$, in that the property of being recessive will be lost. However, the right hand side of equation (5), our approximation for $K_0(z)$, is recessive for all values of θ. So, although we expect $K_0(z) \approx Cz^{-\frac{1}{2}}e^{-z}$ for $\theta = 0$ from equation (5), we would *not* expect the corresponding relation with $\theta = 2\pi$, This must therefore mean that the appropriate approximant for $K_0(z)$ must change as θ increases from 0 to 2π. This phenomenon was discovered by Stokes, and is referred to as the *Stokes phenomenon.*

Broadly, in the very wide literature on asymptotic approximations, there have been three approaches to the two difficulties raised above:

(A) to ignore them, and use the approximations notwithstanding;

(B) to regard the approximations as precisely that—approximations, whose validity was established by establishing bounds (error bounds) for the remainders after truncation;

(C) by formally transforming the asymptotic expansion into a convergent series by the Borel transform and appealing to the properties of convergent series to establish the properties of the asymptotic expansion.

Approach (A), which is by far the most common, is less naive than it may sound. When an approximate method is used by an applied mathematician, there are usually other (typically physical) considerations against which the validity of the answer may be gauged. Nevertheless, however understandable this approach may be, it cannot commend itself to a mathematician exploring the nature of the approximations themselves—the point of view I shall take. Approach (B), the one I adopt here, has been taken to its furthest stage of development in the work of Frank Olver. In particular, his textbook, *Asymptotics and special functions* (Olver, 1974), remains an excellent introduction to the subject, despite the two decades that have elapsed since its publication. Approach (C) is described in a recent textbook by Sternin & Shatalov (1996). It has been developed largely by French workers, notably by Écalle (1981), and was brought to a wide audience by Voros (1984). Many aspects of approach (C) were implicitly used by Dingle (1973) in his interpretational approach to asymptotics. A major task for asymptotic analysts is to bring together and integrate the different insights and techniques of these different approaches, and in particular (B) and (C). In this article, I give some indication of how this may be achieved.

As I have just stated, our approach to asymptotic analysis will be that described by (B) above. A carefully detailed description of how error bounds may be found for functions which satisfy second-order differential equations is provided by Olver (1974, chapter 10). In outline the procedure is as follows. First, one derives a differential equation satisfied by the remainder, after truncation of the asymptotic expansion. Such equations are found to be similar in form to (7), inhomogeneous differential equations, whose inhomogeneous term involves the last retained term in the asymptotic expansion. The solutions for such equations may alternatively described as solutions of Volterra integral equations: the advantage of this formulation is that realistic bounds may be established by the method of successive approximations. (By "realistic bounds", I mean bounds that truly describe the behaviour of the remainder: in other words acceptably sharp bounds.) Realistic bounds enable one to address both of the difficulties raised earlier. First, one never deals with divergent series, but rather with a finite truncated series, complete with its error bound. Secondly, as one varies θ, the (realistic) error bound automatically becomes unacceptably large for values of θ for which the asymptotic approximation is appropriate. This approach has the advantage of mathematical rigour; its disadvantage is that one is limited by exploiting only part of the information available in the asymptotic expansion (the "late terms" have been discarded on truncation.)

1.2 Functions defined by integral representations of steepest-descents type

As well as being defined as the solution of the differential equation (1), the function $K_0(z)$ may also be regarded as defined by various integral representations. In particular, the representation

$$K_0(z) = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-z\cosh w}\,\mathrm{d}w \tag{9}$$

is one to which the classical stationary point methods can be applied. By "stationary point" methods, I mean the techniques and principles described by Laplace's method, Watson's Lemma, the method of stationary phase, the method of steepest descents, and so on. The books of Bleistein & Handelsman (1975) and of Wong (1989) give extended descriptions of these methods. Olver (1974) gives the fullest description available of how error bounds may be found from integral representations. (In general, simply finding the terms in the asymptotic expansions derived from integral representations has proved sufficiently difficult, without the additional challenge of establishing error bounds.)

The central notion behind stationary point methods is that, as the parameter $z \to \infty$, the contribution to the integral overwhelmingly comes from the neighbourhood of a single point on the path of integration. This point is the one at which the value of the argument of the rapidly-varying exponential in the integrand takes its maximum value. At such a point the argument of the exponential is stationary: its derivative is zero. Thus, for example, in the case of the integral in (9), the point $w = 0$ is a stationary point, and the integral should be well-approximated by taking account only of the local behaviour of the integrand:

$$K_0(z) \approx \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-z(1+\frac{w^2}{2})}\,\mathrm{d}w\,, \tag{10}$$

which gives rise to the same approximation (5) as was obtained from the differential equation in § 1.1, but with the value of the constant C now identified as $\sqrt{\pi/2}$. Higher terms in the approximation are available from considering higher approximations to the integrand near the stationary point. However, the effort required becomes increasingly tedious; moreover, realistic and satisfactory bounds have not been readily available in this method.

1.3 Integral representations of Stieltjes-transform type

A different approach to finding asymptotic expansions from that presented in §1.2 is to exploit integral representations of Stieltjes-transform type. Such an approach was adopted for the function $K_0(z)$ by Boyd (1990). In that paper, my starting point was the representation

$$K_0(z) = \pi^{-1} z^{-\frac{1}{2}} \, \mathrm{e}^{-z} \int_0^\infty \frac{K_0(t)\, \mathrm{e}^{-t}\, t^{-\frac{1}{2}}}{1+t/z} \, dt \,. \tag{11}$$

Now for this representation, one argues that as $z \to \infty$, the value of the denominator scarcely differs from 1 unless t is very large indeed: but when this is so, the numerator is very (in fact, exponentially) small. One therefore infers that

$$K_0(z) \approx \pi^{-1} z^{-\frac{1}{2}} \, \mathrm{e}^{-z} \int_0^\infty K_0(t)\, \mathrm{e}^{-t}\, t^{-\frac{1}{2}} \, \mathrm{d}t \,. \tag{12}$$

The integral on the right hand side of (12) may be evaluated exactly (it is a Mellin transform), and so one again arrives at the result (5) was obtained in § 1.1; as in §1.2 the value of the constant C is again identified as $\sqrt{\pi/2}$. So far, there is no gain over the stationary point methods discussed in §1.2. The advantage comes when one considers higher approximations, and error bounds. For large z, the denominator in (11) has as simple an expansion as one could wish:

$$\left(1+\frac{t}{z}\right)^{-1} = 1 - \frac{t}{z} + \frac{t^2}{z^2} - \cdots \, ; \tag{13}$$

formally, therefore the asymptotic expansion of $K_0(z)$ is found to be

$$K_0(z) \sim \pi^{-1} z^{-\frac{1}{2}} \, \mathrm{e}^{-z} \sum_{r=0}^{\infty} (-1)^r \, z^{-r} \int_0^\infty K_0(t)\, \mathrm{e}^{-t}\, t^{r-\frac{1}{2}} \, \mathrm{d}t \,, \tag{14}$$

each term of which is explicitly given and may readily be evaluated. Moreover, the error term on truncation is also available. If one truncates (13) after N terms, the remainder is

$$\left(-\frac{t}{z}\right)^N \left(1+\frac{t}{z}\right)^{-1} , \tag{15}$$

and so if (14) is truncated after N terms, the remainder is

$$(-1)^N \, \pi^{-1} z^{-N-\frac{1}{2}} \, \mathrm{e}^{-z} \int_0^\infty \frac{K_0(t)\, \mathrm{e}^{-t}\, t^{N-\frac{1}{2}}}{1+t/z} \, dt \,, \tag{16}$$

an expression whose properties can readily be found, and in particular, which can readily be bounded. Details are given in Boyd (1990).

There are other advantages, besides simplicity and directness, in using representations of Stieltjes-transform type. Boyd (1990) showed how they could readily be used to find the behaviour of optimal remainders near Stokes lines (now referred to as the Berry smoothing of the Stokes phenomenon), and to find asymptotic expansions for remainder terms (now referred to as hyperasymptotics).

Before I proceed with the rest of my exposition, it is as well to describe the significance of Stokes lines, and the Stokes phenomenon, as understood before 1989, when Michael Berry published a major paper on this subject. The Stokes phenomenon, which I referred to in § 1.1, may be illustrated by means of $K_0(z)$. For $-\frac{3}{2}\pi < \theta < \frac{3}{2}\pi$, one can show that the asymptotic expansion

$$K_0(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \mathrm{e}^{-z} \sum_{r=0}^{\infty} \frac{(-1)^r b_r}{z^r}\,, \tag{17}$$

holds good, where

$$b_r = \frac{(2r)!^2}{r!^3\, 2^{5r}}\,. \tag{18}$$

(By "holds good", I mean, as is conventional in definitions of asymptotic expansions, that for sufficiently large z, the remainder term may be bounded by a constant multiple of the first discarded term after truncation.) However, the exact relation

$$K_0(z) = -K_0(z\mathrm{e}^{-2\pi\mathrm{i}}) + 2K_0(z\mathrm{e}^{-\pi\mathrm{i}}) \tag{19}$$

implies at once (from (17) itself!) that in the sector $\frac{1}{2}\pi < \theta < \frac{5}{2}\pi$,

$$K_0(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \mathrm{e}^{-z} \sum_{r=0}^{\infty} \frac{(-1)^r b_r}{z^r} + 2\,\mathrm{i}\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \mathrm{e}^{z} \sum_{r=0}^{\infty} \frac{b_r}{z^r}\,. \tag{20}$$

Now, in the sector $\frac{1}{2}\pi < \theta < \frac{3}{2}\pi$, *both* (17) and (20) are valid: either may be used. They differ in the presence of the second series, involving the factor e^z, in (20). The conventional explanation by asymptotic analysts has been that the contribution from this series is exponentially small, and therefore irrelevant, in comparison with the first series. In other words, the only difference between the two representations is an exponentially, and so unimportant, small term.

This explanation, while accurate so far as it goes, points up the inadequacy of the classical treatment of asymptotic expansions (for instance, that given by Erdélyi (1956)), In such a treatment, one considers only the order of magnitude of the remainder term. In the description I used earlier in § 1.1, this provides a bound, but not necessarily a *realistic* bound. In fact Olver himself considered this question some years ago (Olver (1980)) by considering realistic bounds, and provided a partial answer. He showed that at optimal truncation, it was better to choose (17) when $\frac{1}{2}\pi < \theta < \pi$, and (20) when $\pi < \theta < \frac{3}{2}\pi$. with a region of uncertainty near the line $\theta = \pi$. This dividing line between different asymptotic expansions for the same function is called a Stokes line.

