数理解析研究所講究録
968巻 1996年 31-48

An Introduction to Hyperasymptotics using Borel-Laplace Transforms

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Abstract

This article will briefly review the work which has been carried out in the field of *hyperasymptotics* over the past few years (1990-1996), give references for further reading and suggest future directions of research. Briefly, hyperasymptotics is the systematic re-expansion of the remainder term in an asymptotic expansion to incorporate non-local contributions and thus generate exponentially improved analytic and numerical accuracy. Here we will examine hyperasymptotics using the Borel Laplace transform, as it appears now that this method can unify the development of asymptotic expansions from a variety of situations, be they differential equations or saddlepoint integrals. Consequently the Borel transform approach may lead to more general results in other areas.

Introduction

The idea of exponentially improved asymptotics predates the development of rigorous algebraic asymptotics by Poincaré in the 1880's. G.G. Stokes (1847) was actually studying exponentially accurate asymptotics for the Airy function Ai(z) as early as the 1840's in the practical context of calculating the supernumerary fringes of the rainbow. It was these studies that gave rise to the discovery of the *Stokes phenomenon*, whereby the asymptotic (divergent) expansions of well behaved functions have apparently exponentially small discontinuities in their form as certain lines or surfaces (Stokes lines or surfaces) in the complex plane of the control parameters are crossed. Stokes never quite mastered the problem to his own satisfaction (Stokes 1864) and published occasional papers on it for the next 50 years until his death. In short, the phenomenon arises because one tries to expand entire functions in terms of multivalued ones. The Stokes phenomenon gives rise to the connection formulae of exact analysis.

Many people worked on the Stokes phenomenon and the associated divergent series. Ideas developed in different directions. What may loosely be termed "the Anglo-American school" was dominated by the need for numerical estimates for functions in physical, applied mathematical and engineering sectors. Continentally, "the French school" (having previously stirred up feeling against divergent series) proceeded to develop elegant theories on universalities associated with formal expansions. A good review of the early twentieth century work can be found in Hardy's 1949 book on *Divergent Series.*

Both these approaches were probably united successfully for the first time in the work of the Australian/Scottish physicist R.B. Dingle in the 1950's. He realised that many of the divergent expansions of the special functions he dealt with in theoretical physics had a similar structure in the late terms: a factorial of the index over a power of an associated function called a *singulant.* He outlined how this universality could be exploited for practical calculations via the techniques of Borel resummation (Dingle 1973).

Dingle's work formed the basis for paper of M.V. Berry (Berry 1989) who again exploited the universal behaviour of the late terms. Berry Borel-summed the late terms of a quite general asymptotic expansion and showed that the Stokes phenomenon is really a sharp but smooth transition in the remainder term of a truncated series. The exponentially small additional contribution is switched on by an error function as the Stokes line is crossed. The error function is only observed if the truncation is at the least term. Subsequently it was widely and rigorously believed (Berry 1990ab, Jones 1990, Olver 1990, 1991ab, Berry 1991ab, Boyd 1990, McLeod 1992, Paris 1992ab, Berry and Howls 1994a) that the error function was universal in occurrence, but a recent paper (Chapman 1996) has demonstrated a subset of cases where this is not so.

Dingle's work was a prenatal form of the ideas of Ecalle (1981-84) who gave this subject the name *resurgence.* The main idea behind resurgence is that the universality of the late terms, the divergence of the series and the Stokes phenomenon are all intimately related to the fact that the original expansions are only locally centred about one expansion point. It is this locality which neglects the possibility of contributions from other non-local behaviour of the function and so causes the imperfect Poincaré asymptotic model. The mathematics rebels by forcing the series to diverge. In order to rectify this, many (in principle all) of the other possible singular expansion points must be considered. The asymptotic remainders must be linked to the nonlocal singularities to allow the function to *resurge*, or to be asymptotically remodelled in the locality of the new singularity. Whilst certainly widely applicable, Ecalle's work is often difficult to understand and contains little or no numerical examples.

The work of Dingle, Berry and Ecalle gave rise to the idea of controlling the remainder terms of asymptotic expansions, linking them to nonlocal behaviour and carefully re-expanding them to give exponentially improved numerical and analytic results.

The first paper on the subject of *hyperasymptotics*, Berry and Howls (1990) discussed the re-expansion near the turning points of Helmholtz-type second order linear ODEs. The approach was formal and non-rigorous, but as often with this type of approach, fairly successful. We were able to re-expand the late terms in the expansion on one formal solution in terms of the early terms of a second. A Borel resummation followed by iteration of the method led to a sequence of finite *hyperseries*. Each hyperseries contained early terms from one of the formal solutions multiplied by certain multiple integrals called *hyperterminants*, which were nevertheless of a universal form. This is the basis of all hyperasymptotic expansions: expand the function locally to finite order, identify the nonlocal contributions, write the remainder in terms of these contributions and re-expand in that distant locality. In this way, for a large parameter $|z|$, we were able to achieve a relative asymptotic improvement of $\exp(-2.386|z|)$ over first term of the original expansion. The latter method relied on truncating each hyperseries at its least term. It was discovered that this meant that each successive hyperseries shortened in length and thus led to the termination of the iterative procedure when one of the hyperseries contained but one term. The remainder term at that stage could not (and cannot) be evaluated. The philosophy of "truncation at the least term" followed from a desire to achieve a monotonically decreasing series expansion. However it was subsequently shown by Olde Daalhuis and Olver (1995) that this approach was not numerically the best scheme and that by a more careful global minimisation of the reminder term at each stage of the process, accuracies of $\exp(-n|z|)$ can be achieved after n iterations of the method.

Several papers followed extending the application of the method to special cases: 1-d saddlepoint integrals (Berry and Howls 1991, hereafter called BH), integrals involving endpoint contributions (Howls 1993), confluent hypergeometric functions (Olde Daalhuis 1992), second order ODEs in the neighbourhood of infinity (Olde Daalhuis & Olver 1994, Olde Daalhuis 1995) higher order ODEs (Olde Daalhuis 1997). In addition, Boyd (1993) rigorously proved the remainder term for integrals derived in Berry and Howls 1991. Each of these methods essentially worked by exploiting particular geometrical, algebraic or equational properties which were particular to the problem. However the results of these calculations were all very similar.