1.4 The Berry smoothing of Stokes discontinuities

In a pioneering paper published in 1989, Michael Berry gave a great impetus to the subject of asymptotics. Berry was concerned to gain further insight into the Stokes phenomenon than was then available. In his paper (Berry (1989)) he indicated by formal calculations based on the Borel transform how the remainder at optimal truncation behaved near a Stokes line. His formal calculations (supported by compelling numerical computations) showed that the transition between (17) and (20) was not abrupt. Rather, the exponentially small series in (20) is to be multiplied by an error function of a certain argument, which has a specified universal form. The argument is such that the error function takes the value $\frac{1}{2}$ on the Stokes line $\theta = \pi$, and tends rapidly to the values 1 (or 0) as θ increases above (or decreases below) π.

Berry's work came to the attention of Olver, who immediately appreciated its seminal importance, and set about developing an analytically rigorous theory to verify Berry's results. Olver's investigations were reported at a conference held in Winnipeg in 1989 (and published in Olver (1990)). Rather than finding that each term in the exponentially small series in (20) was multiplied by the same error function, as Berry had found, Olver showed, in a more accurate analysis, that each term in the exponentially small series in (20) was multiplied by its own characteristic factor. These factors were incomplete gamma functions, whose asymptotic behaviour near the Stokes line and at optimal truncation was given by the error function formula which Berry had discovered. Actually, such factors had been much used by Dingle (1973) in his interpretational account of asymptotic expansions: he referred to them as "terminants", a nomenclature that continues to be used.

In my 1990 paper, I too sought to reproduce the results of Berry in a rigorous fashion for the modified Bessel function $K_0(z)$, taking (11) as my starting point. My approach was not the same as that of Olver, who used a different integral representation. However, an integral representation of Stieltjes-transform type is a very natural route to derive Berry's results, as I now show.

Consider first the nature of optimal truncation in the asymptotic expansion (14). When $|z|$ is large, the value of the index s at or near optimal truncation, is also large. In this circumstance, the integrand in (14) is initially very small as the variable of integration t increases from 0. So the major contribution to the integrand will come from larger values of t, when the modified Bessel function $K_0(t)$ may be very well approximated by its asymptotic approximation (5). Multiplicative constants aside, the integral is therefore very well approximated by

$$\int_0^\infty \mathrm{e}^{-2t}\, t^{r-1}\, \mathrm{d}t\,, \tag{21}$$

that is, $\Gamma(r)/2^r$. By considering the ratio of successive terms, we infer that for large z, the terms of the asymptotic expansion (14) at first steadily decrease in magnitude, level off, and then steadily increase. Optimal truncation occurs therefore when the magnitude of the ratio of successive terms is close to 1: in fact when

$$r \approx 2|z|\,. \tag{22}$$

The integrand in (21) may be expressed as

$$\int_0^\infty \mathrm{e}^{-2t+r\ln t}\, t^{-1}\, \mathrm{d}t\,, \tag{23}$$

in which form Laplace's method is applicable. For large r, the value of the integral is overwhelmingly determined from the behaviour of the integrand when the argument of the exponent takes its maximum value, that is when $t = r/2$, or from (22) when,

$$t \approx |z| \,. \tag{24}$$

This last result is important in two profound ways. These two ways are in fact directly related to the two defects of the asymptotic expansion (8) that I discussed in § 1.1: the divergence, and the Stokes phenomenon, I discuss them in turn.

First, the infinite series expansion (13) converges only when $t < |z|$. Plainly therefore, one should anticipate that a term-by-term integration, as I employed when deriving (14) would (because of its invalidity) lead to difficulties, such as a divergent series. In fact, from (23) we should expect the procedure to be sensible so long as the stationary point $t = r/2$ was less than $|z|$. In other words, we should *expect* the onset of divergence at $r \approx 2|z|$.

Secondly, the integral (16), which represents the the remainder, has interesting behaviour near optimal truncation: the numerator has a stationary point at $t \approx |z|$, while the denominator has a pole when $t = -z$. These points will coincide at optimal truncation when z is real and negative. The asymptotic behaviour of an integral with a coinciding stationary point and pole is well-known, and is described by a uniform asymptotic expansion involving error functions (e.g., Bleistein (1966)). As θ changes, the residue due to the pole in the denominator will cause a disjunction in the behaviour of the remainder when $|z|$ is real and negative. This in fact is the Stokes phenomenon, and the error function behaviour is what Berry found. I outline the detailed calculation next.

At optimal truncation, the major contribution to the integrand in (16) comes from the neighbourhood of $t = |z|$. In this neighbourhood, the function $K_0(t)$ in the integrand is well-approximated by its asymptotic approximation (17); formal substitution of (17) and term-by-term integration in (16) yields

$$2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}z^{-N-\frac{1}{2}}\,\mathrm{e}^{-z}\sum_{r=0}^{\infty}(-1)^{N-r}\,a_r\int_0^{\infty}\frac{\mathrm{e}^{-2t}\,t^{N-r}}{1+t/z}\,\mathrm{d}t\,, \tag{25}$$

or, using (103),

$$2^{\frac{1}{2}}\pi^{\frac{1}{2}}z^{-\frac{1}{2}}\,\mathrm{e}^{z}\sum_{r=0}^{\infty}\frac{a_r}{z^r}(-1)^{N-r}\,F_{N-r}(2z)\,. \tag{26}$$

The functions $F_{N-r}(2z)$ which appear in this equation are the terminants referred to earlier. Now, from (109), provided $|z|$ is large, $N \approx -2z$, and r is not large,

$$F_{N-r}(2z) \approx \frac{1}{2}\,\mathrm{i}\,(-1)^{N-r}\mathrm{erfc}\left(c\sqrt{N/2}\right)\,. \tag{27}$$

Here, the parameter c, which measures the proximity of pole and stationary point, is defined by (equation (106))

$$\tfrac{1}{2}c^2 = 1 + 2z/N + \ln(-2z/N)\,, \tag{28}$$

so that, from (107),

$$c \approx -\mathrm{i}\,(2z/N+1) \approx -(\theta-\pi)\,, \tag{29}$$

where I have ignored quantities $O(1/N)$. Combining (26), (27), and (29), we find

$$2\,\mathrm{i}\sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{z}\sum_{r=0}\frac{a_r}{z^r}\left[\frac{1}{2}\,\mathrm{erfc}\Big(c\sqrt{N/2}\Big)\right]. \tag{30}$$

From the asymptotic behaviour of the complementary error function of large argument, one readily infers that the values of the factors in the square brackets in (30) change swiftly but smoothly from nearly 0 to nearly $+1$ as θ increases and passes through the Stokes line $\theta = \pi$, "switching on" the exponentially small contributions from the second series in (20). (This is the *Berry smoothing* of Stokes discontinuities (Berry (1989)).)

1.5 Further developments

The advantages of using integral representations of Stieltjes-transform type will be evident to the reader: each of the terms in the asymptotic expansion is readily, and independently, given by an explicit formula; likewise, an explicit formula for the remainder is available, from which one may establish error bounds; the form of the remainder is amenable to analysis at optimal truncation. There is another advantage also: the formula for the terms in the expansion (that is, the coefficients) lends itself to analysis when the order of the terms is large (the "late" terms).

These advantages are clear enough. But a perusal of the list of functions which are known to be expressible as Stieltjes transforms (Erdélyi *et al.* (1954)) reveals a disappointingly small number! It might be thought therefore that, interesting and insightful as the Stieltjes-transform approach is, it would be of rather limited applicability. This was certainly my view after I published my 1990 paper. This pessimistic view was shown to be mistaken in two important papers: those of Berry & Howls (1991) and of Olde Daalhuis & Olver (1994). The former paper showed how a wide class of integral representations of steepest-descents type could be expressed as a *sum* of Stieltjes transforms. (It is such a sum that I refer to as an "integral representation of Stieltjes-transform type".) This greatly extended the range of problems to which the techniques described earlier could be applied. The latter paper provided a remarkable extension in a different direction: the authors showed that, starting with connection formulae that link solutions of a second-order linear differential equation with their analytic continuations, one could represent solutions by means of integrals which, for asymptotic purposes, could be regarded as being of Stieltjes-transform type.

The rest of the article is structured as follows. In § 2, I outline my account of Berry & Howls' analysis of steepest-descents integrals. Then in § 3, I illustrate the theory by considering the modified Bessel function $K_0(z)$. In § 4, I describe how the behaviour of the remainders at optimal truncation may be used to characterise the solutions of second-order linear differential equations. In § 5, I describe an important, but technically somewhat unusual application—to the gamma function $\Gamma(z)$. The behaviour of the late coefficients in asymptotic expansions is addressed in § 6. Finally, in § 7, I give a brief discussion. Appendices A and B contain technical statements and results on terminant functions that are needed in the main text.

2 The Berry-Howls representation

Consider how the method of steepest descents would be used to find the asymptotic expansion, as $|z| \to \infty$, of the integral

$$f(z) = \int \mathrm{e}^{-zp(w)}\, q(w)\, \mathrm{d}w\,, \tag{31}$$

defined over some contour in the w-plane. The first stage in applying the method is to locate the zeros of $p'(w)$ (more usually referred to as saddle points or stationary points in this context), and then to deform the contour to a path of steepest descents through one of the saddle points. We shall assume that $p(w)$ and $q(w)$ are holomorphic functions in a domain $\Delta^{(n)}$ which will be defined shortly, and that $p'(w)$ has only simple zeros, located at $w^{(1)}$, $w^{(2)}$, ... I shall use the notation

$$p^{(m)} = p(w^{(m)}), \quad p^{(nm)} = p^{(m)} - p^{(n)}, \quad \theta^{(nm)} = \mathrm{ph}(p^{(nm)})\,. \tag{32}$$

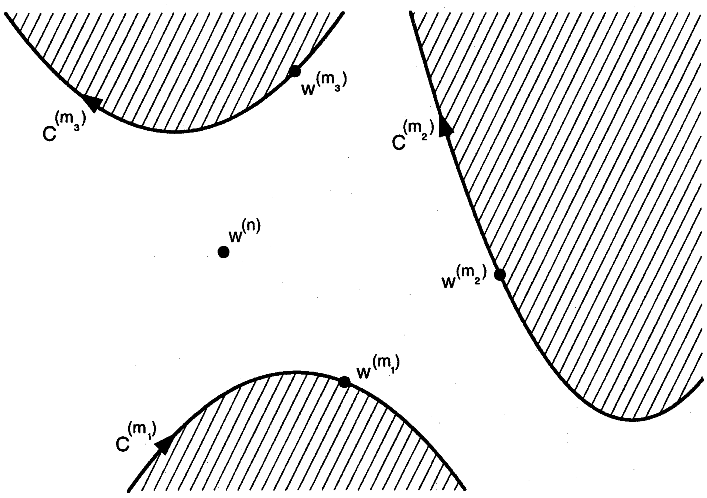

Figure 1. A typical domain associated with the saddle point $w^{(n)}$. (The domain is shown unshaded.) The contours on the boundary of the domain—the figure depicts $C^{(m_1)}$, $C^{(m_2)}$, $C^{(m_3)}$, passing through the adjacent saddle points $w^{(m_1)}$, $w^{(m_2)}$, $w^{(m_3)}$ respectively—are the adjacent contours.

The domain $\Delta^{(n)}$ is defined as follows. Consider all the steepest-descents paths, for different θ, which emanate from the saddle point $w^{(n)}$. Such paths must end either at infinity or at a singularity of $p(w)$; I shall suppose that they all end at infinity. Then a continuity argument shows that the set of all points in the w-plane which can be reached from $w^{(n)}$ by a steepest-descents path forms the closure of a domain. It is this domain that we denote by $\Delta^{(n)}$ (Figure 1). The boundaries of $\Delta^{(n)}$ are themselves paths of steepest descents: they are the paths through the neighbouring saddles $w^{(n)}$ such that

$$\mathrm{ph}\Big(p(w) - p^{(m)}\Big) = \theta^{(nm)}. \tag{33}$$

Berry & Howls refer to such points as *adjacent saddles.* Later, we shall need to use these steepest-descents paths as contours of integration. To do so, we specify that they are to be traced in an anti-clockwise sense with respect to the domain $\Delta^{(n)}$. I refer to such a contour through an adjacent saddle as an *adjacent contour*, and denote it by $C^{(m)}$ (see Figure 1).

Suppose that the contour of integration is deformed into the path of steepest descents through the saddle point $w^{(n)}$. We are thus led to consider the asymptotic expansion of the integral

$$S^{(n)}(z) = z^{\frac{1}{2}} \int_{C^{(n)}(\theta)} \mathrm{e}^{-z[p(w)-p^{(n)}]} \, q(w) \, \mathrm{d}w \,, \tag{34}$$

where $C^{(n)}(\theta)$ is the path of steepest descents through the saddle point $w^{(n)}$ (and where the orientation of $C^{(n)}(\theta)$ would be specified in any particular problem). Here, $S^{(n)}(z)$ is the *slowly-varying* part of the function $f(z)$ as $z \to \infty$. Thus

$$f(z) = z^{-\frac{1}{2}} \, \mathrm{e}^{-zp^{(n)}} S^{(n)}(z) \,, \tag{35}$$

and $S^{(n)}(z)$ has an asymptotic expansion of the form

$$S^{(n)}(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \tag{36}$$

for some constant coefficients a_0, a_1, a_2, ...

On each half of $C^{(n)}$ — before and after the saddle point $w^{(n)}$ — the expression $z[p(w) - p^{(n)}]$ in the exponent of the integrand in (34) increases monotonically from 0 (because $C^{(n)}$ is a path of steepest descent from $w^{(n)}$). I now impose a further condition on $p(w)$: that $|p(w)| \to \infty$ as $|w| \to \infty$ in $\Delta^{(n)}$. Then $z[p(w) - p^{(n)}]$ actually increases monotonically to ∞ on $C^{(n)}(\theta)$. Denote the two halves of $C^{(n)}(\theta)$ by $C_{-}^{(n)}(\theta)$ and $C_{+}^{(n)}(\theta)$. Let us introduce a transformation from w to u defined by

$$z\,[p(w) - p^{(n)}] = u \,. \tag{37}$$

We may assert that for any $u > 0$, there is exactly one value of w on each of $C_{-}^{(n)}(\theta)$ and $C_{+}^{(n)}(\theta)$ such that equation (37) is satisfied. We denote the values by $w_{-}(u)$ and $w_{+}(u)$ respectively. The conditions we have imposed thus far are listed together as *Conditions A(i)–(iv)* in Appendix A; one further condition is given in the Appendix, *Condition A(v)*, which we shall need shortly.

Berry & Howls made the transformation (37) and found

$$S^{(n)}(z) = z^{-\frac{1}{2}} \int_0^\infty \mathrm{e}^{-u} \left\{ \frac{q(w_+(u/z))}{p'(w_+(u/z))} - \frac{q(w_-(u/z))}{p'(w_-(u/z))} \right\} \mathrm{d}u \,. \tag{38}$$

The factor in braces can itself be represented as a contour integral, and so Berry & Howls arrived at

$$S^{(n)}(z) = \frac{z}{2\pi \mathrm{i}} \int_0^\infty u^{-\frac{1}{2}} \mathrm{e}^{-u} \int_{\Gamma^{(n)}} \frac{q(w)\,[p(w) - p^{(n)}]^{\frac{1}{2}}}{z[p(w) - p^{(n)}] - u} \, \mathrm{d}w \, \mathrm{d}u \,, \tag{39}$$

where $\Gamma^{(n)}$ comprises two infinite contours which enclose $C^{(n)}$ in an anti-clockwise loop within $\Delta^{(n)}$. The asymptotic expansion for $S^{(n)}(z)$ can now be found from the representation (39) by expanding in powers of

$$u/z[p(w) - p^{(n)}] \,.$$

One thus finds

$$S^{(n)}(z) = \sum_{r=0}^{N-1} \frac{a_r}{z^r} + R_N(z)\,, \tag{40}$$

where

$$a_r = \frac{\Gamma(r+\frac{1}{2})}{2\pi \mathrm{i}} \oint_{\Gamma^{(n)}} \frac{q(w)}{[p(w)-p^{(n)}]^{r+\frac{1}{2}}}\,\mathrm{d}w \tag{41}$$

and

$$R_N(z) = \frac{1}{2\pi \mathrm{i} z^N} \int_0^\infty u^{N-\frac{1}{2}} \mathrm{e}^{-u} \int_{\Gamma^{(n)}} \frac{q(w)}{[p(w)-p^{(n)}]^{N+\frac{1}{2}}\{1-u/z[p(w)-p^{(n)}]\}}\,\mathrm{d}w\,\mathrm{d}u\,. \tag{42}$$

In (41), the contour of integration $\Gamma^{(n)}$ can be shrunk to a closed contour encircling $w^{(n)}$. To discuss the remainder term $R)N(z)$, Berry & Howls deformed $\Gamma^{(n)}$ to the boundary of $\Delta^{(n)}$, and thus found

$$R_N(z) = \frac{1}{2\pi \mathrm{i} z^N} \int_0^\infty u^{N-\frac{1}{2}} \mathrm{e}^{-u} \sum_m \int_{C^{(m)}} \frac{q(w)}{[p(w)-p^{(n)}]^{N+\frac{1}{2}}\{1-u/z[p(w)-p^{(n)}]\}}\,\mathrm{d}w\,\mathrm{d}u\,, \tag{43}$$

where the summation is over each of the adjacent contours, provided *Condition A(v)* holds. This representation can itself be usefully employed to obtain bounds for the error terms in an asymptotic expansion, as set out in Boyd (1993).

However, Berry & Howls proceeded further with their analysis. By making the change of variable

$$u/[p(w)-p^{(n)}] = v/p^{(nm)}, \tag{44}$$

or equivalently

$$u = v + \left(v/p^{(nm)}\right)\,[p(w)-p^{(n)}]\,,$$

they found

$$R_N(z) = \frac{1}{2\pi \mathrm{i} z^N} \sum_m \frac{1}{[p^{(nm)}]^N} \int_0^\infty \frac{v^{N-\frac{1}{2}}\mathrm{e}^{-v}}{1-v/zp^{(nm)}} \int_{C^{(m)}} q(w) \exp\left(-\frac{v}{p^{(nm)}}[p(w)-p^{(m)}]\right) \mathrm{d}w\mathrm{d}v\,. \tag{45}$$

On defining, in analogy with (34),

$$S^{(m)}(\eta) = \eta^{\frac{1}{2}} \int_{C^{(m)}} \mathrm{e}^{-\eta[p(w)-p^{(n)}]}\, q(w)\,\mathrm{d}w\,, \tag{46}$$

we arrive at the representation

$$R_N(z) = \frac{1}{2\pi \mathrm{i} z^N} \sum_m \frac{1}{[p^{(nm)}]^N} \int_0^\infty \frac{v^{N-1}\mathrm{e}^{-v} S^{(m)}(v/p^{(nm)})}{1-v/zp^{(nm)}}\,\mathrm{d}v\,. \tag{47}$$

In the special and common case where *Condition A(v)* holds good when $N=0$ (i.e., when $q(w)[p(w)]^{-\frac{1}{2}}$ decays sufficiently rapidly at infinity), this result is true also for $N=0$. Then (47) becomes

$$S^{(n)}(z) = \frac{1}{2\pi \mathrm{i}} \sum_m \int_0^\infty \frac{v^{-1}\mathrm{e}^{-v} S^{(m)}(v/p^{(nm)})}{1-v/zp^{(nm)}}\,\mathrm{d}v\,. \tag{48}$$

The representation (48) is is elegant and revealing, and its importance can scarcely be over-emphasised. It demonstrates in a very direct fashion the manner in which the behaviour at one saddle point, $w^{(n)}$, depends on the behaviour at the *adjacent* saddle points, $w^{(m)}$. Even when the relation does not hold good (as for the gamma function—see § 5), essentially the same remarks are appropriate for (47).

3 Application to the modified Bessel function $K_0(z)$

The modified Bessel function $K_0(z)$ enjoys the integral representation (9). The function $\cosh w$ in the integrand of (9) has saddle points at $w^{(n)} = \mathrm{i}n\pi$ for $n = 0, \pm1, \pm2\ldots$.. It is natural to define $p(w) = \cosh w$, $q(w) = 1$, and to express (9) in the form

$$K_0(z) = \tfrac{1}{2} \int_{C^{(0)}(\theta)} \mathrm{e}^{-z\cosh w}\, \mathrm{d}w\,. \tag{49}$$

Here, $C^{(0)}(\theta)$ is the path of steepest descent (running from left to right) passing through the saddle point $w^{(0)} = 0$. In this form, there is no restriction on θ: the contour $C^{(0)}(\theta)$ automatically moves as θ changes. We define

$$S^{(0)}(z) = z^{\frac{1}{2}} \int_{C^{(0)}(\theta)} \mathrm{e}^{-z[\cosh w - 1]}\, \mathrm{d}w\,. \tag{50}$$

The functions K_0 and $S^{(0)}$ are related by

$$K_0(z) = \left(\mathrm{e}^{-z}/2z^{\frac{1}{2}}\right) S^{(0)}(z)\,. \tag{51}$$

In this problem, there are an infinite number of saddle points, but only two of them — $w^{(1)}$ and $w^{(-1)}$ — are adjacent to $w^{(0)}$. We readily establish the validity of *Conditions A* for the domain $\Delta^{(0)}$. For this function, the representation (48) is found to be

$$S^{(0)}(z) = \frac{1}{2\pi\mathrm{i}} \int_0^\infty \frac{v^{-1}\mathrm{e}^{-v} S^{(1)}(-\frac{1}{2}v)}{1 + v/2z}\, \mathrm{d}v + \frac{1}{2\pi\mathrm{i}} \int_0^\infty \frac{v^{-1}\mathrm{e}^{-v} S^{(-1)}(-\frac{1}{2}v)}{1 + v/2z}\, \mathrm{d}v\,. \tag{52}$$

Here, $S^{(\pm1)}$ are defined by

$$S^{(\pm1)}(\eta) = \eta^{\frac{1}{2}} \int_{C^{(\pm1)}(\mp\pi)} \mathrm{e}^{-\eta[\cosh w + 1]}\, \mathrm{d}w\,, \tag{53}$$

and elsewhere by analytic continuation.