Recently, I again looked at the problem. I was trying to extend the existing work to multiple integrals for an application in the eigenvalue spectra associated with certain problems semiclassical mechanics (Berry and Howls 1994b). The geometric approach we had adopted in BH for single integrals could not be extended to the many dimensions of the multiple integrand. A new approach was needed. About this time, Olde Daalhuis (1997) considered the use of the Borel-laplace transform in a general theory of hyperasymptotics of linear ODEs of higher rank and higher order. It was realised that the Borel-Laplace approach was a more general and universal way to attack the problem of hyperasymptotics. In the subsequent sections I will outline the method in the context of 1-d saddlepoint integrals, although I emphasize that (subject to the initial problem of being able to write the function as a Laplace transform in a complex plane) the approach is quite general. Many applications of Borel-Laplace transforms already existed, for example in the work of Balian, Parisi and Voros (1979), but only now are they being used to generate exponentially improved numerical results.

Whilst the need for high precision numerical results generated from asymptotics may have been overtaken by computational power, the importance of asymptotics for checking code and analytic understanding remains of vital importance for applied mathematicians and physicists.

Borel Laplace Hyperasymptotics

The Borel transform of a function is practically its inverse Laplace transform. The idea is that if a function can be represented by a Laplace transform, the associated Borel transform may usually be expanded as a convergent power series in the Laplace integrand variable. The radius of convergence of this expansion is the distance from the expansion point to the nearest singularity in the Borel plane. It is these singularities which are the non-local contributions which the ordinary asymptotic expansion fails to deal with. The action of taking the Laplace transform over an infinite range, and beyond the finite radius of convergence of the Borel transform expansion generates the divergent asymptotic expansion. In order to overcome this, hyperasymptotics effectively analytically continues the Borel transform to the neighbourhood of the distant singularities.

Here we illustrate the method by considering one-dimensional saddlepoint integrals of the type:

$$I^{(n)}(k) = \int_{C_n(\theta_k)} dz\, g(z) \exp\{-kf(z)\} \tag{1}$$

with a large asymptotic parameter $|k|$, with $k \equiv |k|\exp(i\theta_k)$ complex. In general, simple saddlepoints of f exist, situated at $z=z_n$, where $f'(z_n)=0$, $f''(z_n)\neq 0$. These points were labelled n. (Henceforth we shall use the shorthand $f(z_n)=f_n$)

The contour $C_n(\theta_k)$ is the steepest descent path which runs through z_n, between specified asymptotic valleys of $\mathrm{Re}\{k[f(z)\text{-}f_n]\}$ at infinity (de Bruijn 1958 ch. 5, Copson 1965 ch. 7). These paths are given in the z-plane by $\mathrm{Im}\{k[f(z)\text{-}f_n]\}=0$. Without loss of generality we assume that $C_n(\theta_k)$ only initially encounters one saddlepoint, at a finite position. The functions $f(z)$ and $g(z)$ are analytic, at least in a strip including $C_n(\theta_k)$ (although this restriction may be removed later).

We shall exemplify the method by specific application to the Pearcy integral (Pearcey 1946) which arises in the theory of diffraction and more recently in the resolution of Stokes phenomena arising in certain third order differential equations (BH, Uchyiama and Berk Nevins Roberts 1982). Whilst we have already performed the hyperasymptotics for such an integral (BH), it will be useful to reproduce the results with the Borel approach and with better numerical truncations (Olde Daalhuis 1997) for the purposes of comparison. The reader should be able to generalise from there.

The Pearcey integral takes the form

$$P(k;x,y) = \int_C dz \exp\left\{ik\left(\tfrac{1}{4}z^4 + \tfrac{1}{2}xz^2 + yz\right)\right\} \tag{2}$$

where C descends into the valleys at $\infty\exp(i\pi/8)$ and $\infty\exp(5i\pi/8)$. The control parameters x and y normally correspond to real spatial coordinates. Provided the contour passes through just one of the three saddles, this has the form (1), and we could put k=1 without loss of generality (any other k can be reduced to 1 by scaling x and y). The functions f and g are given by g=1 and

$$f(z;x,y) = -i\left(\tfrac{1}{4}z^4 + \tfrac{1}{2}xz^2 + yz\right) \tag{3}$$

For the purposes of exposition, we choose the (non-physical) complex values

$$x = 7, y = 1 + i \tag{4}$$

to ensure that the magnitudes of the *singulants* (differences between saddle heights) are all different.

We start by making the transformation in

$$s = f(z) - f_n \tag{5}$$

in the locality of a saddlepoint z_n of the Pearcey integral, where

$$\left.\frac{\partial f(z;x,y)}{\partial z}\right|_{z=z_n} = -i\left(z_n^3 + xz_n^2 + y\right) = 0$$
$$f(z_n) = f_n \tag{6}$$

This converts integral (2) to the form

$$P(k;x,y) = \frac{\exp(-kf_n)}{\sqrt{k}}\sqrt{k}\int_{C_n} ds\exp(-ks)G(s;x,y)$$
$$G(s;x,y) = 1/-i\left\{[z(s)]^3 + x[z(s)] + y\right\} \tag{7}$$

The integral is now of Laplace transform type from the variable s to the asymptotic parameter k. The function G in the integrand is the Borel transform. The square roots of k have been separated out for future use. Furthermore, we can see that wherever the original saddlepoint condition (6) is satisfied (at points z_n, z_m, $m \neq n$), there will be a corresponding singularity in the Borel plane s at s=0 and $s=F_{nm}=f_m-f_n$ $m \neq n$. The form of this singularity can be deduced by a local analysis to be of square-root type in $s-s_n$ We take cuts in the ks plane as a straight line from the branch point to $+\infty$ to. The integration contours C_n are "Hankel" contours surrounding the appropriate cut (figure 1). The steepest descent path will then correspond to collapsing the Hankel contour C_n onto the cut where $\mathrm{Im}\{k(s-s_n)\}=0$, $\mathrm{Re}\{k(s-s_n)\}>0$, taking account of the discontinuity of G across the relevant cut.