The relation (52) can be simplified. We may express $S^{(1)}(\eta \mathrm{e}^{\mathrm{i}\pi})$ in the form

$$\mathrm{i}\eta^{\frac{1}{2}} \int_{C^{(0)}} \mathrm{e}^{-\eta[\cosh w - 1]}\, \mathrm{d}w\,, \tag{54}$$

and the change of variable from w to $w + \mathrm{i}\pi$ yields

$$-\,\mathrm{i}\eta^{\frac{1}{2}} \int_{C^{(0)}} \mathrm{e}^{-\eta[-\cosh w - 1]}\, \mathrm{d}w\,, \tag{55}$$

which is $-\mathrm{i}S^{(1)}(z)$. Thus

$$S^{(1)}(\eta) = \mathrm{i}S^{(0)}(\eta \mathrm{e}^{\mathrm{i}\pi})\,. \tag{56}$$

Likewise, one can show that

$$S^{(-1)}(\eta) = \mathrm{i}S^{(0)}(\eta \mathrm{e}^{-\mathrm{i}\pi}) \,. \tag{57}$$

So

$$S^{(0)}(z) = \frac{1}{\pi}\int_0^\infty \frac{v^{-1}\mathrm{e}^{-v}S^{(0)}(\frac{1}{2}v)}{1+v/2z}\,\mathrm{d}v \,.$$

and so we conclude that

$$S^{(0)}(z) = \frac{1}{\pi}\int_0^\infty \frac{t^{-1}\mathrm{e}^{-2t}S^{(0)}(t)}{1+t/z}\,\mathrm{d}t \,. \tag{58}$$

When (51) is substituted into this relation, one finds

$$K_0(z) = \pi^{-1}z^{-\frac{1}{2}}\,\mathrm{e}^{-z}\int_0^\infty \frac{K_0(t)\,\mathrm{e}^{-t}\,t^{-\frac{1}{2}}}{1+t/z}\,\mathrm{d}t \,,$$

actually equation (11). This example illustrates how the analysis of Berry & Howls proceeds: the key is the transformation of the original integral representation into one of Stieltjes-transform type, namely (52). In this case, the final result (58) actually is a Stieltjes transform, but this is a special feature of this special problem.

4 Application to solutions of differential equations with integral representations of steepest descents type

In a forthcoming paper (Boyd (1996)), I consider the special but important circumstance that the solutions of a linear differential equation may be represented as contour integrals of an integrand of steepest-descents type. The modified Bessel differential equation of order 0, given in equation (1), is an example of such an equation.

One of the solutions of this differential equation is $K_0(z)$, enjoying the integral representation (49), that is

$$K_0(z) = \tfrac{1}{2}\int_{C^{(0)}(\theta)} \mathrm{e}^{-z\cosh w}\,\mathrm{d}w \,. \tag{59}$$

Here $C^{(0)}(\theta)$, defined for $-\pi < \theta < \pi$ is the steepest-descents path passing through $w^{(0)} = 0$, and running from left to right. Two other solutions are the functions $K_0(z\,\mathrm{e}^{\pm\pi\mathrm{i}})$, which enjoy the integral representations

$$K_0(z\,\mathrm{e}^{\pm\pi\mathrm{i}}) = \mp\tfrac{1}{2}\int_{C^{(\pm1)}(\theta)} \mathrm{e}^{-z\cosh w}\,\mathrm{d}w \,. \tag{60}$$

Here $C^{(\pm1)}(\theta)$ are adjacent contours with respect to $w^{(0)}$: so $C^{(1)}(\theta)$ is defined for $-2\pi < \theta < 0$ to be the steepest-descents path passing through $w^{(1)} = \pi\mathrm{i}$, and running from right to left; $C^{(-1)}(\theta)$ is defined for $0 < \theta < 2\pi$ to be the steepest-descents path passing through $w^{(-1)} = -\pi\mathrm{i}$, and running from left to right.

Each of the three solutions (59) and (60) has a common form: they are all integral representations with the same integrand,

$$\mathrm{e}^{-z\cosh w}, \tag{61}$$

and with contours which run from infinity to infinity. It is easy to see that *any* such integral is a solution of the differential equation (1): conversely therefore, because of the linearity of the differential equation, any solution may be expressed as a linear combination of (at most) two such contours. To discuss the asymptotic behaviour of solutions of the differential equation (1), therefore, it suffices to find that of say, $K_0(z)$ and $K_0(z\,\mathrm{e}^{\pi \mathrm{i}})$. Both of these may be attacked through their representation (48) of Stieltjes-transform type. In fact, in this particular problem, it is obvious that only one of these representations is necessary, that for $K_0(z)$ say.

For example, to discuss the modified Bessel function $I_0(z)$, one would use the relation

$$I_0(z) = \frac{1}{\pi \mathrm{i}}\left(K_0(z) - K_0(z\,\mathrm{e}^{\pi \mathrm{i}})\right) . \tag{62}$$

While of course a description of the behaviour of $I_0(z)$ can thus be made available, it does not yield the strikingly simple behaviour near the a Stokes line at optimal truncation (the complementary error function — erfc — behaviour associated with Berry smoothing) that $K_0(z)$ does. This is only to be expected: after all, the defining feature of the solution $I_0(z)$ is that it is recessive as $z \to 0$. In contrast, the defining feature of the solution $K_0(z)$ is that it is recessive as $z \to \infty$, the asymptotic limit with which we are concerned. In fact, I show in Boyd (1996) that all of the solutions which, at optimal truncation, yield behaviour of erfc type on a Stokes line are solutions which are recessive on a *neighbouring* Stokes line.

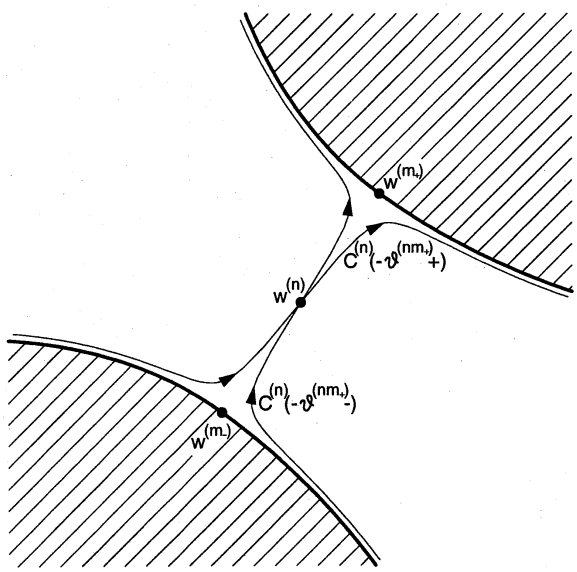

Figure 2. A generic depiction of the "dogleg" contours used to define dominant solutions of Bi-type. The "dogleg" contours $C^{(n)}(-\theta^{(nm_+)}-)$ and $C^{(n)}(-\theta^{(nm_+)}+)$ are associated with the Stokes line $\theta = -\theta^{(nm_+)}$, and are steepest-descents paths through $w^{(n)}$ which respectively turn left and right at the saddle $w^{(m_+)}$, and are the limits of the steepest-descents contours as $\theta \to -\theta^{(nm_+)}$ from below and from above.

It might be imagined, therefore, that only solutions which are recessive at infinity would provide results of interest at optimal truncation on Stokes lines. I show in Boyd (1996) that this is not so. There is a *dominant* solution which also yields results of interest at optimal truncation on Stokes lines, and I shall now define it. So far we have considered solutions which may be represented by a single integral of steepest-descents type. (These in fact are the solutions recessive at infinity for some range of values of θ.) The steepest descents path through a saddle point is uniquely defined except when θ coincides with the phase of a Stokes line. In this situation, the path runs from the saddle point directly to one of the adjacent saddles, and then turns through an angle of $\pi/2$, either to the left or the right. I consider the solution which results from taking the sum of two such "dogleg" contours. In Boyd (1996), I show that for the problems under consideration, the values of $p^{(m)}$ at the adjacent saddles are all the same, and there can be at most two adjacent saddles. A generic depiction of the "dogleg" contours is given in Figure 2. I refer to such solutions as "dominant solutions of Bi-type", because the Bi solution of Airy's differential equation is an archetype of this kind of solution.

I illustrate by considering the modified Bessel differential equation and the Stokes line $\theta = 0$. When $\theta = 0$, paths of steepest descents run from the saddle point $w^{(1)} = \pi\mathrm{i}$ directly to its two adjacent saddles, $w^{(0)} = 0$ and $w^{(2)} = 2\pi\mathrm{i}$. (It would be possible to choose any saddle point $w^{(n)}$ with n odd, but the same results would be found.) Let us then consider the solution $y^{(1)}(z)$ defined by

$$y^{(1)}(z) = \left\{ \int_{C^{(1)}(0-)} + \int_{C^{(1)}(0+)} \right\} \mathrm{e}^{-z\cosh w}\,\mathrm{d}w \tag{63}$$

when $\theta = 0$; elsewhere, $y^{(1)}(z)$ is defined by analytic continuation. Here, $C^{(1)}(0-)$ denotes the contour obtained by continuous deformation of $C^{(1)}(\theta)$ as $\theta \to 0$ through negative values:

$$C^{(1)}(0-) = \lim_{\theta \to 0-} C^{(1)}(\theta)\,. \tag{64}$$

The definition of $C^{(1)}(0+)$ is analogous. We infer from the representations (60) that $y^{(1)}(z)$ may be expressed in terms of $K_0(z\mathrm{e}^{\pi\mathrm{i}})$ and $K_0(z\mathrm{e}^{-\pi\mathrm{i}})$:

$$y^{(1)}(z) = 2\left\{ K_0(z\mathrm{e}^{\pi\mathrm{i}}) - K_0(z\mathrm{e}^{-\pi\mathrm{i}}) \right\}. \tag{65}$$

To derive (65), analytically continue from $\theta = \pm\pi$ to $\theta = 0$ in (60).

If one follows a similar route to the Berry-Howls analysis which leads to the representations of Stieltjes-transform type (as outlined in §2), one finds, for $y^{(1)}(z)$, an integral representation of *Hilbert-transform* type is found:

$$y^{(1)}(z) = \frac{\mathrm{e}^z}{z^{\frac{1}{2}}} \frac{4}{\pi\mathrm{i}} P\int_0^\infty \frac{t^{-\frac{1}{2}}\mathrm{e}^{-t} K_0(t)}{1 - t/z}\,\mathrm{d}t \tag{66}$$

when $\theta = 0$; elsewhere, $y^{(1)}(z)$ is defined by analytic continuation. Here, P denotes the Cauchy principal value integral. The asymptotic expansion of $y^{(1)}(z)$, valid for $-\pi/2 < \theta < \pi/2$, is then found to be

$$y^{(1)}(z) \sim -2\mathrm{i}\,\frac{\mathrm{e}^z}{z^{\frac{1}{2}}} \sum_{r=0}^{\infty} \frac{b_r}{z^r}, \tag{67}$$

where the coefficients b_r continue to be defined by (18). The remainder at optimal truncation, after N terms, is approximated by

$$2\frac{\mathrm{e}^{-z}}{z^{\frac{1}{2}}}\sum_{s=0}\frac{(-1)^s b_s}{z^s}\left[-\operatorname{erf}\left(c\sqrt{\frac{N}{2}}\right)\right] \tag{68}$$

near $\theta = 0$, where in this case c is given, not by (27), but by

$$\tfrac{1}{2}c^2 = 1 - 2z/N + \ln(2z/N)\,, \tag{69}$$

and c is approximated, not by (29), but by

$$c \approx \mathrm{i}\,(2z/N - 1) \approx -\theta\,. \tag{70}$$

We infer that as θ passes through the Stokes line $\theta = 0$, the factors in the square brackets in (68) take values which change rapidly from nearly 0 on the Stokes line itself to nearly $+1$ ($\theta > 0$) and to nearly -1 ($\theta < 0$).

The formal similarity between (68) and (30) should be noted: the square brackets have error function behaviour of erfc and erf type respectively. The error function behaviour in (68) has the effect of rapidly but smoothly switching the signs of the exponentially small contributions of the adjacent saddles as the Stokes line is crossed. As we note at the end of Appendix B, it was just this behaviour which Berry (1989) addressed in his original work on the Stokes phenomenon.

I have shown that solutions of Bi type associated with a Stokes line enjoy the specific error function behaviour (60) near the Stokes line. My definition of "the solution of Bi type associated with a Stokes line", in (61), invokes a special combination of steepest-descents contours. It is natural to speculate that the specific error function behaviour (60) might occur more generally, as for example, when integral representations of solutions are not available. I address this question in § 7.

5 Application to the gamma function

In Boyd (1994), I considered how the Berry-Howls technique could be applied to Euler's integral representation for the gamma function:

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1}\,\mathrm{d}t\,, \tag{71}$$

which is appropriate for $\mathrm{Re}(z) > 0$. The change of variable $w = \ln(t/z)$ transforms Euler's integral to a form more suitable for our purpose:

$$\Gamma(z) = z^z \int_{-\infty}^\infty \mathrm{e}^{-zp(w)}\,\mathrm{d}w\,, \tag{72}$$

again appropriate for $\mathrm{Re}(z) > 0$. Here, the function $p(w)$ is defined by

$$p(w) = \mathrm{e}^w - w\,. \tag{73}$$

It is well-known that $\Gamma(z)$ has the asymptotic expansion (e.g. Abramowitz & Stegun (1965, p. 257) or Wong (1989, p. 62)):

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \cdots \right]. \tag{74}$$

(The approximation given by the leading term is commonly referred to as Stirling's formula.) We are thereby led to define a function $S^{(0)}(z)$ by the relation

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} S^{(0)}(z), \tag{75}$$

so that

$$S^{(0)}(z) = \frac{z^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z[p(w)-p(0)]}\, dw \tag{76}$$

for $\mathrm{Re}(z) > 0$. In both equations (75) and (76), $z^{\frac{1}{2}}$ is defined to be positive when $\theta = 0$, and is defined elsewhere by analytic continuation. We have, as $z \to \infty$,

$$S^{(0)}(z) \sim 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \cdots = \sum_{r=0}^{\infty} \frac{a_r}{z^r}. \tag{77}$$

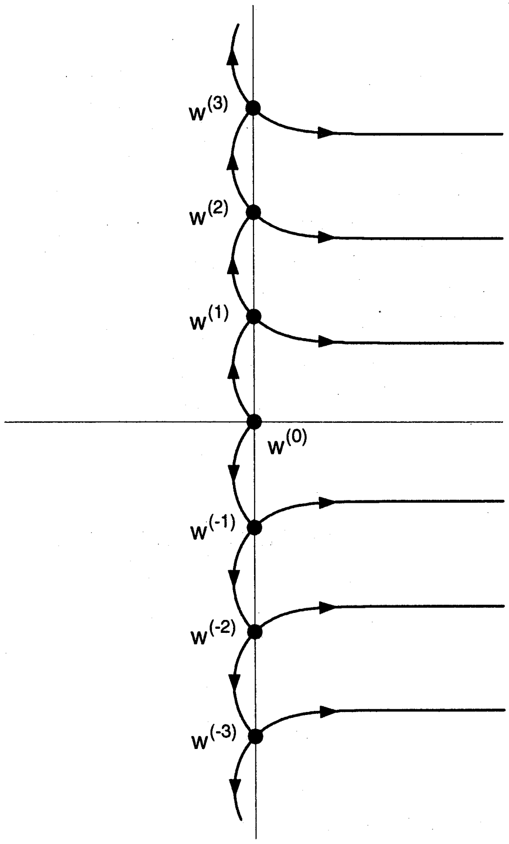

Figure 3. The paths of steepest descents from $w^{(0)}$ to the adjacent saddles $w^{(1)}, w^{(2)}, w^{(3)}, \ldots$ and $w^{(-1)}, w^{(-2)}, w^{(-3)}, \ldots$.

To apply the Berry-Howls technique to the integral representation (76) for $S^{(0)}(z)$, we first find all the saddle points of $p(w)$, namely

$$w^{(m)} = 2m\pi i, \qquad m = 0, \pm 1, \pm 2, \ldots. \tag{78}$$

Next, since the dominant contribution to $S^{(0)}(z)$ for large z comes from the saddle $w^{(0)}$, we would need to determine which saddles are adjacent to $w^{(0)}$. To determine whether the saddle point $w^{(m)}$ is adjacent to $w^{(0)}$, we need to apply the criterion (33) above. It is easy to show that $\theta^{(0m)}$ equals $-\frac{1}{2}\pi$ for each $m = 1, 2, 3, \ldots$ and $\frac{1}{2}\pi$ for each $m = -1, -2, -3, \ldots$. We therefore find (taking account of the values of $\theta^{(0m)}$) that the paths in (15) are the paths of *descent* from $w^{(0)}$ which satisfy

$$\mathrm{Re}(p(w) - 1) = 0\,, \tag{79}$$

or equivalently

$$\cos y = (1 + x)\mathrm{e}^{-x} \tag{80}$$

(writing $w = x + \mathrm{i}y$). The paths in question are depicted in Figure 3. We infer that all the saddle points $w^{(m)}$ $(m \neq 0)$ can be regarded as adjacent to $w^{(0)}$.

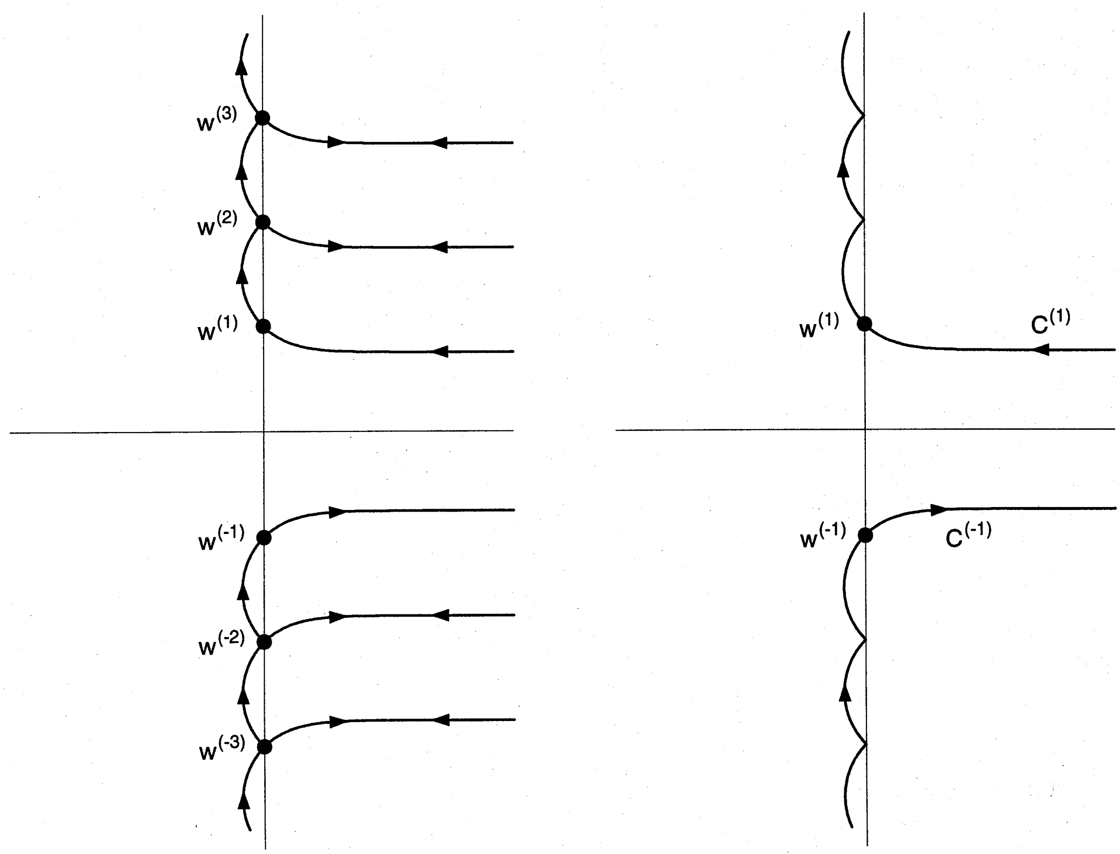

Figure 4. (a) The set of all steepest descent paths $C^{(m)}(\frac{1}{2}\pi)$ for $m > 0$ and $C^{(m)}(-\frac{1}{2}\pi)$ for $m < 0$.
(b) The choice of adjacent contours $C^{(1)}$, $C^{(-1)}$ made in this article; the domain $\Delta^{(0)}$ comprises all the points between $C^{(1)}$ and $C^{(-1)}$.

The adjacent contours $C^{(m)}$ are by definition the paths of steepest descents $C^{(m)}(-\theta^{(0m)})$ which satisfy (17): in this case they are the paths of *descent* from $w^{(m)}$ $(m \neq 0)$ which satisfy (79) (or equivalently (80)). The set of all such paths is shown in Figure 4(a). There is an ambiguity in the choice of $C^{(m)}$ in this problem. Specifically, consider $C^{(1)}$ and in particular that half of the path which leaves $w^{(1)} = 2\pi\mathrm{i}$ upwards and to the left. On reaching the point $w^{(2)} = 4\pi\mathrm{i}$, one may bifurcate either downwards and to the right (taking one to infinity) or else upwards and to the left (taking one to $w^{(3)} = 6\pi\mathrm{i}$). I choose the latter option, and continue to make the same choice as each point $w^{(3)}, w^{(4)}, w^{(5)}, \ldots$ is encountered on the path $C^{(1)}$. Thus our choice of contour $C^{(1)}$ is as shown in Figure 4(b). I likewise choose $C^{(-1)}$ to be as as shown in Figure 4(b).