For the case we study here, we find that we need only to integrate over/around the saddle/singularity n=2. The positions of the cuts here are given by

$s_1=F_{21}(=-F_{12})=-2.429559462904937... + i\ 9.601681152318827...$

$s_2=0$

$s_3=F_{13}=+5.287675788639256... + i\ 5.295388470737003...$

Note that $|F_{12}|=$ 9.904294025047193... and $|F_{13}|=$ 7.483358490796506... so that s_1 is closer to s_3 than to s_2 (figure 1).

Now it can be shown (Balian, Parisi,Voros 1979) that we can invert the series for G about any of the singularities s_n to obtain a Puisieux expansion in $\sqrt{(s-s_n)}$, with non-zero radius of convergence

$$G(s-s_n)=\frac{T_0^{(n)}}{2\sqrt{\pi}}(s-s_n)^{-\frac{1}{2}}+\frac{1}{2}\sum_{r=0}^{\infty}\frac{T_{(r+1)/2}^{(n)}}{\Gamma(r/2+1)}(s-s_n)^{r/2}. \qquad (8)$$

The radius on convergence of each Puisieux series will be determined by the nearest singularity on the *same* Riemann sheet. The coefficients are given in terms of Gegenbauer polynomials (Abramowitz and Stegun 1964)

$$T_r^{(n)}=\frac{-\sqrt{2}\,\mathrm{i}^{r+1/2}(r-1/2)!}{z_n^{4r+1}\left(3+x/z_n^2\right)^{2r+1/2}}\,\mathrm{C}_{2r}^{r+1/2}\left(\left\{\left(3+x/z_n^2\right)/2\right\}^{-1/2}\right) \qquad (9)$$

Hence if we collapse C_2 onto the cut from 0 to infinity and denote the discontinuity of G across a cut by ΔG we have that $\sqrt{s}\Delta G(s)$ is regular on $\mathrm{Im}\{ks\}=0$, $\mathrm{Re}\{ks\}>0$. Thus we may express it as a Cauchy contour integral

$$\sqrt{s}\,\Delta G(s)=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_2(s)}d\zeta\frac{\Delta G(\zeta)\sqrt{\xi}}{\zeta-s} \qquad (10)$$

where ζ-plane is a copy of the z-plane and the loop γ_2 surrounds each point s. (figure 2).

Substituting this result into (7) we obtain the following integral representation

$$P(k;x,y)=\frac{\exp(-kf_2)}{\sqrt{k}}T^{(2)}(k)$$
$$T^{(2)}(k)=\frac{\sqrt{k}}{2\pi\mathrm{i}}\int_0^{\infty}\mathrm{d}s\frac{\exp(-ks)}{\sqrt{s}}\oint_{\Gamma_2(\theta_k)}d\zeta\frac{\Delta G(\zeta)\sqrt{\xi}}{\zeta-s} \qquad (11)$$

The discontinuity of the Borel transform is now expressed as a Cauchy integral over the contour Γ_2, which closely resembles the original integration contour C_2 (figure 2, cf. figure 1). This is the vital step which will enable the remainder term to be bounded and the hyperasymptotic procedure to be implemented.

The next step is to expand binomially the denominator of (11) to finite order to recover a finite number of terms of the asymptotic expansion and a closed form for the remainder term

$$T^{(2)}(k)=\sum_{r=0}^{N-1}\frac{T_r^{(2)}}{k^r}+\frac{1}{2\pi i}\int_0^\infty ds\,\exp(-ks)s^{N-1/2}\oint_{\Gamma_2}d\zeta\,\frac{\Delta G(\zeta)}{\zeta^{N+1/2}(1-s/\zeta)} \tag{12}$$

By observing that for Pearcey the growth condition

$$\left|\frac{\Delta G(\zeta)}{\zeta^{1/2}(1-s/\zeta)}\right|=o\left(\frac{1}{\zeta}\right),\quad |\zeta|\to\infty \tag{13}$$

is satisfied, we may now deform Γ_2 about the other cuts at s_1 and s_3, with vanishing contributions from the arcs at infinity.

A scaling of the form

$$s=\frac{v\zeta}{F_{2m}},\quad m=1,3 \tag{14}$$

is now made and we then recognise that, together with the deformed Γ_2 we have two contributions from ζ integrals which can be rewritten in terms of $T^{(1)},T^{(3)}$ (cf BH).

Thus we recover the fundamental and exact resurgence expression for the truncated expansion

$$\begin{aligned}T^{(2)}(k)=\sum_{r=0}^{N-1}\frac{T_r^{(2)}}{k^r}&+\frac{1}{2\pi i}\frac{1}{(kF_{21})^N}\int_0^\infty\frac{dv\,v^{N-1}\exp(-v)}{(1-v/kF_{21})}T^{(1)}\left(\frac{v}{F_{21}}\right)\\&+\frac{1}{2\pi i}\frac{1}{(kF_{23})^N}\int_0^\infty\frac{dv\,v^{N-1}\exp(-v)}{(1-v/kF_{23})}T^{(3)}\left(\frac{v}{F_{23}}\right)\end{aligned} \tag{15}$$

From here on the procedure for developing hyperasymptotics proceeds exactly as in BH. Each of $T^{(1)},T^{(3)}$ on the RHS of (15) is re-expanded in terms of its own resurgence result of type (15) using scaled "large" parameters

ν/F_{2m}, m=1,3. Iteration of this procedure reproduces the hyperasymptotic results of the original paper.

We have still to discuss two points. The first is related to the question of *adjacency* of the saddlepoints (BH). Briefly a saddlepoint was adjacent to another if a Stokes phenomenon could occur between the two, and the expansion about one saddle could give birth to a contribution from the other. In the Borel plane, this translates to the relative positions of the two singularities on the Riemann sheet structure generated by the cuts. Two singularities n and m are said to be adjacent if the deformation of Γ_n snags on the cut from singularity m. This is equivalent to saying that the two singularities lie on the same Riemann sheet (Voros 1983). If the argument of k is varied, a Stokes phenomenon occurs when the subdominant singularity on an adjacent sheep sweeps through the cut and the contour C_n surrounding it to generate a new Hankel-type, exponentially smaller contribution (see figure 3).