Either of the two choices of steepest-descent path described in the previous paragraph is feasible in specifying the adjacent contours: we chose the latter. Had we chosen the former consistently for all m, we should have found infinite sums (over the adjacent saddles) which, it can be shown, could actually be summed explicitly. However, our choice leads to a more direct calculation. In effect then we consider only two saddles to be adjacent to $w^{(0)}$, namely $w^{(1)}$ and $w^{(-1)}$; the corresponding adjacent contours $C^{(1)}$ and $C^{(-1)}$ are those shown in Figure 4(b). The domain $\Delta^{(0)}$, whose closure comprises all the points in the w-plane which can be reached by a steepest-descent path emanating from the saddle point $w^{(0)}$, lies between $C^{(1)}$ and $C^{(-1)}$ as shown in Figure 4(b).

Proceeding with the analysis of Berry & Howls, appropriately adapted to this problem, we find that the representation of the remainder term $R_N(z)$ (47) is in this case

$$\begin{aligned} R_N(z) = {} & \frac{1}{2\pi\mathrm{i}}\,\frac{1}{(-2\pi\mathrm{i}z)^N}\int_0^\infty \frac{v^{N-1}\mathrm{e}^{-v}S^{(0)}\left(\dfrac{v}{2\pi}\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}\right)}{1+\dfrac{v}{2\pi\mathrm{i}z}}\,\mathrm{d}v \\ & -\frac{1}{2\pi\mathrm{i}}\,\frac{1}{(2\pi\mathrm{i}z)^N}\int_0^\infty \frac{v^{N-1}\mathrm{e}^{-v}S^{(0)}\left(\dfrac{v}{2\pi}\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}\right)}{1-\dfrac{v}{2\pi\mathrm{i}z}}\,\mathrm{d}v \quad , \end{aligned} \tag{81}$$

where $N \geq 1$. Since, for this asymptotic expansion,

$$S^{(0)}(z) = 1 + R_1(z)\,,$$

the special case $N = 1$ leads to the result, valid for $\mathrm{Re}(z) > 0$,

$$\begin{aligned} S^{(0)}(z) = 1 \; & - \; \frac{1}{2\pi\mathrm{i}}\int_0^\infty \frac{\mathrm{e}^{-v}S^{(0)}\left(\dfrac{v}{2\pi}\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}\right)}{v+2\pi\mathrm{i}z}\,\mathrm{d}v \\ & + \frac{1}{2\pi\mathrm{i}}\int_0^\infty \frac{\mathrm{e}^{-v}S^{(0)}\left(\dfrac{v}{2\pi}\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}\right)}{v-2\pi\mathrm{i}z}\,\mathrm{d}v \quad , \end{aligned} \tag{82}$$

a striking integral relation for $S^{(0)}(z)$ and thus, directly from (75), for $\Gamma(z)$. Details of the derivation of these results are given in Boyd (1994).

The integral relations (81) and (82) can be used for a number of purposes. Here we give two.

First, they may be used to provide error bounds for the asymptotic expansion (74). Though this is a classic, and much-used, expansion, such bounds have not been available. Thus, from (81), one finds that for $0 \leq \theta < \frac{1}{2}\pi$ and $N \geq 1$,

$$\begin{aligned} |R_N(z)| \leq \sec\theta\, & \frac{1}{(2\pi)^{N+1}}\,\frac{1}{|z|^N}\int_0^\infty v^{N-1}\mathrm{e}^{-v}\left|S^{(0)}\left(\frac{v}{2\pi}\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}\right)\right|\,\mathrm{d}v \\ & + \frac{1}{(2\pi)^{N+1}}\,\frac{1}{|z|^N}\int_0^\infty v^{N-1}\mathrm{e}^{-v}\left|S^{(0)}\left(\frac{v}{2\pi}\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}\right)\right|\,\mathrm{d}v \quad . \end{aligned} \tag{83}$$

Now, one can easily infer from the reflection formula for the gamma function that for $v > 0$,

$$\left|\Gamma\left(\frac{v}{2\pi}e^{\pm\frac{1}{2}\pi i}\right)\right|^2 = \frac{2\pi^2}{v\sinh\frac{1}{2}v}\,, \tag{84}$$

and so, for $v > 0$,

$$\left|S^{(0)}\left(\frac{v}{2\pi}e^{\pm\frac{1}{2}\pi i}\right)\right| = \frac{1}{\sqrt{1-e^{-v}}}\,. \tag{85}$$

Thus (details in Boyd (1994)), for $0 \le \theta < \frac{1}{2}\pi$, and $N \ge 1$,

$$|R_N(z)| \le (\sec\theta + 1)\frac{C_N\Gamma(N)}{(2\pi)^{N+1}|z|^N} \tag{86}$$

where $C_1 = 2$, and for $N \ge 1$

$$C_N = 1 + \frac{1}{2}\sum_{r=2}^{\infty}\frac{1}{r^N} = \frac{1}{2}(1+\zeta(N))\,. \tag{87}$$

As N increases, the constants C_N decrease monotonically from $\frac{1}{2} + \frac{1}{12}\pi^2$ (≈ 1.32) when $N = 2$ to their limiting value of 1.

A second use that one can make of the integral relations (81) and (82) is to establish the nature of the Berry smoothing for the gamma function. The nature of the Stokes phenomenon for the gamma function is interestingly specific. The Stokes lines are $\theta = \pm\pi/2$. The asymptotic expansion (77) may be shown to hold good for $|\theta| < \pi$; the *same* expansion holds good in $|\theta - \pi| < \pi$ for the function $(1-e^{2\pi iz})S^{(0)}(z)$, that is to say,

$$S^{(0)}(z) \sim \sum_{r=0}^{\infty}\frac{a_r}{z^r} \tag{88}$$

and also

$$(1-e^{2\pi iz})S^{(0)}(z) \sim \sum_{r=0}^{\infty}\frac{a_r}{z^r} \tag{89}$$

for $0 < \theta < \pi$. We see then, taking account of the convergent expansion

$$\left(1-e^{2\pi iz}\right)^{-1} = 1 + e^{2\pi iz} + e^{4\pi iz} + \ldots\,, \tag{90}$$

that the Stokes phenomenon manifests itself on the Stokes line $\theta = \pi/2$ not by the appearance of a single series of exponentially small terms, but rather by the appearance of an *infinite* number of such series.

For the gamma function, the calculation we gave earlier in demonstrating the Berry smoothing (the calculation on (16) given in the final paragraph of § 1.4) needs some modification. If the $S^{(0)}$ expressions in the integrands of (81) are replaced by the approximations (88), we find that the first integral in (81) is approximated by

$$e^{2\pi iz}\sum_{r=0}\frac{a_r}{z^r}\left[\frac{1}{2}\operatorname{erfc}\left(-i\frac{N+2\pi iz}{\sqrt{2N}}\right)\right]\,, \tag{91}$$

and the second by

$$\frac{-i}{2\sqrt{2\pi N}}(-1)^N e^{2\pi iz}\sum_{r=0}(-1)^r\frac{a_r}{z^r}\,. \tag{92}$$

The contribution from (92) is of the same order of magnitude as terms which have been neglected in the approximation (91). We infer therefore that the transitional behaviour of $R_N(z)$ from $\theta \le \frac{1}{2}\pi$ to $\theta \ge \frac{1}{2}\pi$ is essentially described by (91). It describes the Berry smoothing for the first exponentially small term on the right hand side of (90).

In concluding our discussion of the smoothing on Stokes lines, it is worth commenting on the question of whether it is feasible to find higher-order approximations than those we have given. (We have accounted for only one of the exponentially small terms in (90).) Indeed, when considering the large z asymptotics of $\ln\Gamma(z)$, both Berry (1991) and also Paris (1993, equation (A6)) (in extending the results of Paris & Wood (1992)) found that there were *infinitely* many smoothings (or Berry transitions as we have called them) on the Stokes lines. For example, across the Stokes line $\theta = \frac{1}{2}\pi$ they found separate transitions associated with each of $e^{2\pi i z}$, $e^{4\pi i z}$, $e^{6\pi i z}$, ... In contrast, we have only found smoothing associated with $e^{2\pi i z}$. As we discuss in Boyd (1994), this might be feasible, though with some effort: one would find a hyperasymptotic scheme of the kind considered by Berry & Howls (1991). As Berry (1991, p. 471) remarks, the higher-order smoothings which are apparent with $\ln\Gamma(z)$ are obscured if one considers $\Gamma(z)$ instead.

6 On the late coefficients

The behaviour of the coefficients a_r as $r \to \infty$ (the "late coefficients") in a series such as (40) is essential to the analysis presented in this article. The very notion of "optimal truncation", where the terms in the series at first diminish and then steadily diverge, depends on the asymptotic behaviour of the late coefficients. This behaviour is essentially that of a factorial divided by a power. For example, it was this behaviour of the coefficients b_r associated with $K_0(z)$ (given by (18)) that enabled us to determine the optimal truncation given by (22).

It turns out that the methods and techniques used to discuss the large z asymptotics for the integral representation of the function $f(z)$ are also appropriate to the discussion of the large r asymptotics of the late coefficients in the asymptotic expansion of $f(z)$. This connection is discussed in detail in Boyd (1995). Here we present the basic ideas and results of the two approaches that may be adopted.

In the first place, we may exploit the representation (41) for the coefficient a_r. By virtue of *Conditions A* we may deform the contour of integration $\Gamma^{(n)}$ out to the boundary of $\Delta^{(n)}$, and thus obtain

$$a_r = \frac{\Gamma(r+\frac{1}{2})}{2\pi i} \sum_m \int_{C^{(m)}} \exp\left(-(r+\tfrac{1}{2})\ln\left(p(w)-p^{(n)}\right)\right) q(w)\,dw\,, \tag{93}$$

in which a_r is represented as a sum of integrals over the adjacent contours (*cf.* Berry & Howls (1993), equation (15)). Consider how the method of steepest descents may be applied to this integral for $r \to \infty$. First, the stationary points of $\ln(p(w) - p^{(n)})$ have to be located. These occur at the zeros of $p'(w)$, that is, they coincide with the saddle points of $p(w)$. On each contour $C^{(m)}$ in (93) therefore, there is exactly one stationary point. The second stage in the method of steepest descents is to factor out the value of the exponential in each integrand of (93), and then to determine the paths of steepest descents from each saddle point. We thus find

$$a_r = \frac{\Gamma(r+\frac{1}{2})}{2\pi\mathrm{i}} \sum_m \frac{1}{[p^{(nm)}]^{r+\frac{1}{2}}} \int_{C^{(m)}} \exp\left(-(r+\tfrac{1}{2})\ln\left(\frac{p(w)-p^{(n)}}{p^{(nm)}}\right)\right) q(w)\,\mathrm{d}w\,. \tag{94}$$

For each value of m in (94) the argument of the logarithm is real and greater than or equal to 1 for values of w on $C^{(m)}$: *it follows therefore that the adjacent contours are paths of steepest descents for the integrands in (94).*

The method of steepest descents may therefore be applied directly to each of the integrals in (94) to yield, at leading order,

$$a_r = -\mathrm{i}\,\frac{\Gamma(r+\frac{1}{2})}{\sqrt{2\pi(r+\frac{1}{2})}} \sum_m \frac{q(w^{(m)})}{[p^{(nm)}]^r\,[p''(w^{(m)})]^{\frac{1}{2}}} \left[1 + \mathrm{O}\left(\frac{1}{r}\right)\right] \tag{95}$$

as $r \to \infty$. This result is indeed in the form of a "factorial divided by a power" that we referred to above. If one wished to, one could find higher terms involving descending powers of r, but the effort involved quickly makes the effort required (even with computer algebra) unacceptable.

A second way of establishing the asymptotic behaviour of the late terms is to exploit, not the representation (41) for the coefficients a_r, but another representation, derived from the representation (48) of Stieltjes-transform type for $S^{(n)}(z)$. If one expands the denominator of the integrand in (48) in descending powers of z, it is evident that

$$a_r = \frac{1}{2\pi\mathrm{i}} \sum_m \frac{1}{[p^{(nm)}]^r} \int_0^\infty v^{r-1}\,\mathrm{e}^{-v}\,S^{(m)}(v/p^{(nm)})\,\mathrm{d}v\,. \tag{96}$$

Now if $r \gg 1$, one expects only large values of v contribute significantly to the integrals in (96). Consequently, one may reasonably anticipate that the factors $S^{(m)}$ in the integrands of (96) to be well-approximated by the initial terms in (40), and so infer the formal result

$$a_r \sim \frac{1}{2\pi\mathrm{i}} \sum_m \sum_{s=0}^{\infty} \frac{\Gamma(r-s)}{[p^{(nm)}]^{r-s}}\, a_s^{(m)} \tag{97}$$

(*cf.* Berry & Howls (1991), equation (21)). Series expansions of this kind are referred to in the literature as inverse factorial series (see Olver (1994), Olde Daalhuis & Olver (1994), Boyd (1994, § 3.3)). We note that since

$$a_0^{(m)} = \sqrt{\frac{2\pi}{p''(w^{(m)})}}\, q(w^{(m)})\,, \tag{98}$$

the leading terms in (97) and (95) agree.

Of these two approaches—based on (93) and (96)—the latter offers two large advantages. These are that the higher terms in the inverse factorial series expansion (97) are as easy to find as the early ones; and that explicit error bounds for the approximations can readily be found. Neither of these statements is true for the former approach. So, for instance, in the case of the gamma function, we showed in Boyd (1994) that the coefficients a_r appearing in (77) enjoy the approximations

$$a_r = \begin{cases} \dfrac{2(-1)^{\frac{r-1}{2}}}{(2\pi)^{r+1}} \displaystyle\sum_{\substack{s=0 \\ \text{(even } s \text{ only)}}}^{M-2} (-1)^{\frac{s}{2}} a_s (2\pi)^s \Gamma(r-s) + A_M & (r = 1, 3, 5, \ldots,\ M \text{ even}) \\[2ex] \dfrac{2(-1)^{\frac{r}{2}}}{(2\pi)^{r+1}} \displaystyle\sum_{\substack{s=1 \\ \text{(odd } s \text{ only)}}}^{M-2} (-1)^{\frac{s+1}{2}} a_s (2\pi)^s \Gamma(r-s) + A_M & (r = 2, 4, 6, \ldots,\ M \text{ odd}), \end{cases} \tag{99}$$

where

$$|A_M| \le \frac{1}{(2\pi)^{r+1}}\,\pi^{-1}(2\sqrt{M}+1)C_M\Gamma(M)\Gamma(r-M), \tag{100}$$

provided $1 \le M < r$. One readily infers in particular that for large values of r,

$$a_r \sim \begin{cases} \dfrac{2(-1)^{\frac{r-1}{2}}}{(2\pi)^{r+1}}\Gamma(r) & (r \text{ odd}) \\[2ex] -\dfrac{(-1)^{\frac{r}{2}}}{6(2\pi)^r}\Gamma(r-1) & (r \text{ even}). \end{cases} \tag{101}$$

The results given in (101) agree, to leading order as $r \to \infty$, with those given by earlier authors (Dingle (1973, p. 159, equations (18), Diekmann (1975, equation (3.4)). It should be remarked, however, that these authors used approaches different from ours (a formal interpretive approach, and the method of steepest descents respectively).

7 Discussion

In this article, I have deliberately concentrated on my own contributions. But of course, the subject is being actively pursued by several workers. In particular, Olver and Olde Daalhuis have done much work, overlapping some of that presented here, but in their case focusing on the differential equation approach. Their 1994 paper (Olde Daalhuis & Olver (1994)) is a major contribution to our understanding of what has come to be known as "exponential asymptotics". Assuming only the existence of connection formulae, between the solutions of the differential equation, they were able to show that the solutions of a linear second-order differential equation could be expressed by integral representations which, for large values of the independent variable, could be regarded as of Stieltjes-transform type. This very important result shows that the analysis presented in the present article has a wider applicability than may at first be apparent.

The essential aspect of the discussion we gave at the end of § 4 is that the remainder at optimal truncation should enable one to distinguish between different dominant solutions for the class of asymptotic expansions we are considering. In particular, the solution of Bi type, characterised by erf behaviour when crossing its Stokes line, may alternatively be regarded as the dominant solution with the least remainder there (*cf.* Costin & Kruskal (1996)). Our work suggests that, for asymptotic purposes, the solutions of a linear second-order differential equation can be distinguished by their remainders at optimal truncation. It would seem feasible to use the techniques of Olde Daalhuis & Olver (1994) to demonstrate this more widely than I have done in this article.

In what may seem an entirely different direction, there has been much work done over the last two decades or so in what is often described as "exact asymptotics" (usefully summarised in the book of Sternin & Shatalov (1996) referred to earlier). The distinction between the analysis described in the present article and that approach may be simply (perhaps simplistically) described as follows. In the exact asymptotic approach, one takes an asymptotic expansion such as

$$S^{(n)}(z) \sim \sum_{r=0}^{\infty} \frac{a_r}{z^r} \tag{102}$$

as a *starting point*, and then using Borel summation, arrives (perhaps implicitly) at an integral representation of Stieltjes-transform type, such as (48). In contrast, I have taken the Stieltjes-type transform as my starting point. It seems plausible, to say the least, that the different insights and techniques of these different approaches may be brought together in a single theory. This is a task for future research.

Appendix A Conditions on $p(w), q(w)$ in domain $\Delta^{(n)}$

The conditions we shall require of $p(w)$ and $q(w)$ are closely adapted from those given by Boyd (1993, p. 501), as follows.

Conditions A

(i) The functions $p(w)$ and $q(w)$ are analytic at every point in the closure of a domain $\Delta^{(n)}$.

(ii) There is exactly one (interior) point of $\Delta^{(n)}$ which is a saddle point of $p(w)$. At this point, $w^{(n)}$, $p''(w^{(n)}) \neq 0$.

(iii) Any point w is in the closure of $\Delta^{(n)}$ if and only if it can be reached by exactly one path of steepest descents,

$$\mathrm{ph}\,(p(w) - p^{(n)}) = \text{constant},$$

emanating from $w^{(n)}$.

(iv) The boundary of $\Delta^{(n)}$ in the finite w-plane comprises the union of the adjacent contours: the contours $C^{(m)}$, each of which contains exactly one adjacent saddle point $w^{(m)}$ of $p(w)$ and on which

$$\mathrm{ph}\,(p(w) - p(w^{(m)})) = \mathrm{ph}\,(p^{(nm)}).$$

(The right hand side is defined in (10) below.) Furthermore, $p''(w^{(m)}) \neq 0$.

(v) As $|w| \to \infty$ in $\Delta^{(n)}$, $[p(w)]^{-N-\frac{1}{2}}q(w) = \mathrm{o}(w^{-1})$; moreover, $[p(w)]^{-N-\frac{1}{2}}q(w) \to 0$ sufficiently rapidly so that

$$\int_{C^{(m)}} \left| \frac{q(w)\,\mathrm{d}w}{[p(w)]^{N+\frac{1}{2}}} \right|$$

exists for each adjacent contour $C^{(m)}$.

Appendix B The terminant functions $F_p(z)$ and $\overline{F}_p(z)$

Terminant functions arise when one estimates the remainder which occurs after an asymptotic expansion has been terminated. The idea is due to Dingle: a definition substantially the same as (103) below is given by him in his book (1973), p. 407, equation (22).

Olver (1991), equation (2.9), defined the terminant function $F_p(z)$, for $\mathrm{Re}(p) > 0$, by

$$F_p(z) = \frac{1}{2\pi} \frac{\mathrm{e}^{-z}}{z^{p-1}} \int_0^\infty \frac{t^{p-1}\mathrm{e}^{-t}}{t+z}\,\mathrm{d}t \tag{103}$$

when $|\theta| < \pi/2$, and elsewhere by analytic continuation. (As in the main text, I use the notation $\theta = \mathrm{ph}(z)$.) In earlier work, Olver (1990) had defined a terminant function by

$$T_p(z) = \frac{\mathrm{e}^{p\pi\mathrm{i}}}{2\pi\mathrm{i}} \frac{\mathrm{e}^{-z}}{z^{p-1}} \int_0^\infty \frac{t^{p-1}\mathrm{e}^{-t}}{t+z}\,\mathrm{d}t, \tag{104}$$

a definition which differs from (103) only by a constant factor. Although the choice of normalisation constant in (104) is convenient for z near the Stokes line $\theta = \pi$, it is inconvenient for z near the Stokes line $\theta = -\pi$. Definition (103) yields results which are symmetric with respect to the two lines (Olver (1991), § 2.4, (iii)).

Olver (1990) gave asymptotic expansions, elementary and uniform, for $T_p(z)$ when $|z|$ was large and $p \approx |z|$. Subsequently, Olver (1991) gave expansions for $F_p(z)$ in these circumstances: in particular, he gave an expansion which was uniformly valid in a sector which included the Stokes line $\theta = \pi$, and another which was uniformly valid in a sector which included the Stokes line $\theta = -\pi$ (p. 1473, Theorem 1). The latter expansion is obtained by applying complex conjugate operations on the former. The resulting expressions themselves involve complex conjugates, and are not appropriate for our purposes below. Instead, I state the results using notation which is substantially the same as that introduced in an earlier paper (Boyd (1990), Appendix B).