Thus the problem of adjacency is converted to one of resolving the Riemann sheet structure of the transformation $s = f(z) - f_n$. Until recently this was a non-trivial problem. However, using a Borel Laplace approach, Olde Daalhuis (1997) has developed a hyperasymptotic procedure for determining the Stokes constants of expansions arising from ODEs. Due to the similarity in the methods arising from the Borel-Laplace approach, this algorithm has been quite simply adapted to resolve this structure here and in situations involving multiple integrals. It is explained in detail elsewhere (Howls 1997). A feature of the new method is that, unlike in BH, it is quite possible to determine adjacency by purely algebraic means, not resorting to geometric aspects of the problem.

In this case we determine the adjacency of the singularities by considering the simplest re-expansion of the late terms about each singularity.

We write the following suitably truncated formulae for the late $T_r^{(1)}$ (r>>1) (BH eqn 20),

$$T_r^{(1)} \approx \frac{K_{12}}{2\pi i} \sum_{s=0}^{N_{12}-1} \frac{(r-s-1)!}{F_{12}^{\;r-s}} T_s^{(2)} + \frac{K_{13}}{2\pi i} \sum_{s=0}^{N_{13}-1} \frac{(r-s-1)!}{F_{13}^{\;r-s}} T_s^{(3)} \tag{16}$$

in terms of the known $T_r^{(2)}, T_r^{(3)}$ and unknown K_{12}, K_{13}. If $|K_{ij}|$=1, the critical point is adjacent, and if $|K_{ij}|$=0 it is not. Thus if we take two such equations for different (large) values of r, we can arrive at an algebraic system to determine the K_{ij} approximately. With the correct truncations N_{ij} (Olde Daalhuis 1997) we can be sure of the errors in these results. In this case and at this approximation, the maximum errors are all less than 0.5, so that, for example, with r=10 and 11, approximate values of $|K_{12}|$=0.993... and $|K_{13}|=O(10^{-7})$ mean that we can unambiguously assign $|K_{12}|$=1 and $|K_{13}|$=0. Cycling the index n $T_r^{(n)}$ leads to the determination of $|K_{ij}|$ as below

$$|K_{ij}| = \begin{array}{cccc} i\backslash j & 1 & 2 & 3 \\ 1 & - & 1 & 0 \\ 2 & 1 & - & 1 \\ 3 & 0 & 1 & - \end{array} \tag{17}$$

As in BH, we see that even though the singularities s_1 and s_3 appear to be closer to each other than s_1 is to s_2 the late terms of the expansion about s_1 are determined by the presence of s_2 only. (This follows from the zero entries in the matrix (17)). Thus the radius of convergence is $|F_{12}|$ rather than $|F_{13}|$. The reason is that 1 is on a different Riemann sheet to 3. Thus we do not obtain direct contributions to the remainders of expansion about s_1 from the singularity at s_3 and vice versa. (Of course such contributions can and do occur after two iterations of the method, but are exponentially smaller).

The hyperasymptotic method of Berry and Howls (1991) can now be implemented using a globally minimised remainder (see equation 29 BH allowing $N_0, N_1, \ldots, N_s$ to vary at each iteration of the hyper-method). This leads to different truncations at each iteration, of identical to those derived by Olde Daalhuis (1997). The reason for this is again the similarity of the Borel-Laplace approach. The truncations along each scattering path are:

$$0^{\text{th}} \text{ level}: \quad N_2 = \min\{|F_{21}|, |F_{23}|\} = 10$$

$$1^{\text{st}} \text{ level}: \quad N_2 = \min\{|F_{21}| + |F_{12}|, |F_{23}| + |F_{32}|\} = 20$$

$$\begin{cases} N_{21} = \max\{0, N_2 - |F_{21}|\} = 10 \\ N_{23} = \max\{0, N_2 - |F_{23}|\} = 5 \end{cases}$$

$$2^{\text{nd}} \text{ level}: \quad N_2 = \min\begin{Bmatrix} |F_{21}| + |F_{12}| + |F_{21}|, |F_{21}| + |F_{12}| + |F_{23}|, \\ |F_{23}| + |F_{32}| + |F_{21}|, |F_{23}| + |F_{32}| + |F_{23}| \end{Bmatrix} = 30 \tag{18}$$

$$\begin{cases} N_{212} = \max\{0, N_2 - |F_{12}|\} = 20 \\ N_{232} = \max\{0, N_2 - |F_{32}|\} = 15 \end{cases}$$

These values of the truncations should be contrasted with equation (44) of BH. Note that the overall numerical accuracy is much greater with the better truncations, even after the 2nd hyperasymptotic stage. In BH we used three iterations to achieve an accuracy of $O(10^{-12})$ but here we reach $O(10^{-16})$ after the second iteration. Note however that with the new truncations we have used more than twice the number of terms. The hyperterminants have been calculated according to the algorithm of Olde Daalhuis (1997a).

Figures 4, 5 and 6 illustrate the hyperasymptotics stages recalculated here. Figure 7 shows the decay of the terms by modulus.

Generalisations

The calculations performed here have brought the results of BH up to date. They have also illustrated the new general approach to hyperasymptotics using the Borel-Laplace technique. This technique appears to be quite general. Similar results are obtained by Olde Daalhuis (1997) for higher order rank one equations. An application along these lines to eigenvalue asymptotics has also been made (Alvarez, Howls, Silverstone 1997), giving rise to new forms of the hyperterminants. Since the Pearcey integral satisfies a partial differential equation (Connor and Curtis 1984) we tentatively suggest that a careful analysis of certain classes of PDE's in the appropriate complex plane might also generate a hyperasymptotic procedure. Other areas of application include nonlinear differential equations, difference equations and multiple integrals (Howls 1997). It is hoped to use the Borel-Laplace approach to study the coalescence of singularities in greater detail (thereby extending the work of Berry and Howls 1993, 1994a).

Acknowledgements

The author would like to express his thanks to The Royal Society, The Newton Institute, JSPS, RIMS and to Professors Kawai and Takei for their generous hospitality whilst attending this conference.

References

Abramowitz, M. and Stegun, I.A. 1964, *Handbook of mathematical functions* (National Bureau of Standards, Washington).

Alvarez, G., Howls C.J., Silverstone H. 1997, *In preparation.*

Aoki T., Kawai T., Takei Y. 1994, in *Méthodes résurgentes, Analyse algébraique des perturbations singulieres*, ed. L.Boutet de Monvel (Hermann)

Balian, R., Parisi, G. and Voros, A. 1979, *Phys. Rev. Lett.* **41**, 141-144.

Berk, H.L., Nevins, W.M., and Roberts, K.V. 1982, J. Maths. Phys. **23**, 988-1002.

Berry, M.V. 1989, *Proc. Roy. Soc. Lond*, **A422,** 7-21.

Berry, M.V. 1990a, *Proc. Roy. Soc. Lond*, **A427,** 265-280.

Berry, M.V. 1990b, *Proc. Roy. Soc. Lond*, **A429,** 61-72.

Berry, M.V. 1991a, *Proc. Roy. Soc. Lond*, **A434,** 465-472.

Berry, M.V. 1991b, *Proc. Roy. Soc. Lond*, **A435,** 437-444.

Berry, M.V. and Howls, C.J. 1990, *Proc. Roy. Soc. Lond.*, **A430**, 653-668.

Berry, M.V. and Howls, C.J. 1991, *Proc. Roy. Soc. Lond.*, **A434**, 657-675 (Reference BH of the text).

Berry, M.V. and Howls, C.J. 1993, *Proc. Roy. Soc. Lond.*, **A443**, 107-126

Berry, M.V. and Howls, C.J. 1994a, *Proc. Roy. Soc. Lond.*, **A444**, 201-216

Berry, M.V. and Howls, C.J. 1994b, *Proc. Roy. Soc. Lond.*, **A447**, 527-555

Boyd, W.G.C. 1990, *Proc. Roy. Soc. Lond.*, **A429**, 227-246.

Boyd, W.G.C. 1993, *Proc. Roy. Soc. Lond.*, **A440**, 493-518.

Chapman, S. J. 1996, *Proc. Roy. Soc. Lond.* A, in press

Connor J.N.L. and Curtis P.R. 1984, J. Maths. Phys. **25**, 2895-2902

Copson, E.T 1965, *Asymptotic expansions* (Cambridge University Press).

De Bruijn, N.J. 1958, *Asymptotic methods in analysis* (North- Holland, Amsterdam).

Dingle, R.B. 1973, *Asymptotic expansions: their derivation and interpretation* (Academic Press: New York and London).

Écalle, J. 1981, *Les fonctions résurgentes* (3 volumes) Publ. Math. Université de Paris-Sud.

Écalle, J. 1984, *Cinq applications des fonctions résurgentes*, Preprint 84T62, Orsay.

Howls, C. J., 1993, *Proc. Roy. Soc. Lond.*, **A439**, 373-396.

Howls, C. J., 1997, preprint, Brunel University, UK.

Hardy, G. H. 1949, *Divergent Series*, (Clarendon Press Oxford)

Jones, D.S. 1990 in *Proc. Int. Symp.Asymp. & Comp. Anal.* (ed. R. Wong), pp241-264. (New York: Marcel Dekker)

McLeod, J.B. 1992, *Proc. Roy. Soc. Lond.*, **A437**, 343-354.

Olde Daalhuis, A.B 1992, IMA J. Appl. Math. **49**, 203-216

Olde Daalhuis, A.B 1995, Methods Appl. Anal. **2**, 198-211

Olde Daalhuis, A.B 1997, preprint, University of Edinburgh, UK.

Olde Daalhuis (1997a), *Hyperterminants* ,preprint,Univ. of Edinburgh, UK

Olde Daalhuis, A.B. and Olver, F.W.J. 1994, *Proc. Roy. Soc. Lond.* **A445**, 39-56.

Olde Daalhuis, A.B. and Olver, F.W.J. 1995, Methods Appl. Anal. **2**, 173-197.

Olver, F.W.J. 1990, in *Proc. Int. Symp.Asymp. & Comp. Anal.* (ed. R. Wong), pp329-355. (New York: Marcel Dekker).

Olver, F.W.J. 1991a, SIAM J. Math. Anal. **22**, 1460-1474.

Olver, F.W.J. 1991b, SIAM J. Math. Anal. **22**, 1475-1489.

Paris R.B., 1992a *J. Comp.Appl. Math.* **41**, 117-133.

Paris R.B., 1992b , *Proc. Roy. Soc. Lond.*, **A436**, 165-186.

Pearcey, T. 1946 Phil. Mag. **37**, 311-317.

Stokes, G.G. 1847, *Trans. Camb. Phil. Soc.* **9**, 379-407, reprinted in *Mathematical and Physical papers by the late Sir George Gabriel Stokes* (Cambridge University Press 1904) vol II, 329-357.

Stokes, G.G. 1864, *Trans.Camb.Phil.Soc.* **10,** 106-128, reprinted in *Mathematical and Physical papers by the late Sir George Gabriel Stokes* (Cambridge University Press 1904) vol.IV, 77-109.