Specifically, I take p to be a large positive parameter, and change the variable of integration in (103) from t to w by

$$t + z - p\ln(-t/z) = p\left(\tfrac{1}{2}w^2 + \mathrm{i}cw\right), \tag{105}$$

where the logarithm is specified to be real when $z < 0$, and is defined elsewhere by analytic continuation: it is convenient to cut the z-plane along the positive real axis. In (105), both left and right hand sides are zero when $t = z$ and $w = 0$ respectively; it is to be understood that the position of the pole at $t = -z$ is mapped to $w = 0$. The as yet unspecified constant c in (105) is determined (to within a $\pm$ sign) by the requirement that the position of the saddle point at $t = p$ is mapped to $w = -\mathrm{i}c$, that is,

$$\tfrac{1}{2}c^2 = 1 + z/p + \ln(-z/p). \tag{106}$$

Effectively, the complex parameter c (which depends on the ratio z/p) measures the proximity of the saddle at $t = p$ and pole at $t = -z$. I choose that branch of $c(z/p)$

whose Taylor expansion about $z/p = -1$ begins[1]

$$c = -\mathrm{i}\,(z/p+1) - \tfrac{1}{3}\mathrm{i}\,(z/p+1)^2 - \tfrac{7}{36}\mathrm{i}\,(z/p+1)^3 + \cdots. \tag{107}$$

Now apply the change of variable (105) to the integral (103). Recall that we have cut the z-plane along the positive real axis: I consider two cases — when $0 < \theta < 2\pi$ and when $-2\pi < \theta < 0$. We find that

$$F_p(z) = -\frac{1}{2\pi}\frac{\mathrm{e}^{p\ln(-z)}}{z^p}\int_{-\infty}^{\infty} \mathrm{e}^{-p(\frac{1}{2}w^2+\mathrm{i}cw)}\left(\frac{1}{w} + h(c,w)\right)\mathrm{d}w, \tag{108}$$

in which the contour of integration passes above the pole at $w = 0$ when $0 < \theta < 2\pi$, and below when $-2\pi < \theta < 0$. The function $h(c,w)$, which is the same in both cases, is analytic at $w = 0$. Further detailed, but routine, calculations then yield asymptotic expansions in the form

$$F_p(x\mathrm{e}^{\pm\pi\mathrm{i}}) \sim \mathrm{i}\,\mathrm{e}^{\mp p\pi\mathrm{i}}\left[\pm\frac{1}{2}\mathrm{erfc}\left(\pm c\sqrt{\frac{p}{2}}\right) - \mathrm{i}\,\frac{\mathrm{e}^{-\frac{1}{2}pc^2}}{\sqrt{2\pi p}}\sum_{s=0}^{\infty}\frac{h_s(c)}{p^s}\right] \tag{109}$$

as $p \to \infty$ for $|\mathrm{ph}(x)| < \pi$, and where $-x$ replaces z in the previous formulae for c. The upper or lower signs in (109) must be used consistently. These results were proved by Olver (1991), Theorem 1, p. 1473: actually he showed that the results are valid for $p - |x|$ bounded in the sector $|\mathrm{ph}(x)| \le 2\pi - \delta$ for arbitrarily small positive δ, and that this interval is maximal.

Although the case $p > 0$ is that most commonly encountered in applications of $F_p(z)$ to the improvement of asymptotic expansions, there is no difficulty in extending (109) to complex p: the result will be valid for $|p| - |x|$ bounded as $|p| \to \infty$ in the sector $|\mathrm{ph}(x/p)| \le 2\pi - \delta$.

The coefficients $h_s(c)$ in (109) can be found from the Taylor expansion of $h(c,w)$ about the saddle point $w = -\mathrm{i}c$. The coefficients $h_s(c)$ in (109) can be derived from those given by Olver (1991), equations (5.10), (5.6). (2.15), (2.16). Details are given in Boyd (1996).

In the present article, I need to define a second terminant function, $\overline{F}_p(x)$, which is used in special, but commonly-occurring, circumstances. I define $\overline{F}_p(x)$, for $\mathrm{Re}(p) > 0$, by

$$\overline{F}_p(x) = \frac{1}{2\pi}\frac{\mathrm{e}^x}{x^{p-1}}\,P\!\int_0^{\infty}\frac{t^{p-1}\mathrm{e}^{-t}}{t-x}\,\mathrm{d}t \tag{110}$$

when $x > 0$, and elsewhere by analytic continuation; in (110), P denotes the Cauchy principal value. Equivalently,

$$\overline{F}_p(x) = -\tfrac{1}{2}\left(\mathrm{e}^{p\pi\mathrm{i}}F_p(x\mathrm{e}^{\pi\mathrm{i}}) + \mathrm{e}^{-p\pi\mathrm{i}}F_p(x\mathrm{e}^{-\pi\mathrm{i}})\right). \tag{111}$$

Definition (110) is substantially the same as that given by Dingle (1973), p. 407, equation (23), though it should be noted that Dingle employs $\overline{\Lambda}(-x)$ where I use $\overline{F}(x)$. Note that the overbar on the F does *not* denote complex conjugate.

[1]This choice of c differs from that used by Olver (1990) and Boyd (1990), but is consistent with that used by Olver (1991).

Application of the asymptotic expansions (109) to the identity (111) yields

$$\overline{F}_p(x) \sim \tfrac{1}{2}\,\mathrm{i}\,\mathrm{erf}\Big(c\sqrt{\frac{p}{2}}\Big) - \frac{\mathrm{e}^{-\frac{1}{2}pc^2}}{\sqrt{2\pi p}} \sum_{s=0}^{\infty} \frac{h_s(c)}{p^s} \tag{112}$$

as $p \to \infty$ for $|\mathrm{ph}(x)| \leq 2\pi - \delta$ (from Theorem 1 of Olver (1991), referenced below (109)). It should be noted that, appearances notwithstanding, both terms in (112) are real when x is positive: for the first term, note that in this circumstance c is purely imaginary; for the second, note that $h(c, w)$ in (108) is real when $z < 0$ and w is real.

I remark that result (112) was effectively that employed by Berry (1989) in his pioneering work on smoothing across Stokes lines (see Berry's equations (27) and (34), which respectively treated Dawson's integral and the Airy function of the second kind, $\mathrm{Bi}(z)$.) He considered c to be small, and in this circumstance only the error function term in (112) is significant at leading order. (To be more precise, he effectively used a local approximation for small c: that obtained by taking the first term in the Taylor expansion (3.5) of Olver (1991), analogous to our expansion (107).)

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References

Abramowitz, M. & Stegun, I.A. 1965 *Handbook of mathematical functions.*
New York: Dover.

Bender, C.M. & Orszag, S.A. 1978 *Advanced mathematical methods for scientists and engineers.*
New York: McGraw-Hill.

Berry, M.V. 1989 Uniform asymptotic smoothing of Stokes' discontinuities.
Proc. R. Soc. Lond. A **422**, 7–21.

Berry, M.V. & Howls, C.J. 1991 Hyperasymptotics for integrals with saddles.
Proc. R. Soc. Lond. A **434**, 657–675.

Bleistein, N. 1966 Uniform asymptotic expansions of integrals with stationary point near algebraic singularity.
Communs. pure appl. Math. **19**, 353–370.

Bleistein, N. & Handelsman, R.A. 1975 *Asymptotic expansions of integrals.* New York: Holt, Rinehart and Winston.

Boyd, W.G.C. 1990 Stieltjes transforms and the Stokes phenomenon. *Proc. R. Soc. Lond.* A **429**, 227–246.

Boyd, W.G.C. 1993 Error bounds for the method of steepest descents. *Proc. R. Soc. Lond.* A **440**, 493–518.

Boyd, W.G.C. 1994 Gamma function asymptotics by an extension of the method of steepest descents. *Proc. R. Soc. Lond.* A **447**, 609–630.

Boyd, W.G.C. 1995 Asymptotic approximations for the late coefficients in asymptotic expansions arising in the method of steepest descents. *Methods and Applications of Analysis* **2**, 475–489.

Boyd, W.G.C. 1996 Steepest-descents integral representations for dominant solutions of linear second-order differential equations. Accepted for publication by *Methods and Applications of Analysis.*

Copson, E.T. 1965 *Asymptotic expansions.* Cambridge: Cambridge University Press.

Costin, O. & Kruskal, M.D. 1996 Optimal uniform estimates and rigorous asymptotics beyond all orders for a class of ordinary differential equations. *Proc. R. Soc. Lond.* A **452**, 1057–1085.

de Bruijn, N.G. 1958 *Asymptotic methods in analysis.* Amsterdam: North Holland.

Diekmann, O. 1975 *Asymptotic expansion of certain numbers related to the gamma function.* Report TN 80/75, Stichting Mathematisch Centrum, Amsterdam.

Dingle, R.B. 1973 *Asymptotic expansions: their derivation and interpretation.* London: Academic Press.

Écalle, J. 1981–85 *Les fonctions résurgentes, I, II, III.* Paris: Publications Mathématiques d'Orsay.

Erdélyi, A. 1956 *Asymptotic expansions.* New York: Dover.

Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. 1954 *Tables of integral transforms*, vol. 2. New York: McGraw-Hill.

Henrici, P. 1977 *Applied and computational complex analysis.* vol. 2. New York: John Wiley & Sons.

Jeffreys, H. 1962 *Asymptotic approximations.*
Oxford: Oxford University Press.

Olde Daalhuis, A.B. & Olver, F.W.J. 1994 Exponentially improved asymptotic solutions of ordinary differential equations. II Irregular singularities of rank one.
Proc. R. Soc. Lond. A **445**, 39–56.

Olver, F.W.J. 1974 *Asymptotics and special functions.*
New York: Academic Press.

Olver, F.W.J. 1990 On Stokes' phenomenon and converging factors.
In *Proc. Int. Symp. on Asymptotic and Computational Analysis,* Manitoba, Winnipeg, 1989 (ed. R. Wong), pp. 329–355. New York: Marcel Dekker.

Olver, F.W.J. 1991 Uniform, exponentially improved, asymptotic expansions for the generalized exponential integral.
SIAM J. math. Anal. **22**, pp. 1460–1474.

Paris, R.B. & Wood, A.D. 1992 Exponentially-improved asymptotics for the gamma function.
J. comp. appl. Math. **41**, 135–143.

Sternin, B.Y. & Shatalov, V.E. 1996 *Borel-Laplace transform and asymptotic theory,*
Boca Raton: CRC Press.

Voros, A. 1983 The return of the quartic oscillator: the complex WKB method.
Ann. Inst. H. Poincaré **39**, 211–338.

Wong, R. 1989 *Asymptotic approximations of integrals.*
New York: Academic Press.