Uchiyama K. 1994, in *Méthodes résurgentes, Analyse algébraique des perturbations singulieres*, ed. L.Boutet de Monvel (Hermann)

Voros, A. 1983 *Ann. Inst. Henri Poincaré* **39**, 211-338

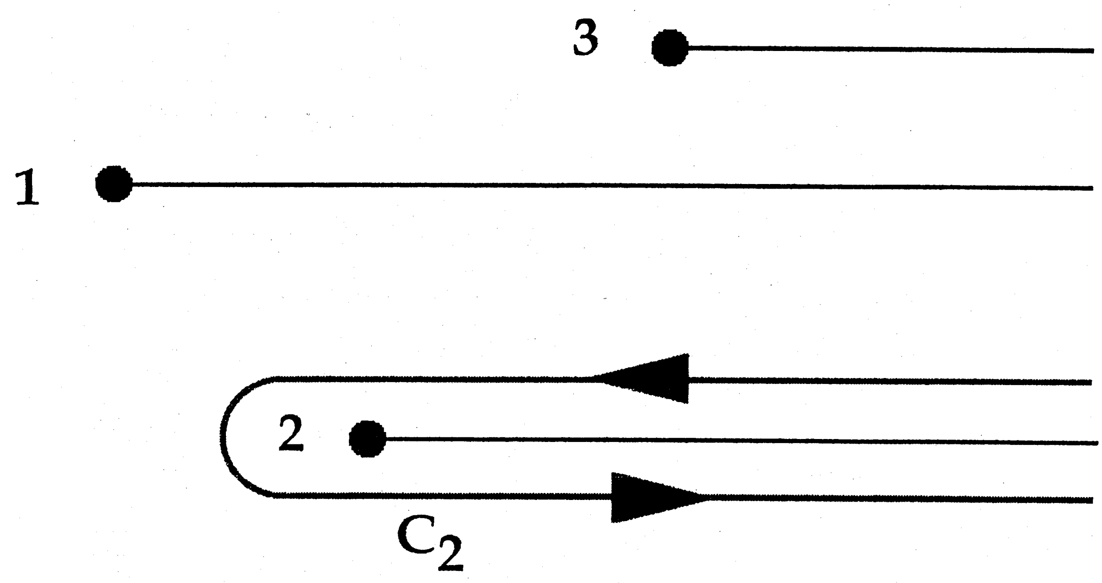

Figure 1

Figure 1: The distribution of singularities 1,2 and 3 in the ks Borel plane, corresponding to the three saddles of BH. The Hankel Contour C_2 defines the Pearcey function.

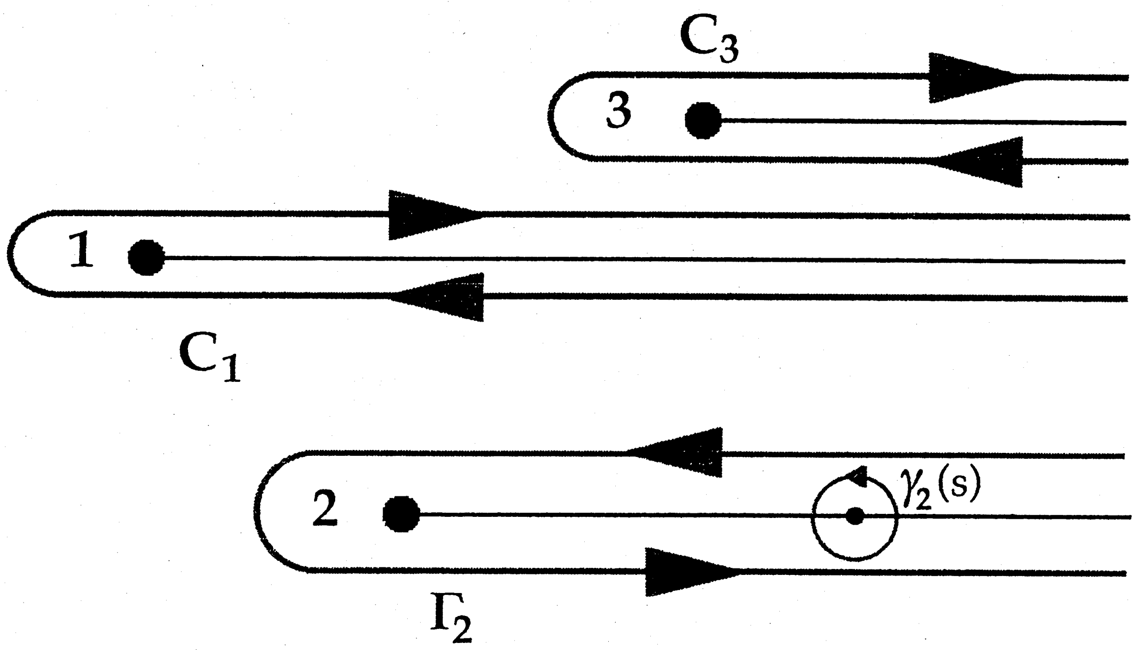

Figure 2

Figure 2: As figure 1, but now indicating the loop contour $\gamma_2(s)$ which is integrated over s to give the contour Γ_2. The latter is deformed as in the text to arcs at infinity and contours C_1 and C_3 encircling the distant singularities 1 and 3.

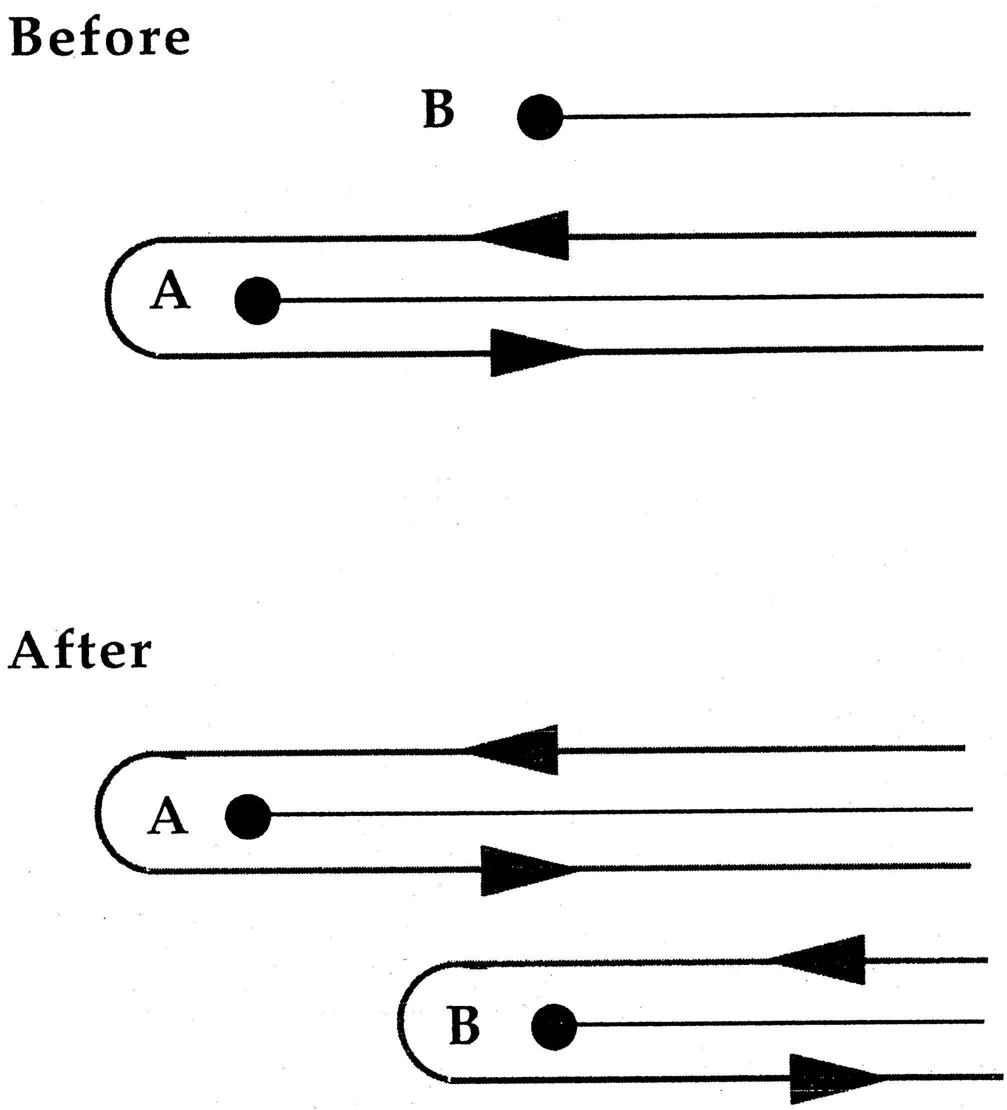

Figure 3

Figure 3: A Stokes phenomenon occurring in a Borel plane. Just before the critical values of the control parameters the only asymptotic contribution to the function comes from a loop integral about A. Singularity B moves relative to A, passes up through the cut from A and pulls with it part of the integration contour. This can be decomposed into a completely separate integration contour about B after the Stokes phenomenon. The contribution from B is necessarily exponentially smaller than that from A.

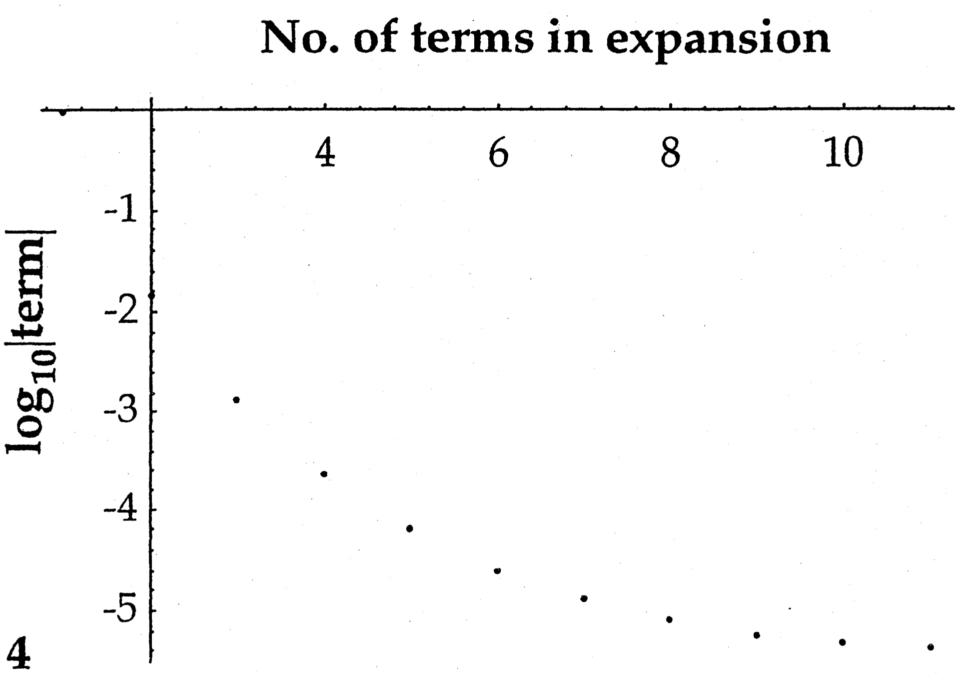

Figure 4

Figure 4: The size of the terms in the Poincaré asymptotic expansion of Pearcey down to the least (for the values of k , x and y used here this is the 10^{th}).

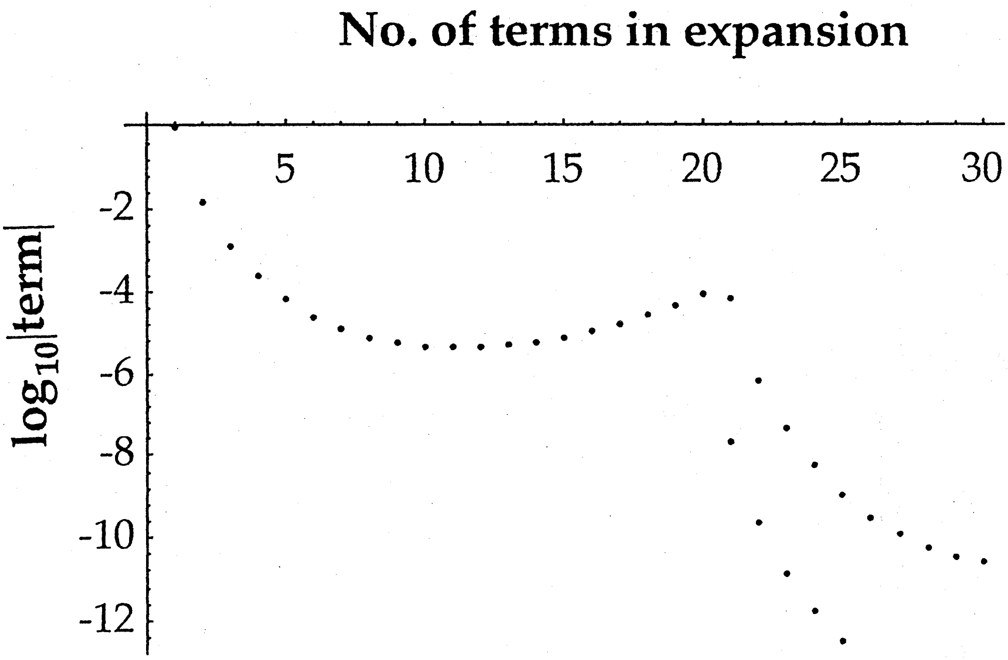

Figure 5

Figure 5: The sizes of the original Poincaré expansion, together with the size of the two hyperseries arising from the re-expansion of the remainder integral at the first hyperasymptotic stage. Both series have been truncated according to 18.

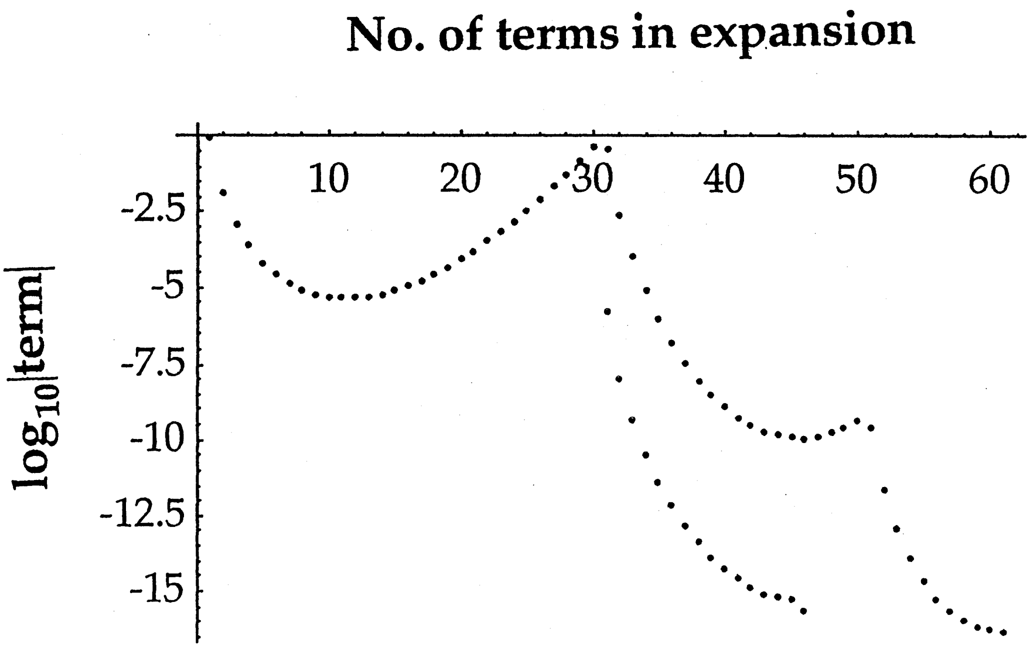

Figure 6

Figure 6: As figure 5, but including the additional hyperseries arising from second stage of hyperasymptotics. All series have been truncated according to 18.

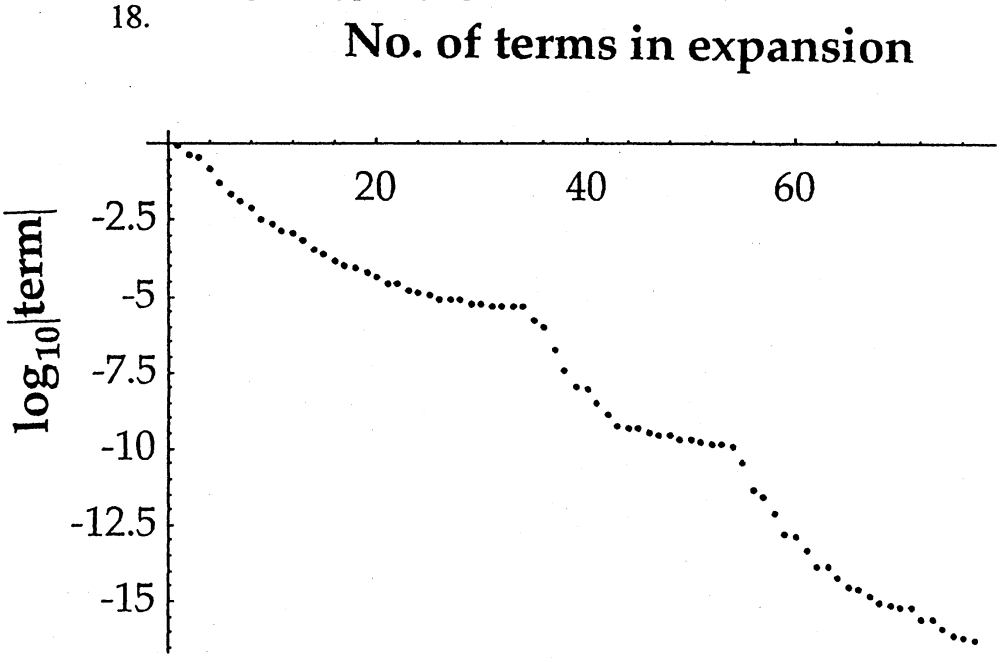

Figure 7

Figure 7: All the 77 terms used in the final calculation, arising from the hyperseries of figure 6, but now ordered sequentially according to size.